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Landau-Khalatnikov two-fluid hydrodynamics of a trapped Bose gas

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Starting from the quantum kinetic equation for the noncondensate atoms and the generalized Gross–Pitaevskii equation for the condensate, we derive the two-fluid hydrodynamic equations of a trapped Bose gas at finite temperatures. We follow the standard Chapman–Enskog procedure, starting from a solution of the kinetic equation corresponding to the complete local equilibrium between the condensate and the noncondensate components. Our hydrodynamic equations are shown to reduce to a form identical to the well-known Landau-Khalatnikov two-fluid equations, with hydrodynamic damping due to the deviation from local equilibrium. The deviation from local equilibrium within the thermal cloud gives rise to dissipation associated with shear viscosity and thermal conduction. In addition, we show that effects due to the deviation from the diffusive local equilibrium between the condensate and the noncondensate (recently considered by Zaremba, Nikuni, and Griffin) can be described by four frequency-dependent second viscosity transport coefficients. We also derive explicit formulas for all the transport coefficients. These results are used to introduce two new characteristic relaxation times associated with hydrodynamic damping. These relaxation times give the rate at which local equilibrium is reached and hence determine whether one is in the two-fluid hydrodynamic region.

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I. INTRODUCTION

At very low temperatures, the dynamics of a trapped Bose gas is described by the time-dependent Gross-Pitaevskii (GP) equation for the macroscopic wave function of the condensate. As discussed in several recent reviews [1,2], there is excellent agreement (within a few percent) between experimental observations of collective modes for $T \lesssim 0.4 T_{\mathrm{BEC}}$ and theoretical calculations based on the $T=0$ GP equation. At elevated temperatures where the condensate is appreciably depleted by thermal excitations, one must consider the coupled motion of the condensate and noncondensate degrees of freedom. In a recent paper, Zaremba, Griffin, and Nikuni (ZGN′) [3] derived a generalized Gross-Pitaevskii equation for the condensate atoms and a quantum kinetic equation for the noncondensate atoms, which can be used to discuss the coupled dynamics of the two components at finite temperatures. These two components are coupled through mean-field interactions as well as collisions between the atoms.

Two limiting cases for the dynamics of the gas correspond to the collisionless and hydrodynamic regimes [2,4]. Up to the present, most experiments on the collective modes of Bose gases are thought to be in the low-density collisionless limit. In this regime, the main effect of the noncondensate (thermal cloud) component is to damp the condensate oscillations. In addition to Landau and Beliaev damping due to the dynamic mean-field interaction between two components, there is damping arising from the lack of diffusive local equilibrium between the condensate and the noncondensate [5]. The latter mechanism of damping has recently been worked out in detail by Williams and Griffin [6].

In contrast, in the regime where collisions between atoms are rapid enough to establish a state of dynamic local equilibrium in the noncondensate gas, the dynamics of the system is described by hydrodynamic equations for a few local variables. This regime contains much new physics and it should be accessible taking advantage of the larger densities now available as well as the large scattering cross sections close to a Feshbach resonance [7]. We have recently given a detailed derivation and discussion of two-fluid hydrodynamic equations for trapped atomic gases at finite temperatures [3,5,8–10], starting from a generalized GP equation coupled with a kinetic equation. These equations were derived under the assumption that the noncondensate atoms are in local thermodynamic equilibrium among themselves but are not in diffusive equilibrium with the condensate atoms. The resulting ZGN′ hydrodynamic equations [9,3] involve a new characteristic relaxation time τ_μ, which is the time scale on which local diffusive equilibrium is established. This equilibration process leads to a novel damping mechanism which is associated with the collisional exchange of atoms between the two components. This ZGN′ hydrodynamics is briefly reviewed in Sec. II. In Ref. [10], we generalized the ZGN′ equations to include the effect of deviations from local equilibrium, and worked out hydrodynamic damping associated with the collisions among the noncondensate atoms. At finite temperatures of interest, this deviation from local equilibrium within the thermal cloud gives rise to damping associated with thermal conductivity and the shear viscosity. Such a generalization was first discussed in Sec. V of Ref. [11] starting from the ZGN′ hydrodynamic equations. We also note that the thermal conductivity and shear viscosity were first derived for a uniform Bose-condensed gas at finite temperatures in pioneering papers by Kirkpatrick and Dorfman [12].

In the present paper, building on our recent work with Zaremba, we present a more complete derivation of two-fluid hydrodynamic equations, including dissipation. In Sec. III, we solve the kinetic equation by expanding the distribution function $f(\mathbf{r},\mathbf{p},t)$ around $f^{(0)}(\mathbf{r},\mathbf{p},t)$, the latter describing complete local equilibrium between the condensate and the noncondensate. We follow the standard Chapman–Enskog method used to derive hydrodynamic damping in the kinetic theory of classical gases. In this treatment, the lowest order hydrodynamic equations involve no dissipative terms. All hydrodynamic damping effects are included by taking into account deviations from the local equilibrium distribution $f^{(0)}$. In Sec. IV, we prove that, with appropriate definitions of various thermodynamic variables, our two-fluid hydrodynamic equations with damping have precisely the structure of those first derived by Landau and Khalatnikov [13,14]. In particular, the damping associated with the collisional exchange of atoms between the two components, which has been discussed at length in our previous work [3,9,10], is now expressed in terms of frequency-dependent second viscosity coefficients. This type of damping is in addition to the usual kind of hydrodynamic damping associated with shear viscosity and thermal conductivity [10,11].

In Sec. V, we also derive explicit expressions for all the transport coefficients that appear in the dissipative terms in our two-fluid hydrodynamic equations. For the purpose of illustration, we evaluate the temperature dependence of all the transport coefficients for the case of a uniform Bose gas. The case of a trapped Bose gas is quite different. Because the condensate density is always much larger than the noncondensate density in the central regions of a trapped Bose gas, we find that the collisions between the condensate and noncondensate atoms are the dominant contribution to all transport coefficients.

The lengthy analysis in Secs. III, IV, and V is, of necessity, very complex and conceptionally quite subtle. Most readers will only be interested in the final conclusions, which we now summarize. We prove that the two-fluid hydrodynamic equations of a trapped Bose gas can be written precisely in the well-known Landau-Khalatnikov form, as summarized in Eq. (86). The new feature which arises in a trapped Bose gas (as compared to superfluid ^{4}He) is that, as noted above, the four second viscosity coefficients can be frequency dependent, as given by Eq. (106). Finally, explicit expressions for all the transport coefficients are given in Eqs. (115), (130), and (139). An important final result of our analysis (see Sec. V) is a precise definition of three relaxation times [see Eqs. (24), (B1), and (B5)] associated with the transport coefficients. Moreover, we show that the collisions between the condensate and noncondensate atoms always play the dominant role in the hydrodynamic damping of trapped Bose gases.

The present paper brings to a natural conclusion our series of papers (with Zaremba) on the two-fluid hydrodynamics of trapped Bose gases [3,9–11]. Much remains to be done solving these hydrodynamic equations and experimentally checking the predictions.

II. ZGN′ HYDRODYNAMIC EQUATIONS

In this section, we derive the most general form of hydrodynamic equations for the condensate and noncondensate at finite temperatures, and then briefly review the ZGN′ two-fluid hydrodynamics. We start with the underlying coupled dynamical equations for the noncondensate and the condensate, as derived in Ref. [3] and recently reviewed in Ref. [5]. The noncondensate atoms are described by the distribution function $f(\mathbf{r},\mathbf{p},t)$, which obeys the quantum kinetic equation

$$\frac{\partial f(\mathbf{r},\mathbf{p},t)}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}_{\mathbf{r}}f(\mathbf{r},\mathbf{p},t)-\boldsymbol{\nabla}_{\mathbf{r}}U\cdot\boldsymbol{\nabla}_{\mathbf{p}}f(\mathbf{r},\mathbf{p},t)=C_{12}[f,\Phi]+C_{22}[f]. \tag{1}$$

Here the effective potential $U(\mathbf{r},t)\equiv U_{\text{ext}}(\mathbf{r})+2g[n_c(\mathbf{r},t)+\tilde{n}(\mathbf{r},t)]$ includes the self-consistent Hartree-Fock (HF) mean field, and as usual, we treat the interatomic interaction in the s-wave approximation with $g=4\pi\hbar^2a/m$. The condensate density is $n_c(\mathbf{r},t)\equiv|\Phi(\mathbf{r},t)|^2$ and the noncondensate density $\tilde{n}(\mathbf{r},t)$ is given by

$$\tilde{n}(\mathbf{r},t)=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f(\mathbf{r},\mathbf{p},t). \tag{2}$$

The two collision terms in Eq. (1) are given by

$$\begin{aligned}C_{22}[f]\equiv\frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4&\\ \times\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\,\delta(\tilde{\varepsilon}_p+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4})&\\ \times[(1+f)(1+f_2)f_3f_4-ff_2(1+f_3)(1+f_4)],&\end{aligned} \tag{3}$$

$$\begin{aligned}C_{12}[f,\Phi]\equiv\frac{2g^2n_c}{(2\pi)^2\hbar^4}\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3&\\ \times\delta(m\mathbf{v}_c+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\,\delta(\varepsilon_c+\tilde{\varepsilon}_{p_1}-\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3})&\\ \times[\delta(\mathbf{p}-\mathbf{p}_1)-\delta(\mathbf{p}-\mathbf{p}_2)-\delta(\mathbf{p}-\mathbf{p}_3)]&\\ \times[(1+f_1)f_2f_3-f_1(1+f_2)(1+f_3)],&\end{aligned} \tag{4}$$

with $f\equiv f(\mathbf{r},\mathbf{p},t)$, $f_i\equiv f(\mathbf{r},\mathbf{p}_i,t)$. The expression in Eq. (4) takes into account the fact that a condensate atom locally has energy $\varepsilon_c(\mathbf{r},t)=\mu_c(\mathbf{r},t)+\frac{1}{2}mv_c^2(\mathbf{r},t)$ and momentum $m\mathbf{v}_c$, where the condensate chemical potential μ_c and velocity $\mathbf{v}_c$ will be defined shortly. On the other hand, in our finite-temperature model, a noncondensate atom locally has the HF energy $\tilde{\varepsilon}_p(\mathbf{r},t)=p^2/2m+U(\mathbf{r},t)$. This particle-like dispersion relation limits our entire analysis to finite temperatures.

The equation of motion for the condensate is given by a generalized Gross–Pitaevskii equation [3] for the macroscopic wave function $\Phi(\mathbf{r},t)$ (see also Ref. [15])

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2\nabla^2}{2m}+U_{\text{ext}}(\mathbf{r})+gn_c(\mathbf{r},t)+2g\tilde{n}(\mathbf{r},t)-iR(\mathbf{r},t)\right]\Phi(\mathbf{r},t), \tag{5}$$

where

$$R(\mathbf{r},t)=\frac{\hbar\Gamma_{12}(\mathbf{r},t)}{2n_c(\mathbf{r},t)}, \tag{6}$$

with

$$\Gamma_{12} \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} C_{12}[f(\mathbf{r},\mathbf{p},t),\Phi(\mathbf{r},t)]. \tag{7}$$

The dissipative term R in Eq. (5) is associated with the exchange of atoms between the condensate and noncondensate, as described by the collision integral $C_{12}[f,\Phi]$ in Eq. (4). We see that Eqs. (1) and (5) must be solved self-consistently. It is customary to rewrite the GP equation (5) in terms of the amplitude and phase of $\Phi(\mathbf{r},t)=\sqrt{n_c(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)}$, which leads to $[\mathbf{v}_c \equiv \hbar\boldsymbol{\nabla}\theta(\mathbf{r},t)/m]$

$$\frac{\partial n_c}{\partial t} + \boldsymbol{\nabla}\cdot(n_c\mathbf{v}_c) = -\Gamma_{12}[f,\Phi], \tag{8a}$$

$$m\left(\frac{\partial}{\partial t} + \mathbf{v}_c\cdot\boldsymbol{\nabla}\right)\mathbf{v}_c = -\boldsymbol{\nabla}\mu_c, \tag{8b}$$

where the condensate chemical potential is given by

$$\mu_c(\mathbf{r},t) = -\frac{\hbar^2\nabla^2\sqrt{n_c(\mathbf{r},t)}}{2m\sqrt{n_c(\mathbf{r},t)}} + U_{\text{ext}}(\mathbf{r}) + gn_c(\mathbf{r},t) + 2g\tilde{n}(\mathbf{r},t). \tag{9}$$

One sees that Γ_{12} in Eq. (8) plays the role of a ‘‘source function’’ in the continuity equation for the condensate, arising from the fact that C_{12} collisions do not conserve the number of condensate atoms [3].

We note that the set of equations (1)–(7) has also been derived using the elegant Kadanoff-Baym formulation [16–18]. More recently, this KB derivation has been extended to cover low temperatures as well [19], by working with a Bogoliubov quasiparticle spectrum instead of the simpler HF spectrum used in the present paper. One could use this extension as the basis for generalizing the present paper.

