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Three-level laser dynamics with squeezed light

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We seek to analyze, employing stochastic differential equations, the squeezing and statistical properties of the light generated by a three-level laser whose cavity contains a parametric amplifier. It is found that the effect of the parametric amplifier is to increase the intracavity squeezing by a maximum of 50%.

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I. INTRODUCTION

It is well known that a degenerate parametric oscillator is a typical source of squeezed light, with a maximum of 50% intracavity noise reduction [1–5]. It has also been established that a three-level laser under certain conditions generates squeezed light [6,7]. We define a three-level laser as a quantum optical system in which three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at a certain rate into a cavity coupled to a vacuum reservoir via a single-port mirror. The squeezing in such a laser is due to the coherent superposition of the top and bottom levels.

It now appears that a highly squeezed light could be generated by a combination of these two quantum optical systems. In view of this, the main objective of this paper is to analyze the squeezing and statistical properties of the light generated by a three-level laser whose cavity contains a degenerate parametric amplifier (see Fig. 1).

Imposing the requirement that the c-number equations of evolution for the first- and second-order moments have the same forms as the corresponding operator equations [8], we obtain stochastic differential equations, associated with the normal ordering, for the dynamical variables of the cavity mode. The solutions of the resulting equations are then used to calculate the quadrature variance and the squeezing spectrum. Applying the same solutions, we also determine the antinormally ordered characteristic function with the aid of which the Q function is obtained. Finally, the Q function is used to calculate the mean photon number and the photon number distribution.

II. THE MASTER EQUATION

A three-level laser consists of a cavity into which three-level atoms in a cascade configuration are injected at a constant rate r_a and removed from the cavity after a certain time τ. We represent the top, middle, and bottom levels by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. In addition, we assume the cavity mode to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. The interaction of a three-level atom with the cavity mode can be described in the interaction picture by the Hamiltonian

$$\hat{H} = ig[\hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|) - \hat{a}(|a\rangle\langle b| + |b\rangle\langle c|)], \quad (1)$$

where g is the coupling constant and $\hat{a}$ is the annihilation operator for the cavity mode. We take the initial state of a three-level atom to be

$$|\psi_A(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle \quad (2)$$

and hence the initial density operator for a single atom has the form

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \quad (3)$$

where $\rho_{aa}^{(0)} = |C_a|^2$, $\rho_{ac}^{(0)} = C_a C_c^*$, $\rho_{ca}^{(0)} = C_c C_a^*$, and $\rho_{cc}^{(0)} = |C_c|^2$. It can be readily established that the equation of evolution of the density operator for the cavity mode has in the linear approximation the form [9]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \tfrac{1}{2} A\rho_{aa}^{(0)}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) \\ &\quad + \tfrac{1}{2}(A\rho_{cc}^{(0)} + \kappa)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) \\ &\quad + \tfrac{1}{2} A\rho_{ac}^{(0)}(\hat{\rho}\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger) \\ &\quad + \tfrac{1}{2} A\rho_{ca}^{(0)}(\hat{\rho}\hat{a}^2 + \hat{a}^2\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}), \end{aligned} \quad (4)$$

where

$$A = 2r_a g^2/\gamma^2 \quad (5)$$

is the linear gain coefficient.

Moreover, with the pump mode treated classically, a degenerate parametric amplifier is describable in the interaction picture by the Hamiltonian

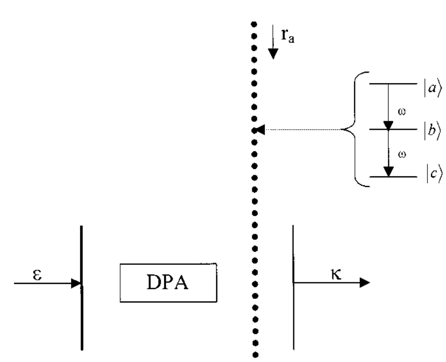

FIG. 1. A three-level laser with a degenerate parametric amplifier (DPA).

$$\hat{H}=\tfrac{1}{2}\,i\varepsilon(\hat{a}^{\dagger 2}-\hat{a}^{2}), \tag{6}$$

where ε, considered to be real and constant, is proportional to the amplitude of the pump mode. The master equation associated with this Hamiltonian has the form

$$\frac{d\hat{\rho}}{dt}=\tfrac{1}{2}\,\varepsilon(\hat{\rho}\hat{a}^{2}-\hat{a}^{2}\hat{\rho}+\hat{a}^{\dagger 2}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger 2}). \tag{7}$$

