Peres criterion for separability through nonextensive entropy

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A bipartite spin-1/2 system having the probabilities $(1+3x)/4$ of being in the Einstein-Podolsky-Rosen (EPR) entangled state $|\Psi^-\rangle \equiv (1/\sqrt{2})(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)$ and $3(1-x)/4$ of being orthogonal is known to admit a local realistic description if and only if $x<1/3$ (Peres criterion). We consider here a more general case where the probabilities of being in the entangled states $|\Phi^\pm\rangle \equiv (1/\sqrt{2})(|\uparrow\rangle_A|\uparrow\rangle_B \pm |\downarrow\rangle_A|\downarrow\rangle_B)$ and $|\Psi^\pm\rangle \equiv (1/\sqrt{2})(|\uparrow\rangle_A|\downarrow\rangle_B \pm |\downarrow\rangle_A|\uparrow\rangle_B)$ (Bell basis) are given, respectively, by $(1-x)/4$, $(1-y)/4$, $(1-z)/4$, and $(1+x+y+z)/4$. Following Abe and Rajagopal, we use the nonextensive entropic form $S_q \equiv (1-\mathrm{Tr}\rho^q)/(q-1)$ $(q\in\mathcal{R}; S_1 = -\mathrm{Tr}\rho\ln\rho)$ which has enabled a current generalization of Boltzmann-Gibbs statistical mechanics, and determine the entire region in the (x,y,z) space where the system is separable. For instance, in the vicinity of the EPR state, separability occurs if and only if $x+y+z<1$, which recovers Peres' criterion when $x=y=z$. In the vicinity of the other three states of the Bell basis, the situation is identical. These results illustrate the computational power of this nonextensive-quantum-information procedure. In addition to this, a critical-phenomenon-like scenario emerges which enrichens the discussion.

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Quantum entanglement is a manifestation of the essential nonlocality of the quantum world, and a most intriguing physical phenomenon. It was first discussed as early as 1935 by Einstein, Podolsky, and Rosen (EPR) [1] and by Schrödinger [2], and has regained intensive interest in recent years due to its remarkable applications in quantum computation, teleportation, and cryptography [3–7], among others, as well as its connections to quantum chaos [8].

Two systems A and B are said to be *uncorrelated* if and only if the density operator ρ_{A+B} can be written as

$$\rho_{A+B} = \rho_A \otimes \rho_B \quad (\mathrm{Tr}_{A+B}\rho_{A+B} = \mathrm{Tr}_A\rho_A = \mathrm{Tr}_B\rho_B = 1), \tag{1}$$

i.e., if and only if

$$\rho_{A+B} = (\mathrm{Tr}_A\rho_{A+B}) \otimes (\mathrm{Tr}_B\rho_{A+B}). \tag{2}$$

Otherwise, A and B are said to be *correlated*. The concept of correlation is not distinctively classical or quantum. There is another concept, more subtle, that can exist only in quantum systems, and that is *(quantum) entanglement*. Two systems A and B are said to be *(quantum) unentangled* (or *separable*, and possibly admitting a *local* description with ‘‘hidden’’ variables, which is sometimes referred to as *local realism*) if and only if the corresponding density operator can be written as

$$\rho_{A+B} = \sum_{i=1}^{W} p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \left(p_i \geq 0\ \forall i, \sum_{i=1}^{W} p_i = 1\right). \tag{3}$$

Otherwise, A and B are said to be *entangled* (or *nonseparable*). Clearly, if A and B are uncorrelated, they are unentangled; the opposite is not true. The definition of entanglement is not necessarily simple to implement, since it might be relatively easy in a specific case to exhibit the form of Eq. (3), but it can be nontrivial to prove that it cannot be presented in that form. Consequently, through the years appreciable effort has been dedicated to the establishment of general operational criteria, preferentially in the form of necessary and sufficient conditions whenever possible. The particular case where A and B are just two simple spins 1/2 is paradigmatic, and illustrates well the relevant points.