Following the standard procedure in the classical kinetic theory of gases [20], we take moments of the kinetic equation (1) with respect to 1, $\mathbf{p}$, and p^2 to derive the most general form of ‘‘hydrodynamic equations’’ for the noncondensate. These moment equations take the form (μ and ν are Cartesian components):

$$\frac{\partial\tilde{n}}{\partial t} + \boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n) = \Gamma_{12}[f,\Phi], \tag{10a}$$

$$m\tilde{n}\left(\frac{\partial}{\partial t} + \mathbf{v}_n\cdot\boldsymbol{\nabla}\right)v_{n\mu} = -\frac{\partial P_{\mu\nu}}{\partial x_\nu} - \tilde{n}\frac{\partial U}{\partial x_\mu} - m(v_{n\mu}-v_{c\mu})\Gamma_{12}[f,\Phi], \tag{10b}$$

$$\frac{\partial\tilde{\epsilon}}{\partial t} + \nabla\cdot(\tilde{\epsilon}\mathbf{v}_n) = -\boldsymbol{\nabla}\cdot\mathbf{Q} - D_{\mu\nu}P_{\mu\nu} + \left[\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2 + \mu_c - U\right]\Gamma_{12}[f,\Phi]. \tag{10c}$$

The noncondensate density was defined earlier in Eq. (2), while the noncondensate local velocity is defined by

$$\tilde{n}(\mathbf{r},t)\mathbf{v}_n(\mathbf{r},t) \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{\mathbf{p}}{m}f(\mathbf{r},\mathbf{p},t). \tag{11}$$

In addition, we have introduced the following quantities:

$$P_{\mu\nu}(\mathbf{r},t) \equiv m\int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(\frac{p_\mu}{m} - v_{n\mu}\right)\left(\frac{p_\nu}{m} - v_{n\nu}\right)f(\mathbf{r},\mathbf{p},t), \tag{12a}$$

$$\mathbf{Q}(\mathbf{r},t) \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2\left(\frac{\mathbf{p}}{m} - \mathbf{v}_n\right)f(\mathbf{r},\mathbf{p},t), \tag{12b}$$

$$\tilde{\epsilon}(\mathbf{r},t) \equiv \int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2 f(\mathbf{r},\mathbf{p},t). \tag{12c}$$

Finally, the symmetric rate-of-strain tensor appearing in Eq. (10) is defined as

$$D_{\mu\nu}(\mathbf{r},t) \equiv \frac{1}{2}\left(\frac{\partial v_{n\mu}}{\partial x_\nu} + \frac{\partial v_{n\nu}}{\partial x_\mu}\right). \tag{13}$$

The hydrodynamic equations in Eq. (10) are formally exact, but obviously are not closed. To proceed, we must specify the conditions under which the dynamics of the system are to be determined. In the ZGN′ hydrodynamics, we consider the situation in which the C_{22} collisions are sufficiently rapid to establish local equilibrium among the noncondensate atoms. This situation is described by the local equilibrium Bose distribution for the thermal cloud,

$$\tilde{f}(\mathbf{r},\mathbf{p},t) = \frac{1}{e^{\beta[(1/2m)(\mathbf{p}-m\mathbf{v}_n)^2 + U - \tilde{\mu}]} - 1}. \tag{14}$$

Here the temperature parameter β, normal fluid velocity $\mathbf{v}_n$, chemical potential $\tilde{\mu}$, and mean field U are all functions of $\mathbf{r}$ and t. One may immediately verify that $\tilde{f}$ has precisely the form such that it satisfies the condition $C_{22}[\tilde{f}]=0$, independent of the value of $\tilde{\mu}$. In contrast, one finds that $C_{12}[\tilde{f},\Phi]$ does not vanish in general, namely,

$$\begin{aligned} C_{12}[\tilde{f}] = {} & \frac{2g^2n_c}{(2\pi)^2\hbar^4}\{1 - e^{-\beta[\tilde{\mu} - (1/2)m(\mathbf{v}_n-\mathbf{v}_c)^2 - \mu_c]}\} \\ & \times\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3\,\delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times\delta(\tilde{\varepsilon}_1 + \varepsilon_c - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) \\ & \times[\delta(\mathbf{p}-\mathbf{p}_1) - \delta(\mathbf{p}-\mathbf{p}_2) - \delta(\mathbf{p}-\mathbf{p}_3)](1+\tilde{f}_1)\tilde{f}_2\tilde{f}_3. \end{aligned} \tag{15}$$

Using the local distribution function (14) to evaluate the moments in Eqs. (2) and (12), we find that the heat current $\mathbf{Q}(\mathbf{r},t)=0$, and that

$$\tilde{n}(\mathbf{r},t)=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\tilde{f}(\mathbf{r},\mathbf{p},t)=\frac{1}{\Lambda^3}g_{3/2}(z), \qquad (16)$$

$$P_{\mu\nu}(\mathbf{r},t)=\delta_{\mu\nu}\tilde{P}(\mathbf{r},t)\equiv\delta_{\mu\nu}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{(\mathbf{p}-m\mathbf{v}_n)^2}{3m}\tilde{f}(\mathbf{r},\mathbf{p},t)$$

$$=\delta_{\mu\nu}\frac{1}{\beta\Lambda^3}g_{5/2}(z). \qquad (17)$$

Here $z(\mathbf{r},t)\equiv e^{\beta[\tilde{\mu}(\mathbf{r},t)-U(\mathbf{r},t)]}$ is the local fugacity, $\Lambda(\mathbf{r},t)\equiv[2\pi\hbar^2/mk_BT(\mathbf{r},t)]^{1/2}$ is the local thermal de Broglie wavelength, and $g_n(z)=\Sigma_{l=1}^{\infty}z^l/l^n$ are the Bose-Einstein functions. The kinetic energy density is given by $\tilde{\epsilon}(\mathbf{r},t)=\frac{3}{2}\tilde{P}(\mathbf{r},t)$.

To summarize, using $f\simeq\tilde{f}$, we obtain the ZGN′ lowest-order hydrodynamic equations for the noncondensate given in Refs. [3,5,9]. In the linearized version of these ZGN′ hydrodynamic equations, the condensate is described by

$$\frac{\partial\delta n_c}{\partial t}+\boldsymbol{\nabla}\cdot(n_{c0}\delta\mathbf{v_c})=-\delta\Gamma_{12}, \qquad (18a)$$

$$m\frac{\partial\delta\mathbf{v}_c}{\partial t}=-\boldsymbol{\nabla}\,\delta\mu_c, \qquad (18b)$$

and the noncondensate variables satisfy

$$\frac{\partial\delta\tilde{n}}{\partial t}+\boldsymbol{\nabla}\cdot(\tilde{n}_0\delta\mathbf{v}_n)=\delta\Gamma_{12}, \qquad (19a)$$

$$m\tilde{n}_0\frac{\partial\delta\mathbf{v}_n}{\partial t}=-\boldsymbol{\nabla}\,\delta\tilde{P}-\delta\tilde{n}\boldsymbol{\nabla}U_0-2g\tilde{n}_0\boldsymbol{\nabla}(\delta\tilde{n}+\delta n_c), \qquad (19b)$$

$$\frac{\partial\delta\tilde{P}}{\partial t}=-\frac{5}{3}\nabla\cdot(\tilde{P}_0\delta\mathbf{v}_n)+\frac{2}{3}\mathbf{v}_n\cdot\boldsymbol{\nabla}\tilde{P}_0-\frac{2}{3}gn_{c0}\delta\Gamma_{12}. \qquad (19c)$$

The fluctuation of the condensate chemical potential is given by

$$\delta\mu_c=g\,\delta n_c+2g\,\delta\tilde{n}. \qquad (20)$$

This assumes the Thomas-Fermi approximation, which means that the first term in Eq. (9), the quantum pressure term, is neglected. The source function $\delta\Gamma_{12}$ can be usefully expressed [3] in terms of the difference between the condensate and noncondensate chemical potentials, namely

$$\delta\Gamma_{12}=-\frac{\beta_0n_{c0}}{\tau_{12}}(\delta\tilde{\mu}-\delta\mu_c)\equiv-\frac{\beta_0n_{c0}}{\tau_{12}}\delta\mu_{\rm diff}, \qquad (21)$$

where $\mu_{\rm diff}\equiv\tilde{\mu}-\mu_c$ and τ_{12} is the mean collision time [3,9] associated with C_{12}:

$$\frac{1}{\tau_{12}}\equiv\frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3$$
$$\times\delta(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\,\delta(\mu_{c0}+\tilde{\varepsilon}_1-\tilde{\varepsilon}_2-\tilde{\varepsilon}_3)$$
$$\times(1+f_{10})f_{20}f_{30}, \qquad (22)$$

where $f_{i0}=[e^{\beta_0(\tilde{\varepsilon}_i-\mu_{c0})}-1]^{-1}$ is the static equilibrium distribution function and $\tilde{\varepsilon}_i=p^2/2m+U_0$.

Since one can show [3,10] that $\delta\mu_{\rm diff}$ can be written as a linear combination of $\delta\tilde{n}$ and $\delta\tilde{P}$, the above coupled hydrodynamic equations in Eqs. (18) and (19) are a closed set for the variables $\delta n_c,\delta\mathbf{v}_c,\delta\tilde{n},\delta\mathbf{v}_n$, and $\delta\tilde{P}$. However, it will be useful later to have an equation of motion for $\delta\mu_{\rm diff}$. This is given by [see Eq. (86) of Ref. [3]]

$$\frac{\partial\delta\mu_{\rm diff}}{\partial t}=-g\boldsymbol{\nabla}\cdot[n_{c0}(\delta\mathbf{v}_c-\delta\mathbf{v}_n)]-\frac{1}{3}gn_{c0}\boldsymbol{\nabla}\cdot\delta\mathbf{v}_n-\frac{\delta\mu_{\rm diff}}{\tau_\mu}. \qquad (23)$$

Here τ_μ is a new relaxation time governing how $\delta\mu_{\rm diff}$ relaxes to zero, i.e., how fast diffusive equilibrium is reached between the condensate and noncondensate components. It is related to the collision time τ_{12} in Eq. (22) by the expression

$$\frac{1}{\tau_\mu(\mathbf{r})}=\frac{\beta_0gn_{c0}}{\sigma_H\tau_{12}}, \qquad (24)$$

where the dimensionless hydrodynamic renormalization factor σ_H is given by

$$\sigma_H(\mathbf{r})=\left[\frac{\frac{5}{2}\tilde{P}_0+2g\tilde{n}_0n_{c0}+\frac{2}{3}g\tilde{\gamma}_0n_{c0}^2}{\frac{5}{2}\tilde{P}_0\tilde{\gamma}_0-\frac{3}{2}g\tilde{n}_0^2}-1\right]^{-1}, \qquad (25)$$

where $\tilde{\gamma}\equiv(g/k_{\rm B}T\Lambda^3)g_{1/2}(z)$.

We can now look for normal mode solutions of the linearized ZGN′ equations in Eqs. (18) and (19). Assuming that these fluctuations have a time dependence $e^{-i\omega t}$, one can solve Eq. (23) for $\delta\mu_{\rm diff}$ to give

$$\delta\mu_{\rm diff}=-\frac{\tau_\mu}{1-i\omega\tau_\mu}\left\{g\boldsymbol{\nabla}\cdot[n_{c0}(\delta\mathbf{v}_c-\delta\mathbf{v}_n)]+\frac{1}{3}gn_{c0}\boldsymbol{\nabla}\cdot\delta\mathbf{v}_n\right\}. \qquad (26)$$

In the limit $\omega\tau_\mu\rightarrow0$, one sees that $\delta\mu_{\rm diff}\rightarrow0$. This situation corresponds to the complete local equilibrium between the condensate and noncondensate components, with $\tilde{\mu}(\mathbf{r},t)=\mu_c(\mathbf{r},t)$. In this limit, one can prove that our ZGN′ hydrodynamics reduces to the Landau two-fluid hydrodynamics without dissipation terms, as discussed in detail in Ref. [3]. It is clear that fluctuation of $\delta\mu_{\rm diff}$ described by Eq. (23) [or equivalently Eq. (26)] gives rise to a new relaxational mode in addition to usual collective oscillations of the condensate and noncondensate (for a uniform superfluid, these are the first and second sound modes). For a uniform Bose gas, the frequency of this new mode is given by $\omega_R = -i/\tau_\mu$ [9]. Thus, in general, our ZGN′ equations predict the existence of a new relaxational mode, associated with the equilibration of the condensate and noncondensate collective variables.

In Ref. [10], we have extended the theory to include small deviations from the local equilibrium distribution $\tilde{f}$ in Eq. (14). These give rise to new dissipative terms in the noncondensate equations associated with the shear viscosity (η) and the thermal conductivity (κ) of the thermal cloud. The damping of first sound, second sound, and the relaxational mode due to the effect of normal fluid transport coefficients was calculated in Ref. [10]. In particular, we showed there that the relaxational mode was strongly coupled to (and renormalized by) fluctuations in the local temperature and hence the thermal conductivity.

In summary, the ZGN′ hydrodynamics exhibit the physics of the coupled dynamics of the condensate and noncondensate atoms in a clear fashion. However, the approach used in the ZGN′ theory has a disadvantage in that it is not based on a small expansion parameter, in contrast to the more systematic Chapman-Enskog procedure used here. In Ref. [10], we only included the effect of C_{22} collisions to discussing the deviation from local equilibrium. This neglect of C_{12} collisions in this connection is only justified when the condensate density is very small compared to the noncondensate density [since the C_{12} term in Eq. (4) is proportional to n_c]. However, in a trapped gas, the C_{12} collision integral is always significant since the condensate is strongly peaked at the trap center, with a density much larger than the noncondensate even at temperatures close to $T_{\rm BEC}$. Thus we must treat both C_{12} and C_{22} when considering deviations from local equilibrium.

In the following section, we present a more systematic derivation of the two-fluid hydrodynamics, by following the standard Chapman-Enskog procedure. This derivation is similar to the work by Kirkpatrick and Dorfman [12] for a uniform Bose gas. As we discuss in Sec. IV, this new approach allows us to show that the extended ZGN′ theory can be written in a form completely equivalent to the Landau–Khalatnikov two-fluid hydrodynamics [14] when we include hydrodynamic damping. This set of equations involves the thermal conductivity, shear viscosity and four frequency-dependent second viscosity coefficients. The latter are shown to arise from the fact that the condensate and noncondensate are not in diffusive equilibrium ($\mu_c \neq \tilde{\mu}$), as discussed by ZGN′ [3,9].