Now taking into account Eqs. (4) and (7), the master equation for the cavity mode of a three-level laser containing a parametric amplifier can be written as

$$\begin{aligned}\frac{d\hat{\rho}}{dt}=&\tfrac{1}{2}\,\varepsilon(\hat{\rho}\hat{a}^{2}-\hat{a}^{2}\hat{\rho}+\hat{a}^{\dagger 2}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger 2})\\&+\tfrac{1}{2}\,A\rho_{aa}^{(0)}(2\hat{a}^{\dagger}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}\hat{a}^{\dagger}-\hat{a}\hat{a}^{\dagger}\hat{\rho})\\&+\tfrac{1}{2}\,(A\rho_{cc}^{(0)}+\kappa)(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{\rho}\hat{a}^{\dagger}\hat{a}-\hat{a}^{\dagger}\hat{a}\hat{\rho})\\&+\tfrac{1}{2}\,A\rho_{ac}^{(0)}(\hat{\rho}\hat{a}^{\dagger 2}+\hat{a}^{\dagger 2}\hat{\rho}-2\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger})\\&+\tfrac{1}{2}\,A\rho_{ca}^{(0)}(\hat{\rho}\hat{a}^{2}+\hat{a}^{2}\hat{\rho}-2\hat{a}\hat{\rho}\hat{a}).\end{aligned} \tag{8}$$

III. STOCHASTIC DIFFERENTIAL EQUATIONS

We next seek to obtain stochastic differential equations for the cavity mode variables. To this end, applying Eq. (8) we readily find

$$\frac{d}{dt}\langle\hat{a}\rangle=-\tfrac{1}{2}\,\mu\langle\hat{a}\rangle+\varepsilon\langle\hat{a}^{\dagger}\rangle, \tag{9a}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle=-\mu\langle\hat{a}^{2}(t)\rangle+2\varepsilon\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle+\varepsilon+A\rho_{ac}^{(0)}, \tag{9b}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle=&-\mu\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle+\varepsilon\langle\hat{a}^{2}(t)\rangle+\varepsilon\langle\hat{a}^{\dagger 2}(t)\rangle\\&+A\rho_{aa}^{(0)},\end{aligned} \tag{9c}$$

in which

$$\mu=A(\rho_{cc}^{(0)}-\rho_{aa}^{(0)})+\kappa. \tag{9d}$$

On the other hand, the c-number equation corresponding to Eq. (9a) can be written as

$$\frac{d\alpha}{dt}=-\tfrac{1}{2}\,\mu\alpha+\varepsilon\alpha^{*}+f(t), \tag{10}$$

where $f(t)$ is a noise force, the properties of which remain to be determined. We see that Eq. (9a) and the expectation value of Eq. (10) will have identical forms if

$$\langle f(t)\rangle=0. \tag{11}$$

In addition, it can be easily verified using Eq. (10) that

$$\frac{d}{dt}\langle\alpha^{2}(t)\rangle=-\mu\langle\alpha^{2}(t)\rangle+2\varepsilon\langle\alpha^{*}(t)\alpha(t)\rangle+2\langle\alpha(t)f(t)\rangle, \tag{12a}$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha^{*}(t)\alpha(t)\rangle=&-\mu\langle\alpha^{*}(t)\alpha(t)\rangle+\varepsilon\langle\alpha^{2}(t)\rangle+\varepsilon\langle\alpha^{*2}(t)\rangle\\&+\langle\alpha(t)f^{*}(t)\rangle+\langle\alpha^{*}(t)f(t)\rangle.\end{aligned} \tag{12b}$$

We note that Eqs. (9b) and (12a) as well as Eqs. (9c) and (12b) will have the same forms if

$$\langle\alpha(t)f(t)\rangle=\tfrac{1}{2}\,(\varepsilon+A\rho_{ac}^{(0)}), \tag{13a}$$

$$\langle\alpha(t)f^{*}(t)\rangle+\langle\alpha^{*}(t)f(t)\rangle=A\rho_{aa}^{(0)}. \tag{13b}$$

A formal solution of Eq. (10) can be written as

$$\alpha(t)=\alpha(0)e^{-\mu t/2}+\int_{0}^{t}e^{-\mu(t-t')/2}[\varepsilon\alpha^{*}(t')+f(t')]dt'. \tag{14}$$