The simplest basis for describing such systems is $|\uparrow\rangle_A|\uparrow\rangle_B$, $|\uparrow\rangle_A|\downarrow\rangle_B$, $|\downarrow\rangle_A|\uparrow\rangle_B$, and $|\downarrow\rangle_A|\downarrow\rangle_B$. All these states clearly are unentangled. Another popular basis (the Bell basis), convenient for a variety of experimental situations, is the *singlet* $|\Psi^-\rangle \equiv (1/\sqrt{2})(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)$ and $|\Psi^+\rangle \equiv (1/\sqrt{2})(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B)$, $|\Phi^\pm\rangle \equiv (1/\sqrt{2})(|\uparrow\rangle_A|\uparrow\rangle_B \pm |\downarrow\rangle_A|\downarrow\rangle_B)$ and $|\Psi^\pm\rangle$. Each state of this basis is fully entangled. The states satisfy

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$$|\Phi^+\rangle\langle\Phi^+|+|\Phi^-\rangle\langle\Phi^-|+|\Psi^+\rangle\langle\Psi^+|+|\Psi^-\rangle\langle\Psi^-| = \hat{1}_{A+B}\equiv\hat{1}_A\otimes\hat{1}_B \tag{4}$$

with $\mathrm{Tr}\hat{1}_{A+B}=2\mathrm{Tr}\hat{1}_A=2\mathrm{Tr}\hat{1}_B=4$.

We assume now that our bipartite system is in the so called Werner-Popescu state [9,10], namely,

$$\rho_{A+B}=\frac{1-x}{4}(|\Phi^+\rangle\langle\Phi^+|+|\Phi^-\rangle\langle\Phi^-|+|\Psi^+\rangle\langle\Psi^+|) +[(1+3x)/4]|\Psi^-\rangle\langle\Psi^-| \tag{5}$$

or equivalently

$$\rho_{A+B}=\frac{1-x}{4}\hat{1}_{A+B}+x|\Psi^-\rangle\langle\Psi^-| \quad (0\leqslant x\leqslant 1), \tag{6}$$

where we have used Eq. (4). For $x=1$ and $x=0$ we have the fully entangled EPR state and the fully random one, respectively. The question arises: up to what value of x is the system separable? The use of the Bell inequality yields that the threshold cannot exceed $1/\sqrt{2}\simeq 0.71$. The use of the α entropic inequality [11] yields a more severe restriction, namely, that it cannot exceed $1/\sqrt{3}\simeq 0.58$. The strongest result, i.e., the necessary and sufficient condition, was finally found (by imposing the non-negativeness of the partial transpose of the density matrix) by Peres [12], and it is $x_c=1/3$. Peres' criterion is known to be a necessary condition for all sytems, and has been shown to be also sufficient for 2×2 and 2×3 systems, whereas it is known to be insufficient for 3×3 and 2×4 or more complex systems [13,14]. In a recent paper, Abe and Rajagopal [15] reobtained Peres' $x_c=1/3$ result in an extremely elegant way. Let us briefly recall it.

Thermostatistically anomalous systems can, in some cases, be handled within a formalism that generalizes Boltzmann-Gibbs statistical mechanics. This formalism, introduced in 1988 [16], has been used in many applications [17] and has already received several verifications [18–23]. It is based on the entropic form

$$S_q=\frac{1-\mathrm{Tr}\rho^q}{q-1} \quad (q\in\mathcal{R}, \mathrm{Tr}\rho=1, S_1=-\mathrm{Tr}\rho\ln\rho). \tag{7}$$

This quantity is non-negative ($\forall q$), concave (convex) for $q>0$ ($q<0$), and satisfies the property:

$$S_q(\rho_A\otimes\rho_B)=S_q(\rho_A)+S_q(\rho_B)+(1-q)S_q(\rho_A)S_q(\rho_B). \tag{8}$$

Consequently, it is *superextensive*, *extensive*, or *subextensive* if $q<1$, $q=1$, or $q>1$, respectively. Also, it is extremal at equiprobability, i.e., $S_q(\hat{1}/W)=(W^{1-q}-1)/(1-q)$ with $S_1(\hat{1}/W)=\ln W$.