III. CHAPMAN–ENSKOG EXPANSION FOR A BOSE-CONDENSED GAS

A. Lowest-order hydrodynamic equations

Following the standard procedure of the Chapman-Enskog expansion [21], we introduce a small expansion parameter α and rewrite the kinetic equation (1) as

$$\frac{\partial f(\mathbf{r},\mathbf{p},t)}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}} f(\mathbf{r},\mathbf{p},t) - \nabla_{\mathbf{r}}U\cdot\nabla_{\mathbf{p}} f(\mathbf{r},\mathbf{p},t) = \frac{1}{\alpha}(C_{12}[f,\Phi]+C_{22}[f]). \tag{27}$$

This expansion parameter α will be eventually taken to be 1, but allows one to develop a perturbative solution of Eq. (27). In order to solve the quantum kinetic equation, we formally expand the distribution function $f(\mathbf{r},\mathbf{p},t)$ in powers of α:

$$f = f^{(0)} + \alpha f^{(1)} + \cdots. \tag{28}$$

Using this expansion (28), we can also expand the various hydrodynamic variables in Eq. (10)

$$\begin{gathered}\tilde{n} = \tilde{n}^{(0)} + \alpha\tilde{n}^{(1)} + \cdots, \quad P_{\mu\nu} = P_{\mu\nu}^{(0)} + \alpha P_{\mu\nu}^{(1)} + \cdots,\\ \mathbf{Q} = \mathbf{Q}^{(0)} + \alpha\mathbf{Q}^{(1)} + \cdots,\\ \tilde{\epsilon} = \tilde{\epsilon}^{(0)} + \alpha\tilde{\epsilon}^{(1)} + \cdots.\end{gathered} \tag{29}$$

The superscript (0) denotes the local equilibrium solution (see below) which is determined by the collision integrals (formally when $\alpha \to 0$). We also redefine the source function Γ_{12} in Eq. (7) as

$$\Gamma_{12} \equiv \frac{1}{\alpha}\int \frac{d\mathbf{p}}{(2\pi\hbar)^3} C_{12}[f,\Phi] = \frac{1}{\alpha}(\Gamma_{12}^{(0)} + \alpha\Gamma_{12}^{(1)} + \alpha^2\Gamma_{12}^{(2)}\cdots). \tag{30}$$

We also have an expansion for the condensate wave function

$$\Phi = \Phi^{(0)} + \alpha\Phi^{(1)} + \cdots. \tag{31}$$

In this expansion, however, we assume that the total local density $n(\equiv n_c + \tilde{n})$ is not altered by the higher order correction terms $f^{(i)}$ $(i \geqslant 1)$ in Eq. (28). That is, we have

$$n_c = n_c^{(0)} + \alpha n_c^{(1)} + \cdots, \tag{32}$$

but

$$n = n_c^{(0)} + \tilde{n}^{(0)}, \quad n_c^{(i)} + \tilde{n}^{(i)} = 0 \quad (i \geqslant 1). \tag{33}$$

We also assume that nonlocal correction terms $f^{(i)}$ make no contribution to the noncondensate velocity fields $\mathbf{v}_n$ or to the phase θ of the condensate wave function (and hence to the condensate velocity $\mathbf{v}_c$). Finally, the condensate chemical potential in Eq. (9) (we work within the Thomas-Fermi approximation) is given in the expansion

$$\mu_c(\mathbf{r},t) = U_{\rm ext}(\mathbf{r}) + g[n(\mathbf{r},t) + \tilde{n}(\mathbf{r},t)] = \mu_c^{(0)}(\mathbf{r},t) + \alpha\mu_c^{(1)}(\mathbf{r},t) + \cdots, \tag{34}$$

with

$$\mu_c^{(0)} \equiv U_{\rm ext} + g(n + \tilde{n}^{(0)}), \quad \mu_c^{(1)} = g\tilde{n}^{(1)}. \tag{35}$$

Using the expansion (28) in the kinetic equation (27), we find that the lowest order solution $f^{(0)}$ is determined from

$$C_{12}[f^{(0)},\Phi^{(0)}]+C_{22}[f^{(0)}]=0. \tag{36}$$

The unique solution of Eq. (36) is given by the ''diffusive local equilibrium'' Bose distribution function, namely

$$f^{(0)}(\mathbf{r},\mathbf{p},t)=\frac{1}{e^{\beta(\mathbf{r},t)[(1/2m)(\mathbf{p}-m\mathbf{v}_n(\mathbf{r},t))^2+U(\mathbf{r},t)-\tilde{\mu}^{(0)}(\mathbf{r},t)]}-1}. \tag{37}$$

Here the local equilibrium noncondensate chemical potential $\tilde{\mu}^{(0)}$ is given by the condition that $C_{12}[f^{(0)},\Phi^{(0)}]=0$, which gives

$$\tilde{\mu}^{(0)}=\mu_c^{(0)}+\frac{m}{2}(\mathbf{v}_n-\mathbf{v}_c)^2. \tag{38}$$

Using Eq. (35), this is equivalent to

$$\begin{aligned}\tilde{\mu}^{(0)}&=\mu_c^{(0)}+\frac{m}{2}(\mathbf{v}_n-\mathbf{v}_c)^2\\&=U_{\text{ext}}+gn+g\tilde{n}^{(0)}+\frac{m}{2}(\mathbf{v}_n-\mathbf{v}_c)^2,\end{aligned} \tag{39}$$

in conjunction with

$$\begin{aligned}\tilde{n}^{(0)}(\mathbf{r},t)&=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f^{(0)}(\mathbf{r},\mathbf{p},t)\\&=\frac{1}{\Lambda^3}g_{3/2}(z^{(0)}).\end{aligned} \tag{40}$$

Here $z^{(0)}(\mathbf{r},t)\equiv e^{\beta[\tilde{\mu}^{(0)}(\mathbf{r},t)-U(\mathbf{r},t)]}$ is the local fugacity in diffusive local equilibrium.

It is important to appreciate that diffusive equilibrium is not defined by the distribution function $f^{(0)}$ alone, but is determined self-consistently with the noncondensate chemical potential as given by Eq. (38). One may immediately verify that $f^{(0)}$ satisfies $C_{22}[f^{(0)}]=0$, independent of the value of $\tilde{\mu}^{(0)}$. In contrast, $C_{12}[f^{(0)},\Phi^{(0)}]=0$ only if the local chemical potential of the thermal cloud is given by Eq. (38) and the condensate and noncondensate densities are determined self-consistently. Of course, it immediately follows that since $C_{12}[f^{(0)},\Phi^{(0)}]=0$, we have $\Gamma_{12}^{(0)}=\Gamma_{12}[f^{(0)},\Phi^{(0)}]=0$ and hence Eq. (30) reduces to

$$\Gamma_{12}=\Gamma_{12}^{(1)}+\alpha\Gamma_{12}^{(2)}+\cdots. \tag{41}$$

Using the local distribution function (37) to evaluate the moments in Eq. (12b), we find that the heat current $\mathbf{Q}^{(0)}(\mathbf{r},t)=0$, and

$$\begin{aligned}P_{\mu\nu}^{(0)}(\mathbf{r},t)&=\delta_{\mu\nu}\tilde{P}^{(0)}(\mathbf{r},t)\\&\equiv\delta_{\mu\nu}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{(\mathbf{p}-m\mathbf{v}_n)^2}{3m}f^{(0)}(\mathbf{r},\mathbf{p},t)\\&=\delta_{\mu\nu}\frac{1}{\beta\Lambda^3}g_{5/2}(z^{(0)}).\end{aligned} \tag{42}$$

The local kinetic energy density is given by $\tilde{\epsilon}^{(0)}(\mathbf{r},t)=\frac{3}{2}\tilde{P}^{(0)}(\mathbf{r},t)$.

To summarize, the lowest-order hydrodynamic equations for the noncondensate are given by

$$\frac{\partial\tilde{n}}{\partial t}+\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n)=\Gamma_{12}^{(1)}, \tag{43a}$$

$$m\tilde{n}\left(\frac{\partial}{\partial t}+\mathbf{v}_n\cdot\boldsymbol{\nabla}\right)\mathbf{v}_n=-\boldsymbol{\nabla}\tilde{P}-\tilde{n}\boldsymbol{\nabla}U-m(\mathbf{v}_n-\mathbf{v}_c)\Gamma_{12}^{(1)}, \tag{43b}$$

$$\begin{aligned}\frac{\partial\tilde{P}}{\partial t}+\nabla\cdot(\tilde{P}\mathbf{v}_n)=&-\frac{2}{3}\tilde{P}\boldsymbol{\nabla}\cdot\mathbf{v}_n\\&+\frac{2}{3}\left[\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2+\mu_c-U\right]\Gamma_{12}^{(1)},\end{aligned} \tag{43c}$$

where $\tilde{n}=\tilde{n}^{(0)}$, $\tilde{P}=\tilde{P}^{(0)}$, and $\mu_c=\mu_c^{(0)}$ are given by Eqs. (40), (42), and (38), respectively. It should be noted that the above equations involve the source term $\Gamma_{12}^{(1)}$. Even though $C_{12}[f^{(0)}]=0$, one sees from Eq. (41) that the lowest-order contribution is in fact given by $\Gamma_{12}^{(1)}$, which involves the contribution from the next order correction $f^{(1)}$. Later we will derive an explicit expression for Γ_{12} when we include the effect of deviations from the local equilibrium distribution and transport processes. Here we only display the result for the lowest-order contribution which enter into Eq. (43) [see also Eq. (78)]:

$$\Gamma_{12}^{(1)}(\mathbf{r},t)=\sigma_H\left\{\boldsymbol{\nabla}\cdot[n_c(\mathbf{v}_c-\mathbf{v}_n)]+\frac{1}{3}n_c\boldsymbol{\nabla}\cdot\mathbf{v}_n\right\}, \tag{44}$$

where σ_H is defined by Eq. (25).

It is important to note that even though $\Gamma_{12}^{(1)}$ involves an integral over the collision integral C_{12} [see Eq. (30)], the expression in Eq. (44) does not involve any collision time. The expression for $\Gamma_{12}^{(1)}$ in Eq. (44) is consistent with the ZGN′ result for $\delta\mu_{\text{diff}}$ given in Eq. (26) in the limit $\omega\tau_\mu\to 0$ [using Eqs. (21) and (25)]. We recall that in this limit, one has $\delta\mu_{\text{diff}}\to 0$ and thus $\tilde{f}$ in Eq. (14) reduces to $f^{(0)}$ in Eq. (37). Therefore the hydrodynamic equations given in Eq. (43) are equivalent, in the $\omega\tau_\mu\to 0$ limit, to those given by the ZGN′ theory. As noted in Ref. [3], these coupled lowest-order hydrodynamic equations in Eqs. (43) and (8) can be combined and also shown to be precisely equivalent to the Landau two-fluid equations without dissipation due to the transport processes. In Sec. IV we prove this equivalence in the more general case when dissipation is included.

B. Two-fluid equations with hydrodynamic dissipation

We next consider the deviation (28) from the local equilibrium distribution function $f^{(0)}$ to first order in the Chapman-Enskog expansion. This deviation $f^{(0)}$ gives rise to additional dissipative terms in the hydrodynamic equations. As usual, in determining the dissipative terms, we restrict ourselves to terms of first order in the velocity fields $\mathbf{v}_n$ and $\mathbf{v}_c$. Following Refs. [12,11, and 10], we write the first correction term in Eq. (28) in the form

$$f^{(1)}=f^{(0)}(\mathbf{r},\mathbf{p},t)[1+f^{(0)}(\mathbf{r},\mathbf{p},t)]\psi(\mathbf{r},\mathbf{p},t), \tag{45}$$

and work with $\psi(\mathbf{r},\mathbf{p},t)$. To first order in α, the C_{22} and C_{12} collision terms in Eq. (2) reduce to ($f=f^{(0)}+\alpha f^{(1)}$).

$$\begin{aligned}\frac{1}{\alpha}C_{22}[f]\simeq C_{22}[f^{(1)}]\simeq{}& \frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\\ &\times\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\,\delta(\tilde{\varepsilon}_{p_1}+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4})\\ &\times f^{(0)}f_2^{(0)}(1+f_3^{(0)})(1+f_4^{(0)})(\psi_3+\psi_4-\psi_2-\psi)\\ \equiv{}&\hat{L}_{22}[\psi].\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{1}{\alpha}C_{12}[f,\Phi]\simeq{}&\frac{2g^2n_c}{(2\pi)^2\hbar^4}\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3\\ &\times\delta(m\mathbf{v}_c+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\\ &\times\delta(\varepsilon_c^{(0)}+\tilde{\varepsilon}_{p_1}-\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3})\\ &\times[\delta(\mathbf{p}-\mathbf{p}_1)-\delta(\mathbf{p}-\mathbf{p}_2)-\delta(\mathbf{p}-\mathbf{p}_3)]\\ &\times(1+f_1^{(0)})f_2^{(0)}f_3^{(0)}(-\beta\mu_c^{(1)}+\psi_2+\psi_3-\psi_1)\\ \equiv{}&-\beta g\tilde{n}^{(1)}\hat{L}_{12}[1]+\hat{L}_{12}[\psi],\end{aligned} \tag{47}$$

where $\varepsilon_c^{(0)}=\mu_c^{(0)}+\frac{1}{2}mv_c^2$ and $\mu_c^{(0)}$ is given by Eq. (35). The linearized $\hat{L}_{12}$ operator is defined by

$$\begin{aligned}\hat{L}_{12}[\psi]\equiv{}&\frac{2g^2n_c}{(2\pi)^2\hbar^4}\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3\\ &\times\delta(m\mathbf{v}_c+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\,\delta(\varepsilon_c^{(0)}+\tilde{\varepsilon}_{p_1}-\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3})\\ &\times[\delta(\mathbf{p}-\mathbf{p}_1)-\delta(\mathbf{p}-\mathbf{p}_2)-\delta(\mathbf{p}-\mathbf{p}_3)]\\ &\times(1+f_1^{(0)})f_2^{(0)}f_3^{(0)}(\psi_2+\psi_3-\psi_1).\end{aligned} \tag{48}$$

Using Eqs. (46)–(48) and expanding the kinetic equation (27) to first order in α, we find that the first nonlocal correction $f^{(1)}$ is determined by the equation

$$\begin{aligned}&\frac{\partial^0 f^{(0)}(\mathbf{r},\mathbf{p},t)}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}_{\mathbf{r}}f^{(0)}(\mathbf{r},\mathbf{p},t)-\boldsymbol{\nabla}_{\mathbf{r}}U\cdot\boldsymbol{\nabla}_{\mathbf{p}}f^{(0)}(\mathbf{r},\mathbf{p},t)\\ &\quad=-\beta g\tilde{n}^{(1)}\hat{L}_{12}[1]+\hat{L}_{12}[\psi]+\hat{L}_{22}[\psi].\end{aligned} \tag{49}$$

Here $\partial^0/\partial t$ means that we use the lowest-order hydrodynamic equations given by (43) in evaluating time derivatives of $\mathbf{v}_n$, $\tilde{\mu}$, T, and U. The resulting linearized equation which determines the function ψ is (for details, see Appendix A)

$$\begin{aligned}&\left\{\frac{\mathbf{u}\cdot\boldsymbol{\nabla}T}{T}\left[\frac{mu^2}{2k_BT}-\frac{5g_{5/2}(z)}{2g_{3/2}(z)}\right]+\frac{m}{k_BT}D_{\mu\nu}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)\right.\\ &\qquad\left.+\left(\sigma_2+\frac{mu^2}{3k_BT}\sigma_1\right)\frac{\Gamma_{12}^{(1)}}{\tilde{n}^{(0)}}\right\}f^{(0)}(1+f^{(0)})\\ &\qquad+\beta g\tilde{n}^{(1)}\hat{L}_{12}[1]=\hat{L}_{12}[\psi]+\hat{L}_{22}[\psi]\equiv\hat{L}[\psi],\end{aligned} \tag{50}$$

where the thermal velocity $\mathbf{u}$ is defined by $m\mathbf{u}\equiv\mathbf{p}-m\mathbf{v}_n$ and $z=z^{(0)}$. The dimensionless thermodynamic functions σ_1, σ_2 in Eq. (50) are defined by

$$\sigma_1(\mathbf{r},t)\equiv\frac{\gamma^{(0)}\tilde{n}^{(0)}[\tilde{\mu}^{(0)}-U]-\frac{3}{2}[\tilde{n}^{(0)}]^2}{\frac{5}{2}\tilde{P}^{(0)}\gamma^{(0)}-\frac{3}{2}[\tilde{n}^{(0)}]^2},$$

$$\sigma_2(\mathbf{r},t)\equiv\beta\frac{\frac{5}{2}\tilde{P}^{(0)}\tilde{n}^{(0)}-[\tilde{n}^{(0)}]^2[\tilde{\mu}^{(0)}-U]}{\frac{5}{2}\tilde{P}^{(0)}\gamma^{(0)}-\frac{3}{2}[\tilde{n}^{(0)}]^2}, \tag{51}$$

where $\gamma^{(0)}(\mathbf{r},t)\equiv(\beta/\Lambda^3)g_{1/2}(z^{(0)}(\mathbf{r},t))=\tilde{\gamma}^{(0)}/g$. We note that C_{12} enters in three separate places in Eq. (50).