We then see that

$$\begin{aligned}\langle\alpha(t)f(t)\rangle=&\langle\alpha(0)f(t)\rangle e^{-\mu t/2}\\&+\int_{0}^{t}e^{-\mu(t-t')/2}[\varepsilon\langle\alpha^{*}(t')f(t)\rangle\\&+\langle f(t)f(t')\rangle]dt'.\end{aligned} \tag{15}$$

Assuming that the noise force f at time t does not affect the cavity mode variables at earlier times and taking into account Eq. (13a), we have

$$\int_{0}^{t}e^{-\mu(t-t')/2}\langle f(t)f(t')\rangle dt'=\tfrac{1}{2}\,(\varepsilon+A\rho_{ac}^{(0)}). \tag{16}$$

One can then write on the basis of this result

$$\langle f(t)f(t')\rangle=(\varepsilon+A\rho_{ac}^{(0)})\,\delta(t-t'). \tag{17a}$$

It can also be established in a similar manner that

$$\langle f^{*}(t)f(t')\rangle=A\rho_{aa}^{(0)}\,\delta(t-t'). \tag{17b}$$

It is worth mentioning that Eqs. (17a) and (17b) describe the correlation properties of the noise force $f(t)$ associated with the normal ordering.

Now introducing a new variable defined by

$$\alpha_{\pm}(t)=\alpha^{*}(t)\pm\alpha(t), \tag{18}$$

one easily gets with the aid of Eq. (10) that

$$\frac{d\alpha_{\pm}}{dt}=-\tfrac{1}{2}\,\lambda_{\mp}\alpha_{\pm}+f^{*}(t)\pm f(t), \tag{19a}$$

where

$$\lambda_{\mp}=\mu\mp 2\varepsilon. \tag{19b}$$

The solution of Eq. (19a) can be written as

$$\alpha_{\pm}(t)=\alpha_{\pm}(0)e^{-\lambda_{\mp}t/2}+\int_0^t e^{-\lambda_{\mp}(t-t')/2}[f^*(t')\pm f(t')]dt'. \tag{20}$$

It then follows that

$$\alpha(t)=A(t)\alpha(0)+B(t)\alpha^*(0)+F(t), \tag{21}$$

in which

$$A(t)=\tfrac{1}{2}\,(e^{-\lambda_- t/2}+e^{-\lambda_+ t/2}), \tag{22a}$$

$$B(t)=\tfrac{1}{2}\,(e^{-\lambda_- t/2}-e^{-\lambda_+ t/2}), \tag{22b}$$

and

$$F(t)=F_+(t)+F_-(t), \tag{23a}$$

with

$$F_{\pm}(t)=\tfrac{1}{2}\int_0^t e^{-\lambda_{\mp}(t-t')/2}[f(t')\pm f^*(t')]dt'. \tag{23b}$$

IV. QUADRATURE FLUCTUATIONS

We now proceed to calculate the quadrature variance and squeezing spectrum for the cavity mode under consideration.

A. Quadrature variance

The variance of the quadrature operators

$$\hat{a}_+=\hat{a}^\dagger+\hat{a} \tag{24a}$$

and

$$\hat{a}_-=i(\hat{a}^\dagger-\hat{a}) \tag{24b}$$

is expressible in terms of c-number variables associated with the normal ordering as

$$\Delta a_{\pm}^2=1\pm\langle\alpha_{\pm}(t),\alpha_{\pm}(t)\rangle, \tag{25}$$

in which $\alpha_{\pm}(t)$ is given by Eq. (18). We consider here the case for which the cavity mode is initially in a vacuum state. Hence on account of Eq. (20) along with Eq. (11), we see that

$$\langle\alpha_{\pm}(t)\rangle=0 \tag{26}$$

and expression (25) takes the form

$$\Delta a_{\pm}^2=1\pm\langle\alpha_{\pm}^2(t)\rangle. \tag{27}$$

Furthermore, one easily gets with the aid of Eq. (19a) that

$$\frac{d}{dt}\langle\alpha_{\pm}^2(t)\rangle=-\lambda_{\mp}\langle\alpha_{\pm}^2(t)\rangle+2\langle\alpha_{\pm}(t)f^*(t)\rangle \pm 2\langle\alpha_{\pm}(t)f(t)\rangle. \tag{28}$$

On account of Eq. (18) along with Eq. (13), we note that

$$\langle\alpha_{\pm}(t)f^*(t)\rangle=\tfrac{1}{2}\,[\varepsilon+A(\rho_{ca}^{(0)}\pm\rho_{aa}^{(0)})], \tag{29a}$$

$$\langle\alpha_{\pm}(t)f(t)\rangle=\tfrac{1}{2}\,[A\rho_{aa}^{(0)}\pm(\varepsilon+A\rho_{ac}^{(0)})]. \tag{29b}$$