Reference [15] defines the following *conditional entropy*:

$$S_q(B|A)\equiv\frac{S_q(A+B)-S_q(A)}{1+(1-q)S_q(A)}, \tag{9}$$

where $S_q(A+B)\equiv S_q(\rho_{A+B})$ and $S_q(A)\equiv S_q(\mathrm{Tr}_B\rho_{A+B})$. The conditional entropy $S_q(A|B)$ is defined in an analogous manner. Consequently, Eq. (9) implies

$$S_q(A+B)=S_q(A)+S_q(B|A)+(1-q)S_q(A)S_q(B|A) \tag{10}$$

$$=S_q(B)+S_q(A|B)+(1-q)S_q(B)S_q(A|B). \tag{11}$$

These expressions generalize [24] Eq. (8), which is recovered as the particular case where $\rho_{A+B}=\rho_A\otimes\rho_B$ and hence $S_q(A|B)=S_q(A)$ and $S_q(B|A)=S_q(B)$. Also, for $q=1$, they reproduce one of the Shannon-Khinchin axioms for obtaining the Boltzmann-Gibbs form for the entropy, i.e., S_1. Finally, it can be shown [15] that, for all values of q, these expressions are consistent with the celebrated Bayes theorem.

The entropies $S_q(A+B)$, $S_q(A)$, and $S_q(B)$ are necessarily non-negative. This is also true for $S_q(A|B)$ and $S_q(B|A)$ if the system is classical, but not necessarily if it is quantum. Therefore, this property can be used as a criterion for separability (see also [25]). The conjecture (at least for some classes of systems) is that the system is separable if and only if both $S_q(A|B)$ and $S_q(B|A)$ are non-negative for all values of q. Let us stress that there is no general reason why $S_q(A|B)$ and $S_q(B|A)$ should be equal, not even in the classical case. Although we have no general proof, it seems plausible that both $S_q(A|B)$ and $S_q(B|A)$ decrease monotonically with q. In that case, the conjecture becomes that the system is separable if and only if both $S_\infty(A|B)$ and $S_\infty(B|A)$ are non-negative. As already stated, Abe and Rajagopal have applied this procedure to the Werner-Popescu state mentioned above, and have successfully recovered Peres' threshold $x_c=1/3$ [15].

In order to illustrate the simplicity of use of the present criterion, we shall assume the following state for the bipartite spin-1/2 system:

$$\rho_{A+B}=\frac{1-x}{4}|\Phi^+\rangle\langle\Phi^+|+\frac{1-y}{4}|\Phi^-\rangle\langle\Phi^-| +\frac{1-z}{4}|\Psi^+\rangle\langle\Psi^+|+\frac{1+x+y+z}{4}|\Psi^-\rangle\langle\Psi^-| \tag{12}$$

or equivalently

$$\rho_{A+B}=\frac{1}{4}\hat{1}_{A+B}-\frac{x}{4}|\Phi^+\rangle\langle\Phi^+|-\frac{y}{4}|\Phi^-\rangle\langle\Phi^-| -\frac{z}{4}|\Psi^+\rangle\langle\Psi^+|+(x+y+z)|\Psi^-\rangle\langle\Psi^-|, \tag{13}$$

with $x,y,z\leqslant 1$. Equations (5) and (6) are reproduced in the $x=y=z$ case. The pure states $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, and $|\Psi^-\rangle$ (EPR state), respectively, correspond to $(x,y,z)=(-3,1,1),(1,-3,1),(1,1,-3)$, and $(1,1,1)$.