The linearized collision operators $\hat{L}_{12}$ and $\hat{L}_{22}$ satisfy the conditions

$$\begin{gathered}\hat{L}_{12}[\mathbf{p}-m\mathbf{v}_c]=0,\quad\hat{L}_{12}[\tilde{\varepsilon}_p-\varepsilon_c^{(0)}]=0,\\ \hat{L}_{22}[1]=0,\quad\hat{L}_{22}[\mathbf{p}]=0,\quad\hat{L}_{22}[\tilde{\varepsilon}_p]=0.\end{gathered} \tag{52}$$

In order to have a unique solution of Eq. (50) for ψ, we impose the following additional constraints:

$$\int d\mathbf{p}\;\mathbf{u}\;f^{(0)}(1+f^{(0)})\psi=0, \tag{53a}$$

$$\begin{aligned}&\int d\mathbf{p}\left(\frac{m}{2}u^2+U-\tilde{\mu}^{(0)}\right)f^{(0)}(1+f^{(0)})\psi\\ &\qquad=\frac{1}{\beta}\int d\mathbf{p}\ln(1+f^{(0)^{-1}})f^{(1)}=0.\end{aligned} \tag{53b}$$

Physically, the constraint (53a) means that the deviation from local equilibrium make no contribution to the local velocity field $\mathbf{v}_n$ defined in Eq. (11). As we discuss in more detail in Sec. IV, the constraint (53b) means that the total energy density and the local entropy density are not altered by the deviation $f^{(1)}$. They have the the same value as given by $f^{(0)}$.

Since Eq. (50) is a linear integral equation for ψ, one may write the most general solution in the following form [21]:

$$\psi(\mathbf{r},\mathbf{p},t)=\frac{\nabla T\cdot\mathbf{u}}{T}A(u)+D_{\mu\nu}\left(u_\mu u_\nu-\frac{1}{3}u^2\delta_{\mu\nu}\right)B(u)+\Gamma_{12}^{(1)}D(u), \tag{54}$$

where the dependence on $(\mathbf{r},t)$ is left implicit and u_μ is a component of the thermal velocity. Here the functions $A(u),B(u)$, and $D(u)$ are given by the solutions to the following three linearized integral equations:

$$\mathbf{u}\left[\frac{mu^2}{2k_BT}-\frac{5g_{5/2}(z)}{2g_{3/2}(z)}\right]f^{(0)}(1+f^{(0)})=\hat{L}[\mathbf{u}A(u)], \tag{55a}$$

$$\frac{m}{k_BT}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)f^{(0)}(1+f^{(0)})=\hat{L}\left[\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)B(u)\right], \tag{55b}$$

$$\left(\sigma_2+\frac{mu^2}{3k_BT}\sigma_1\right)\frac{1}{\tilde{n}^{(0)}}f^{(0)}(1+f^{(0)})+\frac{\beta g\tilde{n}^{(1)}}{\Gamma_{12}^{(1)}}\hat{L}_{12}[1]=\hat{L}[D(u)]. \tag{55c}$$

For the constraints (53) to be satisfied, we also need to require that

$$\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f^{(0)}(1+f^{(0)})u^2A(u)=0, \tag{56a}$$

$$\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f^{(0)}(1+f^{(0)})\left(\frac{mu^2}{2}+U-\tilde{\mu}^{(0)}\right)D(u)=0. \tag{56b}$$

Using the solution for ψ given in Eq. (54), one finds that the corrections due to $f^{(1)}$ in Eq. (45) to the various hydrodynamic variables are given by

$$\tilde{n}^{(1)}=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f^{(0)}(1+f^{(0)})D(u)\Gamma_{12}^{(1)}(\mathbf{r},t)\equiv-\tau\Gamma_{12}^{(1)}, \tag{57}$$

$$P_{\mu\nu}^{(1)}=\delta_{\mu\nu}\tilde{P}^{(1)}-2\eta\left[D_{\mu\nu}-\frac{1}{3}\mathrm{Tr}\,D\,\delta_{\mu\nu}\right], \tag{58}$$

$$\mathbf{Q}^{(1)}=-\kappa\nabla T, \tag{59}$$

with

$$\tilde{P}^{(1)}=\tau\frac{2}{3}(U-\tilde{\mu}^{(0)})\Gamma_{12}^{(1)}\simeq\tau\frac{2}{3}gn_c^{(0)}\Gamma_{12}^{(1)}, \tag{60}$$

$$\tilde{\epsilon}^{(1)}=\frac{3}{2}\tilde{P}^{(1)}. \tag{61}$$

We note that $\tilde{n}$ and $\tilde{P}$ are both altered by an amount proportional to $\Gamma_{12}^{(1)}$. The transport coefficients η and κ are associated with the functions $A(u)$ and $B(u)$,

$$\eta=-\frac{m}{15}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}u^4B(u)f^{(0)}(1+f^{(0)}), \tag{62a}$$

$$\kappa=-\frac{m}{6T}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}u^4A(u)f^{(0)}(1+f^{(0)}). \tag{62b}$$

The relaxation time τ defined in Eq. (57), namely,

$$\tau=-\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f^{(0)}(1+f^{(0)})D(u), \tag{63}$$

plays a crucial role in the subsequent analysis. Using Eq. (57) in Eq. (55c), one can rewrite the integral equation for $D(u)$ in the form

$$\left(\sigma_2+\frac{mu^2}{3k_\mathrm{B}T}\sigma_1\right)\frac{1}{\tilde{n}^{(0)}}f^{(0)}(1+f^{(0)})-\tau\beta g\hat{L}_{12}[1]=\hat{L}[D(u)]. \tag{64}$$

In Sec. V we solve the three linearized equations listed in Eq. (55). It will be shown there that the solution for the function $D(u)$ is

$$D(u)=-\frac{\tau_\mu}{\tilde{n}^{(0)}}\left(\sigma_2+\frac{mu^2}{3k_\mathrm{B}T}\sigma_1\right). \tag{65}$$

Using this, one finds that τ can be identified with the relaxation time τ_μ defined in Eq. (24). In the present discussion, the physical meaning of the relaxation time τ_μ can be clearly seen by writing the source function $\Gamma_{12}^{(1)}$ in the form [see Eqs. (57) and (29)]

$$\Gamma_{12}^{(1)}=-\frac{\tilde{n}^{(1)}}{\tau}=-\frac{\tilde{n}-\tilde{n}^{(0)}}{\tau_\mu}. \tag{66}$$

This kind of relaxation term in the two-fluid hydrodynamic equations such as Eqs. (8a) and (43a) was also discussed in a pioneering paper by Miyake and Yamada [22] in discussing the liquid ^{4}He near the superfluid transition (where a phenomenological relaxation time equivalent to τ_μ was introduced).

In summary, we have obtained the following hydrodynamic equations for the noncondensate including the normal fluid transport coefficients (we now set the expansion parameter $\alpha=1$):

$$\frac{\partial\tilde{n}}{\partial t}+\nabla\cdot(\tilde{n}\mathbf{v}_n)=\Gamma_{12}, \tag{67a}$$

$$\begin{aligned} m\tilde{n}\left(\frac{\partial}{\partial t}+\mathbf{v}_n\cdot\boldsymbol{\nabla}\right)v_{n\mu}&+\frac{\partial\tilde{P}}{\partial x_\mu}+\tilde{n}\frac{\partial U}{\partial x_\mu}\\ &=-m(v_{n\mu}-v_{c\mu})\Gamma_{12}+\frac{\partial}{\partial x_\nu}\left\{2\eta\left[D_{\mu\nu}-\frac{1}{3}(\mathrm{Tr}\,D)\delta_{\mu\nu}\right]\right\}, \end{aligned} \tag{67b}$$

$$\begin{aligned} \frac{\partial\tilde{\epsilon}}{\partial t}&+\boldsymbol{\nabla}\cdot(\tilde{\epsilon}\mathbf{v}_n)+(\boldsymbol{\nabla}\cdot\mathbf{v}_n)\tilde{P}\\ &=\left[\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2+\mu_c-U\right]\Gamma_{12}+\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)\\ &\quad+2\eta\left[D_{\mu\nu}-\frac{1}{3}(\mathrm{Tr}\,D)\delta_{\mu\nu}\right]^2, \end{aligned} \tag{67c}$$

where $\tilde{n}$ and $\tilde{P}$ are given by

$$\tilde{n}=\tilde{n}^{(0)}-\tau_\mu\Gamma_{12}^{(1)}, \tag{68}$$

$$\tilde{P}=\tilde{P}^{(0)}+\tau_\mu\frac{2}{3}gn_c^{(0)}\Gamma_{12}^{(1)}, \tag{69}$$

and $\tilde{\epsilon}=\frac{3}{2}\tilde{P}$. Here $\tilde{n}^{(0)}$ and $\tilde{P}^{(0)}$ are given by Eqs. (40) and (42), respectively. The equivalent ''quantum'' hydrodynamic equations for the condensate are given in Eq. (8), where the condensate chemical potential is given by

$$\mu_c=\mu_c^{(0)}-g\tau_\mu\Gamma_{12}^{(1)}. \tag{70}$$

We now derive an equation for the function Γ_{12}. Using Eqs. (68) and (67a), one obtains

$$\frac{\partial\tilde{n}}{\partial t}=\frac{\partial\tilde{n}^{(0)}}{\partial t}-\tau_\mu\frac{\partial\Gamma_{12}}{\partial t}=-\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n)+\Gamma_{12}. \tag{71}$$

Using the explicit expression for $\tilde{n}^{(0)}$ given in Eq. (40), one obtains

$$\frac{\partial\tilde{n}^{(0)}}{\partial t}=\left(\frac{3}{2}\tilde{n}+g\gamma n_c\right)\frac{1}{T}\frac{\partial T}{\partial t}+g\gamma\left(\frac{\partial\tilde{n}^{(0)}}{\partial t}-\frac{\partial n}{\partial t}\right). \tag{72}$$

Using the continuity equation for the total density n, one finds that Eq. (72) reduces to

$$\frac{\partial\tilde{n}^{(0)}}{\partial t}=\frac{1}{1-g\gamma}\left[\left(\frac{3}{2}\tilde{n}+g\gamma n_c\right)\frac{1}{T}\frac{\partial T}{\partial t}+\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n+n_c\mathbf{v}_c)\right]. \tag{73}$$

Using Eq. (73) in Eq. (71), one finds

$$\begin{aligned} \tau_\mu\frac{\partial\Gamma_{12}}{\partial t}+\Gamma_{12}=\frac{1}{1-g\gamma}&\left[\left(\frac{3}{2}\tilde{n}+g\gamma n_c\right)\frac{1}{T}\frac{\partial T}{\partial t}\right.\\ &\left.+\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n+n_c\mathbf{v}_c)\right]+\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n). \end{aligned} \tag{74}$$

We next use Eq. (69) in the equation for $\tilde{P}$ [given by Eq. (67c)]:

$$\begin{aligned} \frac{\partial\tilde{P}}{\partial t}&=\frac{\partial\tilde{P}^{(0)}}{\partial t}+\tau_\mu\frac{2}{3}gn_c\frac{\partial\Gamma_{12}}{\partial t}\\ &=-\boldsymbol{\nabla}\tilde{P}\cdot\mathbf{v}_n-\frac{5}{3}\tilde{P}\boldsymbol{\nabla}\cdot\mathbf{v}_n-\frac{2}{3}gn_c\Gamma_{12}+\frac{2}{3}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T). \end{aligned} \tag{75}$$

Using the expression for $\tilde{P}^{(0)}$ given in Eq. (42), one finds

$$\frac{\partial\tilde{P}^{(0)}}{\partial t}=\left(\frac{5}{2}\tilde{P}+g\tilde{n}n_c\right)\frac{1}{T}\frac{\partial T}{\partial t}+g\tilde{n}\frac{\partial\tilde{n}^{(0)}}{\partial t}+g\tilde{n}\boldsymbol{\nabla}\cdot(\tilde{n}\mathbf{v}_n+n_c\mathbf{v}_c). \tag{76}$$

Substituting this into Eq. (75) in conjunction with Eq. (71), we obtain

$$\begin{aligned} -\left(\frac{2}{3}gn_c+g\tilde{n}\right)\left(\tau_\mu\frac{\partial\Gamma_{12}}{\partial t}+\Gamma_{12}\right)&=\left(\frac{5}{2}\tilde{P}+g\tilde{n}n_c\right)\frac{1}{T}\frac{\partial T}{\partial t}\\ &\quad+g\tilde{n}\boldsymbol{\nabla}\cdot(n_c\mathbf{v}_c)+\boldsymbol{\nabla}\tilde{P}\cdot\mathbf{v}_n+\frac{5}{3}\tilde{P}\boldsymbol{\nabla}\cdot\mathbf{v}_n-\frac{2}{3}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T). \end{aligned} \tag{77}$$