Therefore, in view of this result, Eq. (28) can be rewritten as

$$\frac{d}{dt}\langle\alpha_{\pm}^2(t)\rangle=-\lambda_{\mp}\langle\alpha_{\pm}^2(t)\rangle+2\varepsilon+A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}\pm 2\rho_{aa}^{(0)}). \tag{30}$$

With the cavity mode initially in a vacuum state, the solution of this equation has the form

$$\langle\alpha_{\pm}^2(t)\rangle=\frac{2\varepsilon+A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}\pm 2\rho_{aa}^{(0)})}{\lambda_{\mp}}[1-e^{-\lambda_{\mp}t}]. \tag{31}$$

It proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)}=\frac{1-\eta}{2}, \tag{32a}$$

so that in view of the fact that

$$\rho_{aa}^{(0)}+\rho_{cc}^{(0)}=1 \tag{32b}$$

and

$$|\rho_{ac}^{(0)}|^2=\rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{32c}$$

one easily finds

$$\rho_{cc}^{(0)}=\frac{1+\eta}{2} \tag{33a}$$

and

$$|\rho_{ac}^{(0)}|=\tfrac{1}{2}\,(1-\eta^2)^{1/2}. \tag{33b}$$

Upon setting

$$\rho_{ac}^{(0)}=|\rho_{ac}^{(0)}|e^{i\theta} \tag{34}$$

and taking into account Eq. (19b) along with Eq. (9d), expression (31) can thus be put in the form

$$\langle\alpha_{\pm}^2(t)\rangle=\frac{2\varepsilon+A[(1-\eta^2)^{1/2}\cos\theta\pm(1-\eta)]}{A\eta+\kappa\mp 2\varepsilon}\times[1-e^{-(A\eta+\kappa\mp 2\varepsilon)t}]. \tag{35}$$

Now a combination of Eqs. (27) and (35) yields

$$\Delta a_{\pm}^2=1\pm\frac{2\varepsilon+A[(1-\eta^2)^{1/2}\cos\theta\pm(1-\eta)]}{A\eta+\kappa\mp 2\varepsilon}\times[1-e^{-(A\eta+\kappa\mp 2\varepsilon)t}], \tag{36}$$

so that at steady state

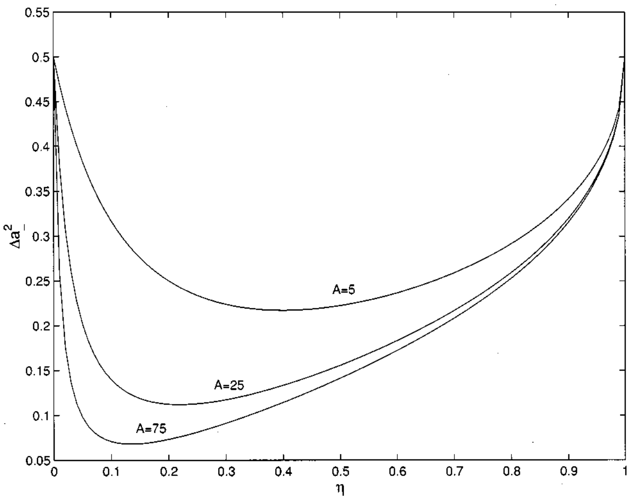

FIG. 2. Plots of the quadrature variance Δa_-^2 vs η [Eq. (38b)] for $\kappa=0.8$, $\theta=0$, and for different values of the linear gain coefficient.

$$\Delta a_+^2 = \frac{\kappa + A[1+(1-\eta^2)^{1/2}\cos\theta]}{A\eta+\kappa-2\varepsilon} \tag{37a}$$

and

$$\Delta a_-^2 = \frac{\kappa + A[1-(1-\eta^2)^{1/2}\cos\theta]}{A\eta+\kappa+2\varepsilon}. \tag{37b}$$

Since no well-behaved solution of Eq. (19a) exists for $(A\eta+\kappa)<2\varepsilon$, we interpret $A\eta+\kappa=2\varepsilon$ as the threshold condition. Hence the solution of this equation given by Eq. (20) is valid for $2\varepsilon<(A\eta+\kappa)$. On the other hand, we note from Eq. (6) that ε is the only parameter representing the parametric amplifier. And inspection of Eq. (37b) shows that the effect of this parameter is to decrease the value of the quadrature variance Δa_-^2. In addition, we see that expressions (37a) and (37b) take at threshold the form

$$\Delta a_+^2 \to \infty \tag{38a}$$

and

$$\Delta a_-^2 = \frac{\kappa + A[1-(1-\eta^2)^{1/2}\cos\theta]}{2(A\eta+\kappa)}. \tag{38b}$$