Let us now calculate $S_q(A+B)$. Equation (12) implies

$$\rho^q_{A+B}=\left(\frac{1-x}{4}\right)^q|\Phi^+\rangle\langle\Phi^+|+\left(\frac{1-y}{4}\right)^q|\Phi^-\rangle\langle\Phi^-|$$
$$+\left(\frac{1-z}{4}\right)^q|\Psi^+\rangle\langle\Psi^+|+\left(\frac{1+x+y+z}{4}\right)^q|\Psi^-\rangle\langle\Psi^-|; \tag{14}$$

hence

$$S_q(A+B)=\frac{1}{1-q}\left[\left(\frac{1-x}{4}\right)^q+\left(\frac{1-y}{4}\right)^q+\left(\frac{1-z}{4}\right)^q\right.$$
$$\left.+\left(\frac{1+x+y+z}{4}\right)^q-1\right]. \tag{15}$$

Let us now calculate $S_q(A|B)$. We need to know $\rho_A=\mathrm{Tr}_B\rho_{A+B}$, i.e.,

$$\rho_A=\frac{1-x}{4}\mathrm{Tr}_B|\Phi^+\rangle\langle\Phi^+|+\frac{1-y}{4}\mathrm{Tr}_B|\Phi^-\rangle\langle\Phi^-|$$
$$+\frac{1-z}{4}\mathrm{Tr}_B|\Psi^+\rangle\langle\Psi^+|+\frac{1+x+y+z}{4}\mathrm{Tr}_B|\Psi^-\rangle\langle\Psi^-|; \tag{16}$$

hence

$$\rho_A=\frac{1}{2}\hat{1}_A, \tag{17}$$

where we have used the fact that $\mathrm{Tr}_B|\Phi^+\rangle\langle\Phi^+|=\mathrm{Tr}_B|\Phi^-\rangle\langle\Phi^-|=\mathrm{Tr}_B|\Psi^+\rangle\langle\Psi^+|=\mathrm{Tr}_B|\Psi^-\rangle\langle\Psi^-|=\frac{1}{2}\hat{1}_A$. Equation (17) implies

$$\rho^q_A=\frac{1}{2^q}\hat{1}_A; \tag{18}$$

hence

$$S_q(A)=\frac{2^{1-q}-1}{1-q}. \tag{19}$$

Substituting expressions (15) and (19) into Eq. (9) we obtain $S_q(A|B)$ as an explicit function of $(x,y,z;q)$ (see Figs. 1 and 2). Both $S_q(A+B)$ and $S_q(A|B)$ are invariant under the transformations $(x,y,z)\to(x,z,y)$, $(x,y,z)\to(-x-y-z,x,y)$, and the analogous ones. $S_q(A|B)=S_q(B|A)=0$ implies

$$\left(\frac{1-x}{4}\right)^q+\left(\frac{1-y}{4}\right)^q+\left(\frac{1-z}{4}\right)^q+\left(\frac{1+x+y+z}{4}\right)^q=\frac{1}{2^{q-1}}. \tag{20}$$

In the limit $q\to\infty$, this relation implies

$$x+y+z=1. \tag{21}$$

In other words, if the present conjecture is correct, separability is impossible in the neighborhood of the $|\Psi^-\rangle$ (EPR)