One may now combine Eq. (74) and (77) to eliminate $\partial T/\partial t$ from these two equations. After a certain amount of rearrangement, we finally obtain our desired equation of motion for Γ_{12}:

$$\begin{aligned} \tau_\mu\frac{\partial\Gamma_{12}}{\partial t}+\Gamma_{12}&=\sigma_H\left\{\boldsymbol{\nabla}\cdot[n_c(\mathbf{v}_c-\mathbf{v}_n)]+\frac{1}{3}n_c\boldsymbol{\nabla}\cdot\mathbf{v}_n\right\}\\ &\quad-\frac{2}{3}\frac{\sigma_H\sigma_1}{g}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T). \end{aligned} \tag{78}$$

If we keep the expansion parameter α and expand Γ_{12} as in Eq. (41), namely $\Gamma_{12}=\Gamma_{12}^{(1)}+\alpha\Gamma_{12}^{(2)}$ (we recall that $\Gamma_{12}^{(0)}=0$), we find $\Gamma_{12}^{(1)}$ is given by Eq. (44) and

$$\Gamma_{12}^{(2)}=-\tau_\mu\frac{\partial}{\partial t}\Gamma_{12}^{(1)}-\frac{2}{3}\frac{\sigma_H\sigma_1}{g\tilde{n}}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T). \tag{79}$$

In closing this section, we discuss the relation between the analysis given in this section and the ZGN′ theory [3] reviewed in Sec. II. In this section, we started with the complete local equilibrium distribution given by Eq. (37). We then included the deviation from local equilibrium, as given by Eq. (45) with Eq. (54). We showed that the deviation from $f^{(0)}$ associated with $D(u)$ in Eq. (54) gives rise to the corrections to the local thermodynamic quantities $\tilde{n}$, $\tilde{P}$, and $\tilde{\epsilon}$. Such corrections did not arise when we included the deviation from $\tilde{f}$ in the ZGN′ hydrodynamics [10]. However, one can show that the type of contribution associated with $D(u)$ is, in fact, already contained in the lowest-order ZGN′ distribution function $\tilde{f}$ given by Eq. (14). To see this, it is convenient to linearize the distribution function around static equilibrium, using $f \simeq f_0 + \delta f$. In the ZGN′ theory [3], one can show that

$$\delta f = \beta_0 f_0 (1+f_0)\left[\frac{\delta\tilde{T}}{T_0}\left(\frac{p^2}{2m}+U_0-\mu_{c0}\right) + \mathbf{p}\cdot\mathbf{v}_n - 2g\,\delta n + \delta\tilde{\mu}\right]. \tag{80}$$

Here we have denoted the temperature fluctuation as $\delta\tilde{T}$ to make a distinction from the temperature defined in the diffusive local equilibrium distribution function (37)(we will find that $\delta\tilde{T}\neq\delta T$). In the present theory, in contrast, one finds [ignoring the terms in Eq. (54) associated with the functions A and B]

$$\delta f = \beta_0 f_0 (1+f_0)\left[\frac{\delta T}{T_0}\left(\frac{p^2}{2m}+U_0-\mu_{c0}\right) + \mathbf{p}\cdot\mathbf{v}_n - 2g\,\delta n + \delta\mu_c^{(0)}\right] + f_0(1+f_0)D(u)\,\delta\Gamma_{12}. \tag{81}$$

The first term in Eq. (81) represents the deviation from f_0 included in $f^{(0)}$ while the second term is due to $f^{(1)}$. Using the explicit solution for $D(u)$ given by Eq. (65) (derived in Sec. V), we find that Eq. (81) can be written as

$$\delta f = \beta_0 f_0 (1+f_0)\left[\left(\frac{\delta T}{T_0} - \frac{2\sigma_1\tau_\mu}{3\tilde{n}_0}\delta\Gamma_{12}\right)\left(\frac{p^2}{2m}+U_0-\mu_{c0}\right) + \mathbf{p}\cdot\mathbf{v}_n - 2g\,\delta n + \delta\mu_c^{(0)} - g(\sigma_H^{-1}+1)\tau_\mu\delta\Gamma_{12}\right]. \tag{82}$$

We note that this linearized distribution function has the same form as the ZGN′ distribution function in Eq. (80), but with a renormalized local temperature

$$\delta\tilde{T} \equiv \delta T - \frac{2}{3}T_0\frac{\sigma_1\tau_\mu}{\tilde{n}_0}\delta\Gamma_{12}, \tag{83}$$

and a renormalized local chemical potential

$$\delta\tilde{\mu} \equiv \delta\mu_c^{(0)} - g(\sigma_H^{-1}+1)\tau_\mu\delta\Gamma_{12}. \tag{84}$$

Using $\delta\mu_c = \delta\mu_c^{(0)} - g\tau_\mu\delta\Gamma_{12}$ [see Eq. (70)] and $\delta\tilde{\mu}$ from Eq. (84), we obtain

$$\begin{aligned}\delta\mu_{\mathrm{diff}} \equiv \delta\tilde{\mu} - \delta\mu_c &= \delta\tilde{\mu} - [\delta\mu_c^{(0)} - g\tau_\mu\delta\Gamma_{12}] \\ &= -g\sigma_H^{-1}\tau_\mu\delta\Gamma_{12}.\end{aligned} \tag{85}$$

This relation between $\delta\mu_{\mathrm{diff}}$ and $\delta\Gamma_{12}$ is precisely that given by Eqs. (21) and (24), as derived in the ZGN′ theory. The physical significance of the renormalized thermodynamic quantities, as given by Eqs. (83) and (84), will become clear in Sec. IV.

IV. EQUIVALENCE TO LANDAU–KHALATNIKOV TWO-FLUID EQUATIONS WITH DISSIPATION

In this section we prove that our hydrodynamic equations in Eqs. (8) and (67) can be written in the form of the Landau-Khalatnikov two-fluid equations. We first display the complete Landau-Khalatnikov two-fluid equations involving dissipative terms [14]:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0, \tag{86a}$$

$$\begin{aligned} m\frac{\partial j_\mu}{\partial t} + \frac{\partial}{\partial x_\nu}(\delta_{\mu\nu}P + m\tilde{n}v_{n\mu}v_{n\nu} + mn_c v_{c\mu}v_{c\nu}) + n\frac{\partial U_{\mathrm{ext}}}{\partial x_\mu} \\ = \frac{\partial}{\partial x_\nu}\left\{2\eta\left[D_{\mu\nu} - \frac{1}{3}\delta_{\mu\nu}(\mathrm{Tr}\,D)\right] + \delta_{\mu\nu}(\zeta_1\boldsymbol{\nabla}\cdot[mn_c(\mathbf{v}_c-\mathbf{v}_n)] + \zeta_2\boldsymbol{\nabla}\cdot\mathbf{v}_n)\right\},\end{aligned} \tag{86b}$$

$$\frac{\partial\mathbf{v}_c}{\partial t} = -\boldsymbol{\nabla}\left\{\frac{\mu}{m} + \frac{v_c^2}{2} - \zeta_3\boldsymbol{\nabla}\cdot[mn_c(\mathbf{v}_c-\mathbf{v}_n)] - \zeta_4\boldsymbol{\nabla}\cdot\mathbf{v}_n\right\}, \tag{86c}$$

$$\frac{\partial s}{\partial t} + \boldsymbol{\nabla}\cdot\left(s\mathbf{v}_n - \frac{\kappa\boldsymbol{\nabla}T}{T}\right) = \frac{R_s}{T}. \tag{86d}$$

The total current is given by $\mathbf{j} = n_c\mathbf{v}_c + \tilde{n}\mathbf{v}_n$ and the dissipative function describing the entropy production rate is given by [14]

$$\begin{aligned} R_s = {} & \zeta_2(\boldsymbol{\nabla}\cdot\mathbf{v}_n)^2 + 2\zeta_1\boldsymbol{\nabla}\cdot\mathbf{v}_n\boldsymbol{\nabla}\cdot[mn_c(\mathbf{v}_c-\mathbf{v}_n)] \\ & + \zeta_3(\boldsymbol{\nabla}\cdot[mn_c(\mathbf{v}_c-\mathbf{v}_n)])^2 + 2\eta\left[D_{\mu\nu} - \frac{1}{3}\delta_{\mu\nu}(\mathrm{Tr}\,D)\right]^2 \\ & + \frac{\kappa}{T}(\boldsymbol{\nabla}T)^2. \end{aligned} \tag{87}$$

As we have discussed in Ref. [3], the normal fluid and the superfluid densities that appear in the standard Landau two-fluid theory can be identified with the corresponding noncondensate and condensate densities, within the context of our finite temperature model based on the HF approximation for single-particle excitations. We have explicitly made use of this correspondence in writing Eq. (86). We also note that in Eqs. (86) and (87), one can write $n_c(\mathbf{v}_c - \mathbf{v}_n)$ in the equivalent form $(\mathbf{j} - n\mathbf{v}_n)$, which is often used.

The thermodynamic functions that appear in these Landau–Khalatnikov (LK) two-fluid equations satisfy the following superfluid local thermodynamic relations:

$$P + \epsilon = \mu n + sT + m\tilde{n}(\mathbf{v}_n - \mathbf{v}_c)^2, \tag{88a}$$

$$dP = n\,d\mu + s\,dT - m\tilde{n}(\mathbf{v}_n - \mathbf{v}_c)\cdot d(\mathbf{v}_n - \mathbf{v}_c), \tag{88b}$$

$$d\epsilon = \mu\,dn + T\,ds + (\mathbf{v}_n - \mathbf{v}_c)\cdot d[m\tilde{n}(\mathbf{v}_n - \mathbf{v}_c)]. \tag{88c}$$

The various local thermodynamic functions which appear in the LK theory have to be carefully defined so that they satisfy the relations in Eq. (88). The local entropy is defined by (as in Ref. [3])

$$s=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}[(1+f)\ln(1+f)-f\ln f]. \tag{89}$$

Using $f=f^{(0)}+f^{(1)}$ and working to first order in $f^{(1)}$, one finds

$$\begin{aligned}s=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}&[(1+f^{(0)})\ln(1+f^{(0)})-f^{(0)}\ln f^{(0)}\\&+\ln(1+f^{(0)^{-1}})f^{(1)}].\end{aligned} \tag{90}$$

From the constraint on $f^{(1)}$ given by Eq. (53b), one sees that the last term, which arises from $f^{(1)}$, makes no contribution to the local entropy. One thus obtains

$$\begin{aligned}s&=\frac{1}{T}\left[\frac{5}{2}\tilde{P}^{(0)}-\tilde{n}^{(0)}(\tilde{\mu}^{(0)}-U)\right]\\&\simeq\frac{1}{T}\left[\frac{5}{2}\tilde{P}^{(0)}-\tilde{n}^{(0)}(\mu_c^{(0)}-U)-\frac{m\tilde{n}}{2}(\mathbf{v}_n-\mathbf{v}_c)^2\right],\end{aligned} \tag{91}$$

where we have used $\tilde{n}^{(0)}=\tilde{n}+O(v_n,v_c)$.

The local energy density ϵ in the Landau–Khalatnikov theory is defined in the local frame where $\mathbf{v}_c=0$ [23]. In the context of the present theory, this is given by

$$\epsilon=\tilde{\epsilon}+nU_{\text{ext}}+\frac{g}{2}(n^2+2n\tilde{n}-\tilde{n}^2)+\frac{m}{2}\tilde{n}(\mathbf{v}_n-\mathbf{v}_c)^2, \tag{92}$$

while the local energy density in the original lab frame is given by

$$\epsilon_{\text{lab}}=\epsilon+m\tilde{n}(\mathbf{v}_n-\mathbf{v}_c)\cdot\mathbf{v}_c+\frac{mn}{2}v_c^2. \tag{93}$$

Using Eqs. (68) and (69) in Eq. (92), one finds that the first-order corrections from Γ_{12} cancel out, leaving

$$\epsilon=\tilde{\epsilon}^{(0)}+nU_{\text{ext}}+\frac{g}{2}(n^2+2n\tilde{n}^{(0)}-[\tilde{n}^{(0)}]^2)+\frac{1}{2}m\tilde{n}(\mathbf{v}_n-\mathbf{v}_c)^2. \tag{94}$$

We conclude that both the local entropy density and the local energy density are determined by the diffusive local equilibrium distribution function $f^{(0)}$ alone, and are not altered by the deviation $f^{(1)}$.

In contrast, as we now show, the total pressure and the chemical potential must be carefully defined so that they satisfy the superfluid thermodynamic relations in Eq. (88). We first define the nonequilibrium pressure by

$$P'\equiv\tilde{P}+\frac{g}{2}(n^2+2n\tilde{n}-\tilde{n}^2). \tag{95}$$

Using Eqs. (68) and (69) and working to first order in $\Gamma_{12}^{(1)}$, one obtains

$$P'=P-\tau_\mu\frac{gn_c}{3}\Gamma_{12}^{(1)}, \tag{96}$$

where P is the (diffusive) local equilibrium pressure defined as

$$P\equiv\tilde{P}^{(0)}+\frac{g}{2}(n^2+2n\tilde{n}^{(0)}-[\tilde{n}^{(0)}]^2). \tag{97}$$

We find that the LK thermodynamic relations given in Eq. (88) are not satisfied if we assume that P' is the pressure (P) and μ_c is the chemical potential (μ). Extra terms appear which are associated with Γ_{12}. This means that the above identification of the thermodynamic variables is only valid in the lowest-order hydrodynamic equations, where there is no dissipation (see Sec. III A).

We recall that in deriving the Landau equations from the ZGN$'$ equations in Ref. [3], we defined the total pressure by Eq. (95) and $\mu=\mu_c$, and also found extra terms in the thermodynamic relations proportional to $\delta\mu_{\text{diff}}$ (see Eq. (71) of Ref. [3]). Therefore the precise equivalence between the ZGN$'$ hydrodynamics and the Landau theory shown in Ref. [3] was restricted in the limit $\omega\tau_\mu\to0$, i.e., when $\delta\mu_{\text{diff}}\to0$. In contrast, if the pressure P is defined to be Eq. (97) and $\mu=\mu_c^{(0)}$, we can show that the superfluid thermodynamic relations in Eq. (88) are satisfied. Therefore we conclude that the local equilibrium pressure defined in Eq. (97) and the local equilibrium chemical potential $\mu_c^{(0)}$ given by Eq. (35) are, in fact, the correct variables to be used in the Landau–Khalatnikov equations. We will show later that the corrections to the total pressure and the chemical potential actually give rise to the additional damping terms associated with the four second viscosity coefficients ζ_i in Eq. (86).