Now upon setting $\varepsilon=0$ in Eq. (37b) and comparing the resulting expression with Eq. (38b), we observe that the effect of the parametric amplifier is to increase the intracavity squeezing by a maximum of 50%. Moreover, Fig. 2 clearly shows that the degree of squeezing increases with the linear gain coefficient and it appears that almost perfect squeezing can be achieved for sufficiently large values of the linear gain coefficient.

B. Squeezing spectrum

The squeezing spectrum of a single-mode light is expressible in terms of c-number variables associated with the normal ordering as

$$S_\pm^{\text{out}}(\omega) = 1 \pm 2\,\text{Re}\int_0^\infty \langle \alpha_\pm^{\text{out}}(t), \alpha_\pm^{\text{out}}(t+\tau)\rangle_{\text{ss}} e^{i\omega\tau} d\tau, \tag{39a}$$

where the subscript ''ss'' stands for steady state and

$$\alpha_\pm^{\text{out}}(t) = \alpha_{\text{out}}^*(t) \pm \alpha_{\text{out}}(t). \tag{39b}$$

We note that for a cavity mode coupled to a vacuum reservoir, the output and intracavity variables are related by

$$\alpha_\pm^{\text{out}}(t) = \sqrt{\kappa}\,\alpha_\pm(t). \tag{40}$$

Therefore, in view of Eqs. (26) and (40), the squeezing spectrum can be put in the form

$$S_\pm^{\text{out}}(\omega) = 1 \pm 2\kappa\,\text{Re}\int_0^\infty \langle \alpha_\pm(t)\alpha_\pm(t+\tau)\rangle_{\text{ss}} e^{i\omega\tau} d\tau. \tag{41}$$

Furthermore, the solution of the expectation value of Eq. (19a) can be written as

$$\langle \alpha_\pm(t+\tau)\rangle = \langle \alpha_\pm(t)\rangle e^{-\lambda_\mp \tau/2}, \tag{42}$$

so that on account of the quantum regression theorem, we have

$$\langle \alpha_\pm(t)\alpha_\pm(t+\tau)\rangle = \langle \alpha_\pm^2(t)\rangle e^{-\lambda_\mp \tau/2}. \tag{43}$$

Now with the aid of Eq. (43) together with Eq. (35), the squeezing spectrum is found to be

$$S_\pm^{\text{out}}(\omega) = 1 \pm \frac{2\kappa\varepsilon + \kappa A[(1-\eta^2)^{1/2}\cos\theta \pm (1-\eta)]}{\omega^2 + [\frac{1}{2}(A\eta+\kappa\mp 2\varepsilon)]^2}. \tag{44}$$

It is easy to see that at threshold

$$S_+^{\text{out}}(\omega) = \frac{\omega^2+\kappa^2+\kappa A[1+(1-\eta^2)\cos\theta]}{\omega^2} \tag{45a}$$

and

$$S_-^{\text{out}}(\omega) = \frac{\omega^2+A^2\eta^2+\kappa A[1-(1-\eta^2)\cos\theta]}{\omega^2+[A\eta+\kappa]^2}. \tag{45b}$$

Figure 3 shows that there is perfect squeezing at zero frequency for any value of A and for $\eta=0$.

V. PHOTON STATISTICS

We finally wish to calculate, using the Q function, the mean photon number and the photon number distribution for the cavity mode. According to the derivation presented in the Appendix, the Q function for the cavity mode has the form

$$Q(\alpha^*,\alpha,t) = \frac{[u^2 - vv^*]^{1/2}}{\pi}\exp[-u\alpha^*\alpha + (v\alpha^2 + v^*\alpha^{*2})/2], \tag{46}$$