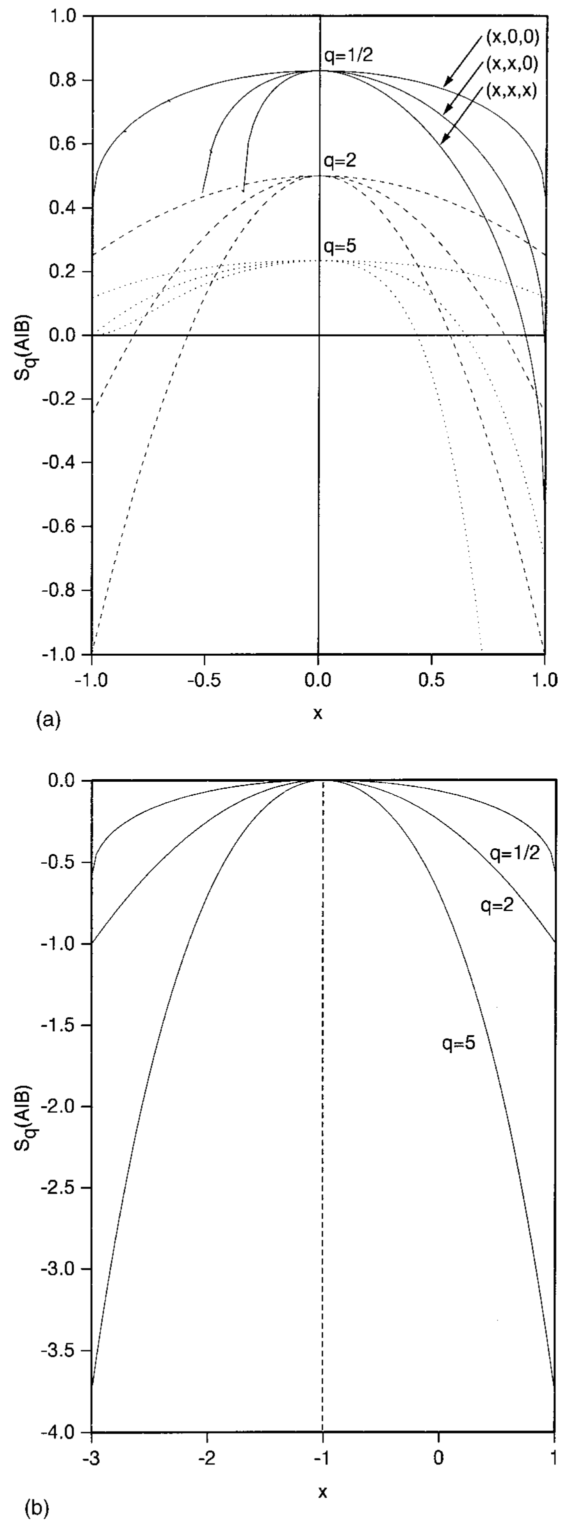

FIG. 1. $S_q(B|A)=S_q(A|B)$ versus (x,y,z) for typical values of q: (a) $q=1/2$ for the solid lines, $q=2$ for the dashed lines, and $q=5$ for the dotted lines, along the directions $(x,0,0)$, $(x,x,0)$, and (x,x,x) from top to bottom; (b) for (x,y,z) along the edge joining $|\Phi^+\rangle$ and $|\Psi^-\rangle$ or, equivalently, $|\Phi^+\rangle$ and $|\Phi^-\rangle$ (notice the symmetry with regard to the $x=-1$ axis). In fact $S_q(B|A)$ varies in the same way along the six edges of the big tetrahedron indicated in Fig. 3 below.