We now proceed to derive the LK equations from our microscopic theory, one by one. Our continuity equations for n_c and $\tilde{n}$ are given by Eqs. (8a) and (67a). Adding them, we obtain the continuity equation for the total density (86a). To derive the equation (86b) for the total current $\mathbf{j}$, we combine our two continuity equations and the two velocity equations (8b) and (67b) to give

$$\begin{aligned}m\frac{\partial j_\mu}{\partial t}&+\frac{\partial}{\partial x_\nu}(\delta_{\mu\nu}P'+m\tilde{n}v_{n\mu}v_{n\nu}+mn_cv_{c\mu}v_{c\nu})+n\frac{\partial U_{\text{ext}}}{\partial x_\mu}\\&=\frac{\partial}{\partial x_\nu}\left\{2\eta\left[D_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}(\text{Tr}\,D)\right]\right\}.\end{aligned} \tag{98}$$

Using Eq. (96), we find

$$\begin{aligned}m\frac{\partial j_\mu}{\partial t}&+\frac{\partial}{\partial x_\nu}(\delta_{\mu\nu}P+m\tilde{n}v_{n\mu}v_{n\nu}+mn_cv_{c\mu}v_{c\nu})+n\frac{\partial U_{\text{ext}}}{\partial x_\mu}\\&=\frac{\partial}{\partial x_\nu}\left\{2\eta\left[D_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}(\text{Tr}\,D)\right]+\delta_{\mu\nu}\tau\frac{gn_c}{3}\Gamma_{12}^{(1)}\right\}.\end{aligned} \tag{99}$$

To consistently include damping due to the first-order correction term in the Chapman-Enskog expansion, we use $\Gamma_{12}=\Gamma_{12}^{(1)}$ as given in Eq. (44). We then find that Eq. (99) is identical with the LK equation (86b) with the second viscosity coefficients ζ_1 and ζ_2 given by

$$\zeta_1=\frac{gn_c}{3m}\tau_\mu\sigma_H, \quad \zeta_2=\frac{gn_c^2}{9}\tau_\mu\sigma_H. \tag{100}$$

Using $\mu_c=\mu_c^{(0)}+g\tilde{n}^{(1)}=\mu-g\tau_\mu\Gamma_{12}^{(1)}$ [see Eq. (35)] and the expression for $\Gamma_{12}^{(1)}$ in our equation for the condensate velocity given in Eq. (8b), we find the latter can be written precisely in the LK form (86c). Comparison between the two equations shows that the second viscosity coefficients ζ_3 and ζ_4 are given by

$$\zeta_3=\frac{g}{m^2}\tau_\mu\sigma_H, \quad \zeta_4=\frac{gn_c}{3m}\tau_\mu\sigma_H. \tag{101}$$

We note that our results for the second viscosities satisfy the Onsager reciprocal relation $\zeta_1=\zeta_4$ (this equality follows quite generally, as shown by Eq. (4.28) of Ref. [23]).

Finally, we derive the equation for the local entropy. Using Eq. (88c), we have

$$T\frac{\partial s}{\partial t}=\frac{\partial \epsilon}{\partial t}-\mu\frac{\partial n}{\partial t}-m(\mathbf{v}_n-\mathbf{v}_c)\frac{\partial}{\partial t}[\tilde{n}(\mathbf{v}_n-\mathbf{v}_c)]. \tag{102}$$

With the expression for the local energy density ϵ given in Eq. (92), we find Eq. (102) reduces to

$$\begin{aligned}\frac{\partial s}{\partial t}&=\frac{\partial\tilde{\epsilon}}{\partial t}+[U_{\rm ext}-\mu+g(n+\tilde{n})]\frac{\partial n}{\partial t}+gn_c\frac{\partial\tilde{n}}{\partial t}\\&\quad-\frac{\partial n}{\partial t}\frac{m}{2}(\mathbf{v}_n-\mathbf{v}_c)^2\\&=\frac{\partial\tilde{\epsilon}}{\partial t}+g\tilde{n}^{(1)}\frac{\partial n}{\partial t}+gn_c\frac{\partial\tilde{n}}{\partial t}.\end{aligned} \tag{103}$$

Here we have neglected the last term in the first line, since it is of third order in the local velocities. Using our hydrodynamic equations (67), we find Eq. (103) reduces to the form (86d), assuming the entropy production rate R_s is given by

$$\begin{aligned}R_s=\tau_\mu g\Gamma_{12}\left\{g\boldsymbol{\nabla}\cdot[n_c(\mathbf{v}_c-\mathbf{v}_n)]+\frac{1}{3}n_c\boldsymbol{\nabla}\cdot\mathbf{v}_n\right\}\\+2\eta\left[D_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}(\mathrm{Tr}D)\right]^2+\frac{\kappa}{T}(\boldsymbol{\nabla}T)^2.\end{aligned} \tag{104}$$

Using Eq. (44) and the expression in Eqs. (100) and (101), we see that Eq. (104) is equivalent to the Landau-Khalatnikov [14] expression given in Eq. (87).

We have thus shown that our equations based on a microscopic theory built on Bose condensation can be written in a form precisely identical to the phenomenological Landau-Khalatnikov two-fluid equations including the damping associated with the shear viscosity, thermal conductivity, and four second viscosity coefficients. An analogous derivation of the Landau-Khalatnikov equations for a uniform Bose gas was first given by Kirkpatrick and Dorfman [12]. However, at finite temperatures, where the dominant excitations are particle-like Hartree-Fock excitations, Kirkpatrick and Dorfman did not obtain the second viscosities since they neglected the source term Γ_{12} associated with deviation from local equilibrium produced by the C_{12} collisions. We have shown that the second viscosity coefficients are directly related to the Γ_{12} term first discussed in Ref. [3], which represents the collisional exchange of atoms between the condensate and noncondensate.

In the above derivation of the second viscosity terms, we used $\Gamma_{12}=\Gamma_{12}^{(1)}$. This restricts the validity of the results to the case $\omega\tau_\mu\ll 1$ when we consider collective fluctuations with frequency ω. However, our discussion can be easily extended to the situation when $\omega\tau_\mu$ is not small, by using [see Eq. (26)]

$$\Gamma_{12}(\omega)=\frac{\sigma_H}{1-i\omega\tau_\mu}\left\{\boldsymbol{\nabla}\cdot[n_c(\mathbf{v}_c-\mathbf{v}_n)]+\frac{1}{3}n_c\boldsymbol{\nabla}\cdot\mathbf{v}_n\right\}. \tag{105}$$

Using this expression, we can still write our equations in the Landau-Khalatnikov form, but now with the frequency-dependent second viscosity coefficients

$$\zeta_i(\omega)=\frac{\zeta_i}{1-i\omega\tau_\mu}. \tag{106}$$

Everything else in our derivation goes through.

The expression (106) for the frequency-dependent second viscosity coefficients has in fact the expected form, as derived from general considerations [24]. The second viscosity, such as associated with compression and expansion, arises when a gas is coupled to an internal relaxation process (for example, the transfer of energy from the translational degrees of freedom of a molecule to the vibrational degrees of freedom). If the relaxation time of the internal process is denoted by $\tau_{\rm R}$, the frequency-dependent second viscosity coefficient is given by $\zeta(\omega)=\zeta_0/(1-i\omega\tau_{\rm R})$, where $\zeta_0\propto\tau_{\rm R}$. In a Bose-condensed gas at finite temperatures, the noncondensate atoms are coupled to the condensate degree of freedom, and we have shown that the characteristic relaxation time for the equilibration between the two components is given by τ_μ. In this connection, we might recall that in the superfluid ^{4}He, the second viscosities are due to the fact that the total number of phonons and rotons is not conserved [14]. Above $T_{\rm BEC}$ (when $n_c=0$), all the second viscosity coefficients ζ_i in Eqs. (100) and (101) vanish, as expected in a normal dilute single-component gas.

We finally note that the Landau-Khalatnikov equations could have been derived from the ZGN′ hydrodynamic equations if we simply identified the total pressure P with that by Eqs. (95) and (96), and took the chemical potential to be $\mu=\mu_c+g\tau_\mu\delta\Gamma_{12}$ [see Eq. (70)]. This leads more naturally to frequency-dependent second viscosities. On the other hand, the physical meaning of this choice of the local equilibrium pressure and chemical potential is not made clear.

V. CALCULATION OF TRANSPORT COEFFICIENTS

In this section, we solve the linearized equation for functions A, B, and D in Eq. (55) which determine the deviation from local equilibrium as described by ψ in Eqs. (45) and (54). We can then calculate the transport coefficients η and κ as given in Eq. (62). We follow the standard procedure in the Chapman-Enskog method, as reviewed in [21]. In this approach, one solves the linearized equation by expanding ψ in a basis set of polynomial functions. These polynomial functions are chosen to satisfy the constraints such as Eq. (53) which the solution ψ must satisfy. In a classical gas, one uses Sonine polynomials [21]. One can also define analogous polynomials for a degenerate Bose gas [25]. As usual, we calculate the transport coefficients using the lowest-order polynomial approximation, which usually gives very accurate results for the transport coefficients. For a more detailed mathematical discussion which is easily generalized to Bose condensed gases, we refer to Refs. [21,25].

A. The thermal conductivity

In evaluating the thermal conductivity, it is convenient to rewrite Eq. (62b) as

$$\kappa=-\frac{1}{3}k_B\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\mathbf{u}A(u)\cdot\mathbf{u}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]f_0 \times(1+f_0)=-\frac{1}{3}k_B\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\mathbf{u}A(u)\cdot\hat{L}[\mathbf{u}A(u)]. \tag{107}$$

Here we explicitly use the static equilibrium distribution function f_0 to evaluate the transport coefficients. To solve the linear integral equation (55a) for $A(u)$, we introduce a simple ansatz of the form [11,25]:

$$A(u)=A\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]. \tag{108}$$

This is the lowest-order polynomial function that satisfies the constraint given by Eq. (56a). The constant A in Eq. (108) is determined by multiplying Eq. (55a) by $\mathbf{u}[mu^2/2k_\mathrm{B}T-5g_{5/2}(z_0)/2g_{3/2}(z_0)]$ and integrating over $\mathbf{p}$, giving

$$A=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}u^2\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]^2f_0(1+f_0) \times\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]\mathbf{u}\cdot\hat{L} \times\left[\left\{\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right\}\mathbf{u}\right]\right\}^{-1}. \tag{109}$$

Using Eqs. (108) and (109) in Eq. (107), we find

$$\kappa=-\frac{k_B}{3}\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}u^2\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]^2f_0(1+f_0)\right\}^2 \times\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]\mathbf{u}\cdot\hat{L} \times\left[\left\{\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right\}\mathbf{u}\right]\right\}^{-1}. \tag{110}$$

To evaluate the $\mathbf{p}$ integrals in Eq. (110), it is convenient to introduce the dimensionless velocity variable by

$$\mathbf{x}\equiv\left(\frac{m}{2k_BT_0}\right)^{1/2}\mathbf{u}=\left(\frac{1}{2mk_BT_0}\right)^{1/2}\mathbf{p}. \tag{111}$$

With this new variable, we can rewrite the linearized collision operator as

$$\hat{L}[\psi]=\frac{8m(k_BT_0)^2a^2}{\pi^3\hbar^3}(\hat{L}'_{22}[\psi]+\pi^{3/2}n_{c0}\Lambda_0^3\hat{L}'_{12}[\psi]), \tag{112}$$

where the dimensionless collision operators $\hat{L}'_{22}$ and $\hat{L}'_{12}$ are defined by

$$\hat{L}'_{22}[\psi]\equiv\int d\mathbf{x}_2\int d\mathbf{x}_3\int d\mathbf{x}_4\,\delta(\mathbf{x}+\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4) \times\delta(x^2+x_2^2-x_3^2-x_4^2)f_{10}f_{20}(1+f_{30})(1+f_{40}) \times(\psi_3+\psi_4-\psi_2-\psi), \tag{113}$$

$$\hat{L}'_{12}[\psi]\equiv\int d\mathbf{x}_1\int d\mathbf{x}_2\int d\mathbf{x}_3\,\delta(\mathbf{x}-\mathbf{x}_2-\mathbf{x}_3) \times\delta(x_1^2-\beta_0gn_{c0}-x_2^2-x_3^2)[\delta(\mathbf{x}-\mathbf{x}_1)-\delta(\mathbf{x}-\mathbf{x}_2) -\delta(\mathbf{x}-\mathbf{x}_3)](1+f_{10})f_{20}f_{30}(\psi_2+\psi_3-\psi), \tag{114}$$

where $f_{i0}=(z_0^{-1}e^{x_i^2}-1)^{-1}$. Carrying out the $\mathbf{p}$ (or $\mathbf{u}$) integrals in Eq. (110), one finds

$$\kappa=\frac{75k_B}{64a^2m}\left(\frac{mk_BT}{\pi}\right)^{1/2}\frac{\pi^{1/2}}{I_{22}^{\kappa}(z_0)+\Lambda_0^3n_{c0}I_{12}^{\kappa}(z_0)}\left[\frac{7}{2}g_{7/2}(z_0) -\frac{5g_{5/2}^2(z_0)}{2g_{3/2}(z_0)}\right]^2, \tag{115}$$

where the functions $I_{22}^{\kappa}(z_0)$ and $I_{12}^{\kappa}(z_0)$ are defined by

$$I_{22}^{\kappa}\equiv-\int d\mathbf{x}\;\;\mathbf{x}x^2\cdot\hat{L}'_{22}[\mathbf{x}x^2], \tag{116}$$

$$I_{12}^{\kappa}\equiv-\pi^{3/2}\int d\mathbf{x}\;\;\mathbf{x}x^2\cdot\hat{L}'_{22}[\mathbf{x}x^2]. \tag{117}$$

In Ref. [11], we derived a convenient formula for the integral I_{22}^{κ}, namely

$$I^{\kappa}_{22}(z_0)=\sqrt{2}\pi^3=\int_0^\infty dx_0\int_0^\infty dx_r\int_{-1}^1 dy \times\int_{-1}^1 dy' F_{22}(x_0,x_r,y,y';z_0) \times x_0^4x_r^7(y^2+y'^2-2y^2y'^2), \tag{118}$$

where

$$F_{22}(x_0,x_r,y,y';z_0) =\frac{z_0^2 e^{-x_0^2-x_r^2}}{(1-z_0e^{-x_1^2})(1-z_0e^{-x_2^2})(1-z_0e^{-x_3^2})(1-z_0e^{-x_4^2})}, \tag{119}$$

with

$$x_1^2=\tfrac{1}{2}(x_0^2+2x_0x_ry+x_r^2),\quad x_2^2=\tfrac{1}{2}(x_0^2-2x_0x_ry+x_r^2),$$
$$x_3^2=\tfrac{1}{2}(x_0^2+2x_0x_ry'+x_r^2),\quad x_4^2=\tfrac{1}{2}(x_0^2-2x_0x_ry'+x_r^2). \tag{120}$$