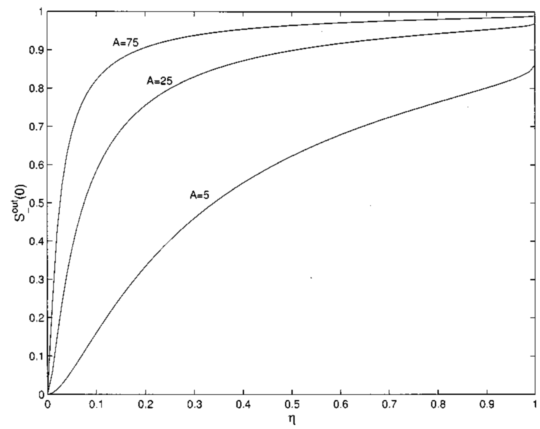

FIG. 3. Plots of the squeezing spectrum $S_-^{\rm out}(0)$ vs η [Eq. (45b)] for $\kappa=0.8$, $\theta=0$, and for different values of the linear gain coefficient.

in which

$$u=\frac{a}{a^2-bb^*},\tag{47a}$$

$$v=\frac{b}{a^2-bb^*},\tag{47b}$$

with

$$a=1+\frac{2\varepsilon+A[1-\eta+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa-2\varepsilon)}[1-e^{-(A\eta+\kappa-2\varepsilon)t}]-\frac{2\varepsilon+A[\eta-1+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa+2\varepsilon)}[1-e^{-(A\eta+\kappa+2\varepsilon)t}],\tag{48a}$$

$$b=\frac{2\varepsilon+A[1-\eta+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa-2\varepsilon)}[1-e^{-(A\eta+\kappa-2\varepsilon)t}]+\frac{2\varepsilon+A[\eta-1+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa+2\varepsilon)}[1-e^{-(A\eta+\kappa+2\varepsilon)t}]+\frac{iA(1-\eta^2)^{1/2}\sin\theta}{2(A\eta+\kappa)}[1-e^{-(A\eta+\kappa)t}].\tag{48b}$$

The mean photon number can be written employing the Q function (46) as

$$\langle\hat a^\dagger\hat a\rangle=-\frac{1}{\pi}[u^2-vv^*]^{1/2}\frac{d}{du}\int d^2\alpha\,\exp[-u\alpha^*\alpha+(v^*\alpha^{*2}+v\alpha^2)/2]-1,\tag{49}$$

so that on performing the integration, there follows

$$\langle\hat a^\dagger\hat a\rangle=-[u^2-vv^*]^{1/2}\frac{d}{du}\left[\frac{1}{u^2-vv^*}\right]^{1/2}-1.\tag{50}$$

Therefore, carrying out the differentiation and taking into account Eq. (47) along with Eq. (48a), one readily obtains

$$\langle\hat a^\dagger\hat a\rangle=\frac{2\varepsilon+A[1-\eta+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa-2\varepsilon)}\times[1-e^{-(A\eta+\kappa-2\varepsilon)t}]-\frac{2\varepsilon+A[\eta-1+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa+2\varepsilon)}\times[1-e^{-(A\eta+\kappa+2\varepsilon)t}].\tag{51}$$

It is not hard to observe that the parametric amplifier contributes significantly to the mean photon number when the system is operating particularly near threshold.

Furthermore, the photon number distribution for a single-mode light is expressible in terms of the Q function as [4,5]

$$P(n,t)=\frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}[Q(\alpha^*,\alpha,t)e^{\alpha^*\alpha}]_{\alpha=\alpha^*=0}.\tag{52}$$

Thus with the aid of Eqs. (46) and (52), the photon number distribution for the cavity mode can be written in the form

$$P(n,t)=\frac{1}{n!}[u^2-vv^*]^{1/2}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\exp[(1-u)\alpha^*\alpha+(v^*\alpha^{*2}+v\alpha^2)/2]_{\alpha^*=\alpha=0}.\tag{53}$$

Now expanding the exponential functions in power series, we have

$$P(n,t)=\frac{1}{n!}[u^2-vv^*]^{1/2}\sum_{klm}\frac{(1-u)^k v^{*l}v^m}{2^{l+m}k!l!m!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\times[(\alpha^*)^{k+2l}\alpha^{k+2m}]_{\alpha^*=\alpha=0},\tag{54}$$

so that on carrying out the differentiation and applying the condition $\alpha=\alpha^*=0$, there follows

$$P(n,t)=\frac{1}{n!}[u^2-vv^*]^{1/2}\sum_{klm}\times\frac{(1-u)^k v^{*l}v^m(k+2l)!(k+2m)!}{2^{l+m}k!l!m!(k+2l-n)!(k+2m-n)!}\times\delta_{k+2l,n}\delta_{k+2m,n}.\tag{55}$$

Finally, on account of the result that $m=l$ and $k=n-2l$, the photon number distribution can be written as

$$P(n,t)=[u^2-vv^*]^{1/2}\sum_{l=0}^{[n]}n!\frac{(1-u)^{n-2l}(vv^*)^l}{2^{2l}l!^2(n-2l)!},\tag{56}$$

where $[n]=n/2$ for even n and $[n]=(n-1)/2$ for odd n.