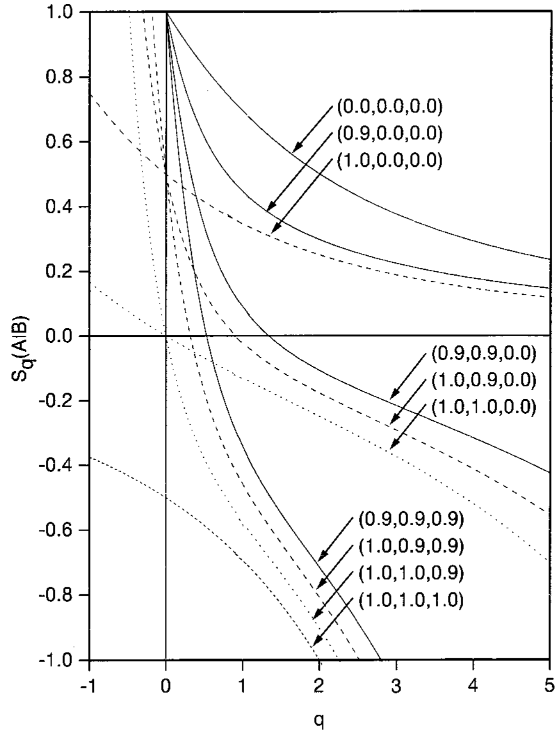

FIG. 2. $S_q(B|A)=S_q(A|B)$ versus q, for typical values of (x,y,z). The curve that for $q>0$ is the uppermost is given by $(2^{1-q}-1)/(1-q)$. The lowest curve is given by $-(2^{q-1}-1)/(q-1)$. Notice that six interesting nonuniform convergences occur at $q=0$, namely, when (i) the $(x,0,0)$ curves approach, for $x\to 1$, the (1,0,0) curve; (ii) the $(x,x,0)$ curves approach, for $x\to 1$, the (1,1,0) curve; (iii) the (x,x,x) curves approach, for $x\to 1$, the (1,1,1) curve; (iv) the $(1,x,0)$ curves approach, for $x\to 1$, the (1,1,0) curve; (v) the $(1,x,x)$ curves approach, for $x\to 1$, the (1,1,1) curve; (vi) the $(1,1,x)$ curves approach, for $x\to 1$, the (1,1,1) curve. For $q<0$, all curves, excepting the (1,1,1) one, have positive values and curvatures. The (1,1,1) curve is everywhere negative in both value and curvature.

state if and only if $x+y+z>1$. If $x=y=z$ we recover $x_c=1/3$ (Peres criterion). If all the symmetries of the problem are used, we obtain Fig. 3. We see there that the physical space is a tetrahedron included in a $4\times 4\times 4$ cube. The vertices of the tetrahedron correspond to the four states of the Bell basis. Each of these vertices is also the outer vertex of a smaller tetrahedron, inside which no separability is possible. These four smaller tetrahedra delimit an octahedron surrounding the origin $x=y=z=0$ (state of full randomness). Separability is possible if and only if (x,y,z) belongs to this octahedron. This geometry coincides with that obtained in [26] from a quite different standpoint.

Let us focus on the vicinity of the EPR state. If we observe Fig. 2 carefully we can see that all the curves such that $1<x+y+z<3$ exhibit, as functions of q, an inflection point, hereafter referred to as q_I. The inflection point runs to infinity when we approach the plane $x+y+z=1$ from above (see Fig. 3), and runs to unity when we approach the point $x=y=z=1$, varying continuously in between. In all cases where the inflection point exists, we notice that for $q>q_I$ the conditional entropy $S_q(A|B)=S_q(B|A)$ bends quickly toward

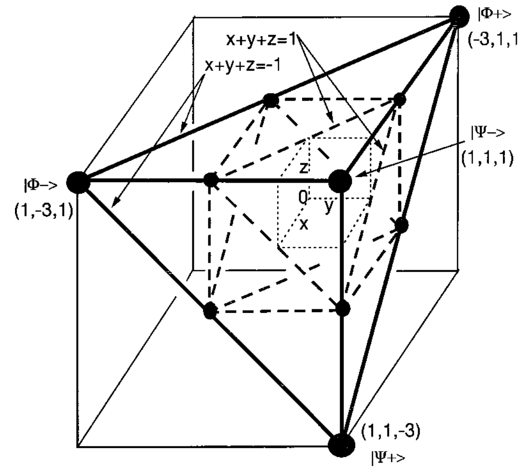

FIG. 3. The physical space of the mixed state considered in the present paper is the tetrahedron determined by the four big circles. Every big circle and its three neighboring small circles determine a region (small tetrahedron) where no separability is possible. The four small tetrahedra delimit a central octahedron where the system is separable. The $x+y+z=1$ plane (dashed) generalizes the $x_c=1/3$ Peres criterion, and plays the role of a critical surface. The entanglement ''order parameter'' $\eta\equiv 1/q_I$ is zero inside the central octahedron, and continuously increases when we approach the four vertices of the big tetrahedron, where $\eta=1$.