We note that $I^{\kappa}_{22}(z_0)$ is a universal function of the equilibrium fugacity z_0, where $z_0=e^{-\beta g n_{c0}(\mathbf{r})}$. To derive a similar expression for I^{κ}_{12}, we introduce the transformation

$$\mathbf{x}_2=\tfrac{1}{2}(\mathbf{x}_0+\mathbf{x}_r),\qquad \mathbf{x}_3=\tfrac{1}{2}(\mathbf{x}_0-\mathbf{x}_r). \tag{121}$$

We then express $\mathbf{x}_r$ in the polar coordinate (x_r,θ,ϕ) where θ is the azimuthal angle with respect to the vector $\mathbf{x}_0$, i.e., $\mathbf{x}_r\cdot\mathbf{x}_0=x_rx_0\cos\theta$. With these new variables, one obtains the following expression for I^{κ}_{12}:

$$I^{\kappa}_{12}(z_0)=8\pi^{7/2}\int_0^\infty dx_r\int_{-1}^1 dy F_{12}(x_r,y;z_0)x_r^2(x_r^2+\beta_0gn_{c0})^{3/2} \times\left[x_r^2(x_r^2+3\beta_0gn_{c0})(1-y^2)+\frac{9}{4}(\beta_0gn_{c0})^2\right], \tag{122}$$

where $y=\cos\theta$ and

$$F_{12}(x_r,y;z_0)=\frac{z_0e^{-x_1^2}}{(1-z_0e^{-x_1^2})(1-z_0e^{-x_2^2})(1-z_0e^{-x_3^2})}, \tag{123}$$

with

$$x_1^2=2(x_r^2+\beta_0gn_{c0}),$$
$$x_2^2=x_r^2+x_ry\sqrt{x_r^2+\beta_0gn_{c0}}+\tfrac{1}{2}\beta_0gn_{c0}, \tag{124}$$
$$x_3^2=x_r^2-x_ry\sqrt{x_r^2+\beta_0gn_{c0}}+\tfrac{1}{2}\beta_0gn_{c0}.$$

The formula in Eq. (115) gives the thermal conductivity κ as a universal function of $gn_{c0}(\mathbf{r})$ or equivalently in terms of the local fugacity $z_0(\mathbf{r})=e^{-\beta_0gn_{c0}(\mathbf{r})}$. If we ignore the contribution from C_{12} collisions, i.e., set I^{κ}_{12} to zero, Eq. (115) reduces to the expression for κ derived in our earlier work [11,10].

One can also write the expression for κ in Eq. (115) in the following useful form:

$$\kappa=\frac{5\sqrt{2}}{\pi^3}\left(\tau_\kappa\frac{\tilde{n}_0k_B^2T_0}{m}\right)\left\{\frac{7g_{7/2}(z_0)}{2g_{5/2}(z_0)}-\left[\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]^2\right\}, \tag{125}$$

where τ_κ is the ‘‘thermal relaxation time’’ associated with the thermal conductivity, as defined in Appendix B. In turn, one can also write the reciprocal of this relaxation time τ_κ as the sum of contributions from C_{12} and C_{22} collisions,

$$\frac{1}{\tau_\kappa}=\frac{1}{\tau_{\kappa,12}}+\frac{1}{\tau_{\kappa,22}}, \tag{126}$$

where these relaxation times are given explicitly in Appendix B. The physical meaning of this τ_κ relaxation time is discussed in Appendix B, using a simple relaxation time approximation for the collision integrals in Eq. (1).

B. The shear viscosity

In evaluating the shear viscosity η, it is convenient to rewrite Eq. (62a) as

$$\eta=-\frac{m}{10}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)B(u) \times\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)f_0(1+f_0)$$
$$=-\frac{k_BT_0}{10}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}B(u)\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right) \times\hat{L}\left[B(u)\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)\right]. \tag{127}$$

To solve Eq. (127), the simplest consistent ansatz [11,25] is to use $B(u)\equiv B$. The constant B can be determined by multiplying Eq. (55b) by $(u_\mu u_\nu-\delta_{\mu\nu}u^2/3)$ and integrating over $\mathbf{p}$ to give

$$B=\frac{m}{k_BT_0}\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)^2 f_0(1+f_0)\right\} \times\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right) \times\hat{L}\left[u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right]\right\}^{-1}. \tag{128}$$

Using this in Eq. (127) we obtain

$$\eta=-\frac{m^2}{10k_BT_0}\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)^2 f_0\right.$$
$$\left.\times(1+f_0)\right\}^2\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)\right.$$
$$\left.\times\hat{L}\left[u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right]\right\}^{-1}. \tag{129}$$

With the dimensionless variable defined in Eq. (111), this expression for the shear viscosity η can be rewritten as

$$\eta=\frac{5\pi^3}{32\sqrt{2}a^2}(mk_{\mathrm{B}}T_0)^{1/2}\frac{g_{5/2}^2(z_0)}{I_{22}^{\eta}(z_0)+\Lambda_0^3 n_{c0}I_{12}^{\eta}(z_0)}, \tag{130}$$

where

$$I_{22}^{\eta}\equiv-\int d\mathbf{x}\left(x_\mu x_\nu-\frac{1}{3}\delta_{\mu\nu}x^2\right)\hat{L}'_{22}\left[x_\mu x_\nu-\frac{1}{3}\delta_{\mu\nu}x^2\right]$$
$$=\frac{\pi^3}{\sqrt{2}}\int_0^\infty dx_0\int_0^\infty dx_r\int_{-1}^{1}dy\int_{-1}^{1}dy'\,F_{12}(x_0,x_r,y,y';z_0)$$
$$\times x_0^2x_r^7(1+y^2+y'^2-3y^2y'^2), \tag{131}$$

and

$$I_{12}^{\eta}\equiv-\pi^{3/2}\int d\mathbf{x}\left(x_\mu x_\nu-\frac{1}{3}\delta_{\mu\nu}x^2\right)\hat{L}'_{12}\left[x_\mu x_\nu-\frac{1}{3}\delta_{\mu\nu}x^2\right]$$
$$=8\pi^{7/2}\int_0^\infty dx_r\int_{-1}^{1}dy F_{12}(x_r,y;z_0)$$
$$\times x_r^2\sqrt{x_r^2+\beta_0 g n_{c0}}\Big[x_r^2(x_r^2+\beta_0 g n_{c0})(1-y^2)$$
$$+\frac{1}{3}(\beta_0 g n_{c0})^2\Big]. \tag{132}$$

These expressions involve the same functions F_{22} and F_{12} defined earlier in Eqs. (119) and (123).

Analogous to Eq. (125) for the thermal conductivity, one can also write the expression for η in Eq. (130) in the following form:

$$\eta=\tau_\eta\tilde{n}_0k_BT_0\left[\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right], \tag{133}$$

where the viscous relaxation time τ_η is defined in Appendix B. Again one can write the reciprocal of the relaxation time τ_η as

$$\frac{1}{\tau_\eta}=\frac{1}{\tau_{\eta,12}}+\frac{1}{\tau_{\eta,22}}, \tag{134}$$

where these C_{12} and C_{22} relaxation times are given in Appendix B. As with τ_κ, the relaxation time τ_η can be understood in terms of a simple relaxation time approximation for the collision term in the kinetic equation.

C. The second viscosity coefficients

To find the expression for τ as defined in Eq. (63), we use the simple ansatz for the form of the solution for $D(u)$ of Eq. (55c),

$$D(u)=D\left(\sigma_2+\frac{mu^2}{3k_BT_0}\sigma_1\right), \tag{135}$$

where σ_1 and σ_2 are defined in Eq. (51). As usual, we leave the dependence on $(\mathbf{r},t)$ implicit. One easily verifies that Eq. (135) satisfies the constraint (56b). The constant D can be determined by integrating Eq. (55c) over $\mathbf{p}$:

$$D\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\hat{L}_{12}\left[\sigma_2+\frac{mu^2}{3k_BT_0}\sigma_1\right]$$
$$=\frac{1}{\tilde{n}^{(0)}}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(\sigma_2+\frac{mu^2}{3k_BT_0}\right)$$
$$\times f_0(1+f_0)-\beta_0 g\tau\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\hat{L}_{12}[1]. \tag{136}$$

Using $\hat{L}_{12}[mu^2/2]=\beta_0 g n_{c0}\hat{L}_{12}[1]$ and

$$\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\hat{L}_{12}[1]=-\frac{n_{c0}}{\tau_{12}},$$

where τ_{12} is defined in Eq. (22), Eq. (136) gives finally

$$D=-\frac{(\tau_{12}+\beta_0 g n_{c0}\tau)}{n_{c0}\left[\sigma_2+\frac{2}{3}\sigma_1\beta(\mu_{c0}-U_0)\right]}. \tag{137}$$

Using Eqs. (137) and (135) in the expression for τ given by Eq. (63), we can solve to give an explicit expression for τ, namely

$$\frac{1}{\tau}=\frac{1}{\tau_{12}}\left\{\frac{n_{c0}}{\tilde{n}_0}\left[\sigma_2+\frac{2}{3}\sigma_1\beta_0(\mu_{c0}-U_0)\right]-\beta_0 g n_{c0}\right\}$$
$$=\frac{\beta_0 g n_{c0}}{\sigma_H\tau_{12}}=\frac{1}{\tau_\mu}. \tag{138}$$

We thus see that τ is precisely the relaxation time τ_μ first introduced in the ZGN′ two-fluid hydrodynamics. We can now express the four second viscosity coefficients in Eqs. (100) and (101) in terms of the τ_{12} collision time defined in Eq. (22):

$$\zeta_1=\frac{k_BT}{3m}\sigma_H^2\tau_{12},\quad \zeta_2=\frac{n_ck_BT}{9}\sigma_H^2\tau_{12},\quad \zeta_3=\frac{k_BT}{m^2n_c}\sigma_H^2\tau_{12},$$
$$\zeta_4=\zeta_1. \tag{139}$$

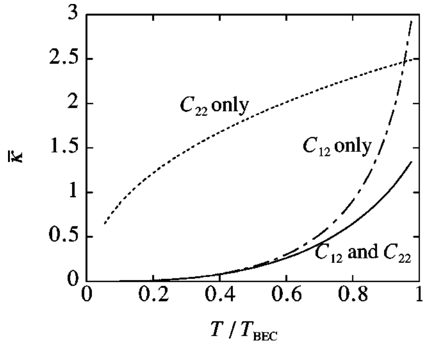

FIG. 1. Plot of the thermal conductivity κ [in units defined in Eq. (140)] in a uniform gas for $gn=0.2k_BT_{\rm BEC}$ as a function of temperature. We also plot the results by taking $I_{12}^{\kappa}=0$ (dashed line) and $I_{22}^{\kappa}=0$ (dot-dashed line).

D. Numerical results for a uniform Bose gas

For illustration, we calculate the transport coefficients in Eqs. (115) and (130) for a uniform Bose gas ($U_{\rm ext}=0$). As in Ref. [10] we choose $gn/k_BT_{\rm BEC}=0.2$. In Figs. 1 and 2 we plot the temperature dependence of the dimensionless transport coefficients $\overline{\kappa}$ and $\overline{\eta}$, defined by

$$\overline{\kappa}\equiv\kappa/nv_{\rm cl}^2\tau_0k_B, \quad \overline{\eta}\equiv\eta/nv_{\rm cl}m\tau_0. \tag{140}$$

Here $v_{\rm cl}\equiv(5k_BT_{\rm BEC}/3m)^{1/2}$ is the hydrodynamic sound velocity of a classical gas at $T=T_{\rm BEC}$ and $\tau_0^{-1}\equiv\sqrt{2}(8\pi a^2)n(8k_BT_{\rm BEC}/\pi m)^{1/2}$ is the classical mean collision time evaluated at $T=T_{\rm BEC}$. To see the separate effects of the C_{22} collisions and C_{12} collisions, we also plot the results obtained by taking either $I_{12}^{\kappa,\eta}=0$ or $I_{22}^{\kappa,\eta}=0$. In Ref. [10] we neglected C_{12}. We see that both κ and η are reduced when we include the C_{12} collisions. At lower temperatures $T\lesssim0.5T_{\rm BEC}$, both κ and η are dominated by the C_{12} collision integral.

In Fig. 3 we plot the four second viscosity coefficients given in Eq. (139) for a uniform Bose gas. We recall that in Fig. 1 of Ref. [9] we gave the temperature dependence of τ_μ and τ_{12} for $gn/k_{\rm B}T_{\rm BEC}=0.1$. In Fig. 3 we use the dimensionless second viscosity coefficients, defined by

$$\overline{\zeta_1}\equiv\zeta_1/v_{\rm cl}^2\tau_0, \quad \overline{\zeta_2}\equiv\zeta_2m/nv_{\rm cl}^2\tau_0, \quad \overline{\zeta_3}\equiv\zeta_2n/mv_{\rm cl}^2\tau_0. \tag{141}$$

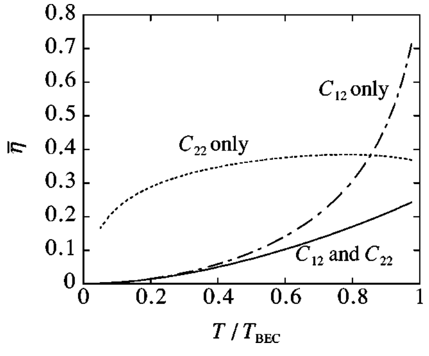

FIG. 2. Plot of the shear viscosity coefficient in a uniform gas for $gn=0.2k_BT_{\rm BEC}$ as a function of temperature η [in units defined in Eq. (140)]. We also plot the results by taking $I_{12}^{\eta}=0$ (dashed line) and $I_{22}^{\eta}=0$ (dot-dashed line).

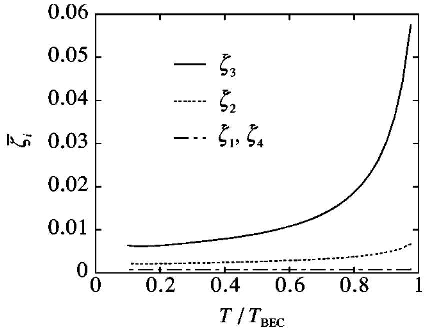

FIG. 3. Plot of the second viscosity coefficients ζ_i [in units defined in Eq. (141)] in a uniform gas for $gn=0.2k_BT_{\rm BEC}$ as a function of temperature.