Figure 4 indicates that the probability of finding an even number of photons is greater than the probability of finding an odd number of photons, whether the light is produced by a three-level laser with or without a parametric amplifier. This is because the photons are always generated in pairs and the existence of some finite probability to find an odd number of photons is due to damping of the cavity mode. We

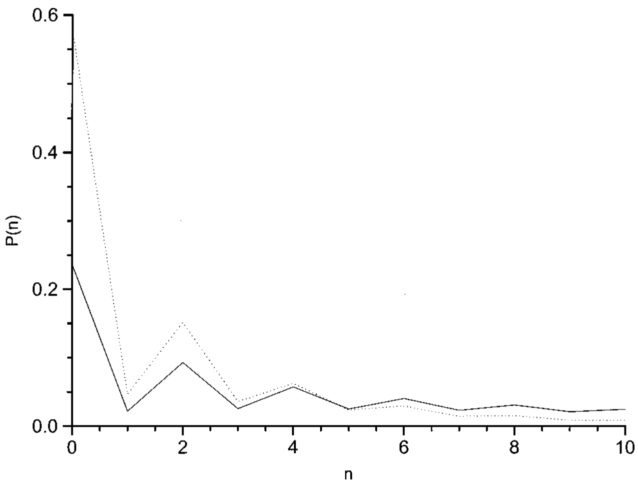

FIG. 4. Plots of the steady-state photon number distribution $P(n)$ for $A=25$, $\eta=0.2$, $\kappa=0.8$, $\theta=0$, and for $\varepsilon=0$ (dotted curve) and 2.5 (solid curve).

also see that the probability of finding n photons, with $n\leqslant 4$, is smaller for the light generated by the three-level laser with a parametric amplifier than for that produced without a parametric amplifier. And the opposite of this holds for $n\geqslant 5$.

VI. CONCLUSION

We have obtained stochastic differential equations, by demanding that the c-number equations of evolution for the first- and second-order moments have the same forms as the corresponding operator equations for the cavity mode of a three-level laser with a parametric amplifier. Applying the solutions of these equations, we have calculated the quadrature variance and the squeezing spectrum. We have also obtained, with the aid of the same solutions, the mean photon number and the photon number distribution.

We have shown that the effect of the parametric amplifier is to increase the intracavity squeezing by a maximum of 50%. We have also seen that the amount of squeezing increases with the linear gain coefficient A and almost perfect squeezing can be obtained for substantially large values of A. Moreover, the squeezing spectrum of the output light indicates that perfect squeezing can be achieved at zero frequency for any value of A and for $\eta=0$. Since the presence of the parametric amplifier also leads to a significant increase in the mean photon number, the system under consideration can produce a bright and highly squeezed light.

APPENDIX: THE Q FUNCTION

In this appendix, we calculate the Q function for the cavity mode under consideration. The Q function is expressible in the form

$$Q(\alpha^*,\alpha,t)=\frac{1}{\pi^2}\int d^2z\ \phi(z^*,z,t)\exp(z^*\alpha-z\alpha^*), \tag{A1}$$

where the antinormally ordered characteristic function $\phi(z^*,z,t)$ is defined in the Heisenberg picture by

$$\phi(z^*,z,t)=\mathrm{Tr}(\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}). \tag{A2}$$

Applying the identity

$$e^{\hat{A}}e^{\hat{B}}=e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{A3}$$

the expression for the characteristic function can be written in terms of c-number variables associated with the normal ordering as

$$\phi(z^*,z,t)=e^{-z^*z}\langle\exp(z\alpha^*-z^*\alpha)\rangle, \tag{A4}$$

so that employing Eq. (21) and assuming that $\alpha(0)$ is independent of the noise force $F(t)$, we get

$$\begin{aligned}\phi(z^*,z,t)=e^{-z^*z}&\langle\exp[(zA-z^*B)\alpha^*(0)\\&+(zB-z^*A)\alpha(0)]\rangle\\&\times\langle\exp(zF^*-z^*F)\rangle.\end{aligned} \tag{A5}$$