minus infinity. Consequently, this point is an intrinsic characteristic of the quantum entanglement between the subsystems A and B. Moreover, for convenience we can define the quantity $\eta\equiv 1/q_I\in[0,1]$, which plays a role analogous to an order parameter in standard critical phenomena. Indeed, in the whole region $0\leqslant x+y+z\leqslant 1$ we have $\eta=0$ (''separable'' phase); the region $1<x+y+z\leqslant 3$ corresponds therefore to the ''nonseparable'' phase, the entanglement ''order parameter'' η reaching unity at the (1,1,1) corner of the cube in Fig. 3. If we consider now the entire physical region (Fig. 3), we see that η vanishes inside the central octahedron described above, and is different from zero inside the four tetrahedra neighboring, respectively, the four pure states of the Bell basis; for these states it is unity. At the present stage, this critical-phenomenon-like scenario is but a suggestive analogy. Indeed, η (or any other convenient quantity related to q_I) cannot be considered as an order parameter in the thermodynamical sense unless several other properties are clearly understood, such as the symmetry that is broken if any, and the parameter thermodynamically conjugate to the order parameter (the associated susceptibility would diverge at the critical surface, i.e., the faces of the central octahedron). Further studies are needed for better understanding the implications and degree of generality of the present scheme. In particular, it is still early for dismissing the alternative possibility that the η vs x curve plays a different role, namely, that of a critical line; if this turns out to be the case, the order parameter remains to be appropriately defined.

Summarizing, we make the following points

(i) We have used the zero of the Abe-Rajagopal conditional entropy [15] [i.e., the condition $S_q(A+B)=S_q(A)$] as a criterion for separability in a bipartite spin-1/2 system in the quite general state (12), and have obtained Eq. (21). The calculation itself is in fact very simple [as simple as the non-negativity of the partially transposed density matrix [12], which can also be shown [27] to yield Eq. (21)]. It is known that Peres' criterion constitutes for all systems a necessary condition for separability. For 2×2 and 2×3 systems, it also constitutes a sufficient condition. However, for more complex systems [12,28], such as 3×3 and 2×4, it is known to be not sufficient [13,14]. The discussion of other systems (e.g., generic $M\times N$ ones, harmonic oscillators, and others) and/or of more general 2×2 states would certainly be enlightening. In particular, this would clarify the degree of generality of the Abe-Rajagopal method for determining necessary and sufficient conditions for separability [29].

(ii) We have exhibited that, through the inflection point q_I, quantum separability presents some analogies with standard critical phenomena ($1/q_I$ vanishes in the separable region and is positive in the nonseparable region, achieving its maximum value, namely unity, in the fully entangled states, such as the EPR one). Since entropies $S_q(B|A)$ with values of q above (below) q_I can be considered as strongly (weakly) entangledlike, it is quite suggestive that, in the extremely entangled cases (e.g., $x=y=z=1$), $q_I=1$, i.e., the nonextensivity effects penetrate all the way down to the Boltzmann-Gibbs-Shannon entropy ($q=1$), whereas in the separable phase (e.g., $0\leqslant x=y=z<1/3$) those effects are ''dismissed'' out to $q\to\infty$.