Considering the cavity mode to be initially in a vacuum state, we see that

$$\langle\exp[(zA-z^*B)\alpha^*(0)+(zB-z^*A)\alpha(0)]\rangle=1 \tag{A6}$$

and hence

$$\phi(z^*,z,t)=e^{-z^*z}\langle\exp(zF^*-z^*F)\rangle. \tag{A7}$$

On account of the fact that F is a Gaussian random variable, one can express Eq. (A7) in the form [10]

$$\phi(z^*,z,t)=e^{-z^*z}\exp(\tfrac{1}{2}\langle[zF^*-z^*F]^2\rangle). \tag{A8}$$

It then follows that

$$\phi(z^*,z,t)=e^{-z^*z}\exp(\tfrac{1}{2}\langle[z^2F^{*2}+z^{*2}F^2-2z^*zF^*F]\rangle). \tag{A9}$$

Furthermore, from Eq. (23) one easily gets

$$\langle F^2\rangle=\langle F_+^2\rangle+\langle F_-^2\rangle+2\langle F_+F_-\rangle, \tag{A10}$$

$$\langle F^*F\rangle=\langle F_+^2\rangle-\langle F_-^2\rangle. \tag{A11}$$

Applying Eq. (23b) along with Eq. (17), it can be easily established that

$$\langle F_+^2\rangle=\frac{2\varepsilon+A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}+2\rho_{aa}^{(0)})}{4\lambda_-}[1-e^{-\lambda_-t}], \tag{A12}$$

$$\langle F_-^2\rangle=\frac{2\varepsilon+A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}-2\rho_{aa}^{(0)})}{4\lambda_+}[1-e^{-\lambda_+t}], \tag{A13}$$

$$\langle F_+F_-\rangle = \frac{A(\rho_{ac}^{(0)}-\rho_{ca}^{(0)})}{4\mu}[1-e^{-\mu t}], \tag{A14}$$

so that in view of these results, there follows

$$\langle F^2\rangle = \frac{2\varepsilon + A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}+2\rho_{aa}^{(0)})}{4\lambda_-}[1-e^{-\lambda_- t}] + \frac{2\varepsilon + A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}-2\rho_{aa}^{(0)})}{4\lambda_+}[1-e^{-\lambda_+ t}] + \frac{A(\rho_{ac}^{(0)}-\rho_{ca}^{(0)})}{2\mu}[1-e^{-\mu t}], \tag{A15}$$

$$\langle F^*F\rangle = \frac{2\varepsilon + A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}+2\rho_{aa}^{(0)})}{4\lambda_-}[1-e^{-\lambda_- t}] - \frac{2\varepsilon + A(\rho_{ac}^{(0)}+\rho_{ca}^{(0)}-2\rho_{aa}^{(0)})}{4\lambda_+}[1-e^{-\lambda_+ t}]. \tag{A16}$$

Now on account of Eqs. (A15) and (A16), the characteristic function (A9) can be written as

$$\phi(z^*,z,t) = \exp[-az^*z + (bz^2 + b^*z^{*2})/2], \tag{A17}$$

where the coefficients are expressible in terms of the parameter η as

$$a = 1 + \frac{2\varepsilon + A[1-\eta+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa-2\varepsilon)} \times [1-e^{-(A\eta+\kappa-2\varepsilon)t}] - \frac{2\varepsilon + A[\eta-1+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa+2\varepsilon)} \times [1-e^{-(A\eta+\kappa+2\varepsilon)t}], \tag{A18}$$

$$b = \frac{2\varepsilon + A[1-\eta+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa-2\varepsilon)}[1-e^{-(A\eta+\kappa-2\varepsilon)t}] + \frac{2\varepsilon + A[\eta-1+(1-\eta^2)^{1/2}\cos\theta]}{4(A\eta+\kappa+2\varepsilon)}[1-e^{-(A\eta+\kappa+2\varepsilon)t}] + \frac{iA(1-\eta^2)^{1/2}\sin\theta}{2(A\eta+\kappa)}[1-e^{-(A\eta+\kappa)t}]. \tag{A19}$$

Finally, introducing Eq. (A17) into Eq. (A1) and carrying out the integration, the Q function for the cavity mode is found to be

$$Q(\alpha^*,\alpha,t) = \frac{[u^2-vv^*]^{1/2}}{\pi}\exp[-u\alpha^*\alpha + (v\alpha^2+v^*\alpha^{*2})/2], \tag{A20}$$

in which

$$u = \frac{a}{a^2-bb^*}, \tag{A21}$$

$$v = \frac{b}{a^2-bb^*}. \tag{A22}$$

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