Generally speaking, the present work reinforces the now common understanding [11,29–36] that the connections and analogies between quantum entanglement and (nonextensive) thermodynamics are deep and fruitful.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555 (1935).
[3] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[4] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[5] A. Barenco, D. Deutsch, A. Ekert, and R. Jozsa, Phys. Rev. Lett. **74**, 4083 (1995).
[6] R. Laflamme, C. Miquel, J.P. Paz, and W.H. Zurek, Phys. Rev. Lett. **77**, 198 (1996).
[7] H. Touchette and S. Lloyd, Phys. Rev. Lett. **84**, 1156 (2000).
[8] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
[9] R.F. Werner, Phys. Rev. A **40**, 4277 (1989).
[10] S. Popescu, Phys. Rev. Lett. **72**, 797 (1994).
[11] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **210**, 377 (1996).
[12] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[13] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[14] D. Bruss and A. Peres, Phys. Rev. A **61**, 030301 (2000).
[15] S. Abe and A.K. Rajagopal, Physica A **289**, 157 (2001).
[16] C. Tsallis, J. Stat. Phys. **52**, 479 (1988); E.M.F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); **24**, 3187(E) (1991); **25**, 1019(E) (1992); C. Tsallis, R.S. Mendes, and A.R. Plastino, Physica A **261**, 534 (1998); A regularly updated bibliography on the subject is accessible at http://tsallis.cat.cbpf.br/biblio.htm.
[17] C. Tsallis, in *Nonextensive Statistical Mechanics and Thermodynamics*, edited by S.R.A. Salinas and C. Tsallis, special issue of Braz. J. Phys. **29**, 1 (1999); in *Nonextensive Statistical Mechanics and Its Applications*, edited by S. Abe and Y. Okamoto, Lecture Notes in Physics (Springer-Verlag, Berlin, 2001).
[18] P.A. Alemany and D.H. Zanette, Phys. Rev. E **49**, R956 (1994); D.H. Zanette and P.A. Alemany, Phys. Rev. Lett. **75**, 366 (1995); C. Tsallis, S.V.F Levy, A.M.C. de Souza, and R. Maynard, *ibid.* **75**, 3589 (1995); **77**, 5442(E) (1996); M. Buiatti, P. Grigolini, and A. Montagnini, *ibid.* **82**, 3383 (1999); D. Prato and C. Tsallis, Phys. Rev. E **60**, 2398 (1999); C. Budde, D. Prato, and M. Re, e-print cond-mat/0007038; A. Robledo, Phys. Rev. Lett. **83**, 2289 (1999).
[19] C. Tsallis, A.R. Plastino, and W.-M. Zheng, Chaos, Solitons Fractals **8**, 885 (1997); U.M.S. Costa, M.L. Lyra, A.R. Plastino, and C. Tsallis, Phys. Rev. E **56**, 245 (1997); M.L. Lyra and C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998); U. Tirnakli, C. Tsallis, and M.L. Lyra, Eur. Phys. J. B **10**, 309 (1999); C.R. da Silva, H.R. da Cruz, and M.L. Lyra, Braz. J. Phys. **29**, 144 (1999); V. Latora, M. Baranger, A. Rapisarda, and C. Tsallis, Phys. Lett. A **273**, 97 (2000).
[20] D.B. Walton and J. Rafelski, Phys. Rev. Lett. **84**, 31 (2000).
[21] C. Beck, Physica A **277**, 115 (2000); T. Arimitsu and N. Arimitsu, Phys. Rev. E **61**, 3237 (2000); J. Phys. A **33**, L235 (2000); C. Beck, G.S. Lewis, and H.L. Swinney, Phys. Rev. E **63**, 035303 (2001).
[22] M.L.D. Ion and D.B. Ion, Phys. Lett. B **482**, 57 (2000).
[23] I. Bediaga, E.M.F. Curado, and J. Miranda, Physica A **286**, 156 (2000); C. Beck, *ibid.* **286**, 164 (2000).
[24] S. Abe, Phys. Lett. A **271**, 74 (2000).
[25] N.J. Cerf and C. Adami, Phys. Rev. Lett. **79**, 5194 (1997); Phys. Rev. A **60**, 893 (1999); A.K. Rajagopal, in *Nonextensive Statistical Mechanics and its Applications* (Ref. [17]).
[26] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).
[27] F.C. Alcaraz (private communications).
[28] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in *Bell's theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 73; D.M. Greenberger, M.A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[29] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[30] W.H. Zurek, S. Habib, and J.P. Paz, Phys. Rev. Lett. **70**, 1187 (1993).
[31] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).
[32] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[33] D. Vitali and P. Grigolini, Phys. Lett. A **249**, 248 (1998).
[34] S. Abe and A.K. Rajagopal, Phys. Rev. A **60**, 3461 (1999).
[35] A. Vidiella-Barranco, Phys. Lett. A **260**, 335 (1999).
[36] C. Brukner and A. Zeilinger, Phys. Rev. A (to be published).