The transport coefficients in a trapped Bose gas behave quite differently from those of a uniform Bose gas. In particular, since $n_c\gg\tilde{n}$ holds in the central region of the trap at all temperatures, the contribution of C_{12} collisions dominates over the contribution of the C_{22} collisions at all temperatures below $T_{\rm BEC}$. We will discuss the implications of this at the end of Sec. VI.

VI. CONCLUSIONS

In this paper we have derived two-fluid hydrodynamic equations starting from the quantum kinetic equation and the generalized GP equation derived in [3,9,17]. However, to complement and extend our earlier work [3,9], we started from the complete local equilibrium single-particle distribution $f^{(0)}$ as given by Eqs. (37) and (38). Using the Chapman–Enskog approach, we then included the effects of a small deviation from this local equilibrium form. This deviation from local equilibrium within the thermal cloud brings in the usual kind of hydrodynamic damping due to the thermal conductivity and shear viscosity of the thermal cloud. A summary of our major results is given in the final paragraph of Sec. I.

In addition, we showed that there is additional dissipation associated with the collisional exchange of atoms between the condensate and noncondensate components. When we write our hydrodynamic equations in the Landau–Khalatnikov [14,23] form given by Eq. (86), this damping is described in terms of the usual four second viscosity coefficients for a Bose superfluid. The appearance of the second viscosity coefficients in the equations for the total current **j** in Eq. (86b) and for the superfluid velocity $\mathbf{v}_c$ in Eq. (86c) is due to the deviation of the total pressure and the chemical potential from their local equilibrium values. We might also recall that Khalatnikov [14] discusses a specific model for the second viscosity coefficients in superfluid ^{4}He by introducing ''local chemical potentials'' for the phonons ($\mu_{\rm ph}$) and rotons ($\mu_{\rm r}$). These describe a situation where such excitations (describing the normal fluid) are out of local equilibrium with the superfluid component. Clearly this discussion has connections with our calculations based on the condensate and noncondensate not being in diffusive equilibrium.

The frequency-dependence of the second viscosity coefficients is a result of the fact that our two-fluid hydrodynamics deals with the dynamics of the condensate and noncondensate components as separate degrees of freedom. This feature is made more explicit in our recent papers [3,9,10]. In particular, it gives rise to a new relaxational zero-frequency mode. As mentioned at the end of Sec. II, and more explicitly in Ref. [10], this mode may be viewed as the (renormalized) version of the zero-frequency thermal diffusion mode [20] above T_{BEC}. The presence of this new mode below T_{BEC} is somewhat hidden in the formulation in terms of the LK two-fluid equations given in Eq. (86).

In Sec. V we derived explicit formulas for all the transport coefficients within our model. In Ref. [10] we only took into account the deviations from local equilibrium due to the C_{22} collision integral in calculating the shear viscosity and the thermal conductivity coefficients. In the present paper we have also included the contribution to these quantities from the C_{12} collision integral. From Eqs. (125) and (133) we see that both κ and η are given in a form proportional to characteristic relaxation times $\tau_\kappa=(\tau^{-1}_{\kappa,12}+\tau^{-1}_{\kappa,22})^{-1}$ and $\tau_\eta=(\tau^{-1}_{\eta,12}+\tau^{-1}_{\eta,22})^{-1}$, respectively, which are defined and motivated in Appendix B.

In a rough estimate we find $\tau_{\kappa,22},\tau_{\eta,22}\sim\tau_{\mathrm{cl}}\sim 1/\tilde{n}$ and $\tau_{\kappa,12},\tau_{\eta,12}\sim(\tilde{n}/n_c)\tau_{\mathrm{cl}}\sim 1/n_c$, where τ_{cl} is the classical collision time defined in Eq. (B2). We therefore observe that the effect of C_{12} collisions reduces the magnitude of both κ and η by a factor $\sim 1/(1+n_c/\tilde{n})$, a result also noted in Ref. [12] for a uniform gas. The contribution of the C_{12} collisions is always important in a trapped gas, since the condensate density at the central region of a trap is much larger than the density of the thermal cloud even at temperatures very close to T_{BEC}. In a trapped Bose gas we find the effect of the C_{12} collisions is enhanced by a large factor $n_c/\tilde{n}\gg 1$. This means that κ and η are always dominated by the contribution of the C_{12} collisions. Since the effective relaxation times $\tau_{\kappa,12}$ and $\tau_{\eta,12}$ are smaller than the classical collision time τ_{cl} by a factor $\tilde{n}/n_c\ll 1$, this implies that in a trapped Bose gas, the hydrodynamic region may be much easier to reach at finite temperatures than expected from naive considerations based on using the classical collision time (i.e., $\omega\tau_{\mathrm{cl}}\ll 1$). That is to say, one might easily have $\omega\tau_{\kappa,12}\ll 1$ and $\omega\tau_{\eta,12}\ll 1$, even though $\omega\tau_{\mathrm{cl}}\gg 1$. This has very important implications in deciding if one is in the collisionless or the hydrodynamic region.

One problem not dealt with in this paper is the fact that in a trapped Bose gas, the decreasing density in the tail of the thermal cloud means that the hydrodynamic description breakdowns eventually. This problem enters the evaluation of the expressions for the η and κ transport coefficients given in Sec. V. In recent papers dealing with the case above T_{BEC} [26,11], this problem was handled in an *ad hoc* manner by introducing a spatial cutoff in the thermal cloud. In a future paper we give an alternative approach based on starting with an improved solution of the kinetic equation, which naturally includes the crossover to the noninteracting gas in the thermal gas tail.

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APPENDIX A

We briefly sketch the derivation of the first-order kinetic equation given in Eq. (50). Using Eq. (37) in Eq. (49), one has

$$\left[\frac{\partial^0}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}_{\mathbf{r}}-\boldsymbol{\nabla}_{\mathbf{r}}U(\mathbf{r},t)\cdot\boldsymbol{\nabla}_{\mathbf{p}}\right]f^{(0)}(\mathbf{r},\mathbf{p},t)$$
$$=\left[\frac{1}{z}\left(\frac{\partial^0}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)z+\frac{mu^2}{2k_BT^2}\left(\frac{\partial^0}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)T\right.$$
$$\left.+\frac{m\mathbf{u}}{k_BT}\cdot\left(\frac{\partial^0}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)\mathbf{v}_n+\frac{\boldsymbol{\nabla}U(\mathbf{r},t)}{k_BT}\cdot\mathbf{u}\right]f^{(0)}(1+f^{(0)}). \tag{A1}$$

The notation $\partial^0/\partial t$ is explained below Eq. (49). Using the expressions for the density $\tilde{n}^{(0)}$ given by Eq. (40) and the pressure $\tilde{P}^{(0)}$ in Eq. (42), one finds

$$\frac{\partial^0\tilde{n}^{(0)}}{\partial t}=\frac{3\tilde{n}^{(0)}}{2T}\frac{\partial^0T}{\partial t}+\frac{\gamma k_BT}{z}\frac{\partial^0z}{\partial t},$$
$$\frac{\partial^0\tilde{P}^{(0)}}{\partial t}=\frac{5\tilde{P}^{(0)}}{2T}\frac{\partial^0T}{\partial t}+\frac{\tilde{n}^{(0)}k_BT}{z}\frac{\partial^0z}{\partial t}, \tag{A2}$$

where γ is the variable introduced after Eq. (51) and $z=z^{(0)}$ as defined below Eq. (40). One may combine these equations with Eqs. (43a) and (43c) to show that the equations in Eq. (A2) reduce to

$$\frac{\partial^0T}{\partial t}=-\frac{2}{3}T(\boldsymbol{\nabla}\cdot\mathbf{v}_n)-\mathbf{v}_n\cdot\boldsymbol{\nabla}T+\frac{2T}{3\tilde{n}^{(0)}}\sigma_1\Gamma_{12},$$
$$\frac{\partial^0z}{\partial t}=-\mathbf{v}_n\cdot\boldsymbol{\nabla}z+\sigma_2z\frac{\Gamma_{12}}{\tilde{n}^{(0)}}. \tag{A3}$$

The analogous equation for $\partial^0\mathbf{v}_n/\partial t$ is given directly by Eq. (43b). Using these results in Eq. (A1), one finds that it reduces to

$$\begin{aligned}\left(\frac{\partial^0}{\partial t}+\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}_{\mathbf{r}}-\boldsymbol{\nabla}_{\mathbf{r}}U\cdot\boldsymbol{\nabla}_{\mathbf{p}}\right)f^{(0)}\\ =\Bigg\{\frac{1}{T}\mathbf{u}\cdot\boldsymbol{\nabla}T\left(\frac{mu^2}{2k_BT}-\frac{5\tilde{P}^{(0)}}{2\tilde{n}^{(0)}k_BT}\right)\\ +\frac{m}{k_BT}\left[\mathbf{u}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}_n-\frac{u^2}{3}\boldsymbol{\nabla}\cdot\mathbf{v}_n\right]\\ +\left[\sigma_2+\frac{mu^2}{3k_BT}\sigma_1+\frac{m}{k_BT}\mathbf{u}\cdot(\mathbf{v}_c-\mathbf{v}_n)\right]\frac{\Gamma_{12}}{\tilde{n}^{(0)}}\Bigg\}f^{(0)}\\ \times(1+f^{(0)}),\end{aligned}\tag{A4}$$

where we recall $\mathbf{u}\equiv\mathbf{p}/m-\mathbf{v}_n$. In calculating the dissipative terms, we only consider terms to first order in the velocity fields $\mathbf{v}_n$ and $\mathbf{v}_c$. Since Γ_{12} is proportional to $\mathbf{v}_n$ and $\mathbf{v}_c$ [see Eq. (44)], we can neglect the last term (proportional to $\mathbf{v}_c-\mathbf{v}_n$) in Eq. (A4). This linearized version of Eq. (A4) can be rewritten in the form shown on the left-hand side of Eq. (50).

APPENDIX B

The relaxation time τ_κ in Eq. (125) is defined by

$$\begin{aligned}\tau_\kappa\equiv-\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]^2u^2f_0(1+f_0)\Bigg\}\\ \times\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]\mathbf{u}\\ \times\hat{L}\left[\left\{\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right\}\mathbf{u}\right]\Bigg\}^{-1}\\ =\frac{15\pi^{9/2}\hbar^3}{8m(k_BT_0)^2a^2}\left[\frac{\frac{7}{2}g_{7/2}(z_0)-\frac{5}{2}g_{5/2}^2(z_0)/g_{3/2}(z_0)}{I_{22}^{\kappa}(z_0)+n_{c0}\Lambda_0^3I_{12}^{\kappa}(z_0)}\right]\\ =\frac{15\sqrt{2}\pi^{7/2}}{4}\tau_{\rm cl}\left[\frac{\frac{7}{2}g_{7/2}(z_0)g_{3/2}(z_0)-\frac{5}{2}g_{5/2}^2(z_0)}{I_{22}^{\kappa}(z_0)+n_{c0}\Lambda_0^3I_{12}^{\kappa}(z_0)}\right].\end{aligned}\tag{B1}$$

Here

$$\tau_{\rm cl}^{-1}\equiv\sqrt{2}(8\pi a^2)\tilde{n}_0(\mathbf{r})(8k_BT_0/\pi m)^{1/2},\tag{B2}$$

is the collision time of a classical gas with density $\tilde{n}_0$ and quantum cross section $\sigma=8\pi a^2$. In turn, the relaxation times in Eq. (126) are defined by

$$\begin{aligned}\tau_{\kappa,22}\equiv-\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]^2u^2f_0(1+f_0)\Bigg\}\\ \times\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]\mathbf{u}\\ \times\hat{L}_{22}\left[\left\{\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right\}\mathbf{u}\right]\Bigg\}^{-1}\\ =\frac{15\pi^{9/2}\hbar^3}{8m(k_BT_0)^2a^2}\left[\frac{\frac{7}{2}g_{7/2}(z_0)-\frac{5}{2}g_{5/2}^2(z_0)/g_{3/2}(z_0)}{I_{22}^{\kappa}(z_0)}\right]\\ =\frac{15\sqrt{2}\pi^{7/2}}{4}\tau_{\rm cl}\left[\frac{\frac{7}{2}g_{7/2}(z_0)g_{3/2}(z_0)-\frac{5}{2}g_{5/2}^2(z_0)}{I_{22}^{\kappa}(z_0)}\right],\end{aligned}\tag{B3}$$

and

$$\begin{aligned}\tau_{\kappa,12}\equiv-\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]^2u^2f_0(1+f_0)\Bigg\}\\ \times\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right]\mathbf{u}\\ \times\hat{L}_{12}\left[\left\{\frac{mu^2}{2k_BT_0}-\frac{5g_{5/2}(z_0)}{2g_{3/2}(z_0)}\right\}\mathbf{u}\right]\Bigg\}^{-1}\\ =\frac{15\pi^{9/2}\hbar^3}{8m(k_BT_0)^2a^2}\left[\frac{\frac{7}{2}g_{7/2}(z_0)-\frac{5}{2}g_{5/2}^2(z_0)/g_{3/2}(z_0)}{n_{c0}\Lambda_0^3I_{12}^{\kappa}(z_0)}\right]\\ =\frac{15\sqrt{2}\pi^{7/2}}{4}\tau_{\rm cl}\left[\frac{\frac{7}{2}g_{7/2}(z_0)g_{3/2}(z_0)-\frac{5}{2}g_{5/2}^2(z_0)}{n_{c0}\Lambda_0^3I_{12}^{\kappa}(z_0)}\right].\end{aligned}\tag{B4}$$

In an analogous way, the relaxation time τ_η in Eq. (133) is defined by

$$\begin{aligned}\tau_\eta\equiv-\left[\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)^2f_0(1+f_0)\right]\\ \times\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)\\ \times\hat{L}\left[u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right]\Bigg\}^{-1}\\ =\frac{5\pi^{9/2}\hbar^3}{16m(k_BT_0)^2a^2}\left[\frac{g_{5/2}(z_0)}{I_{22}^{\eta}(z_0)+n_{c0}\Lambda_0^3I_{12}^{\eta}(z_0)}\right]\\ =\frac{5\sqrt{2}\pi^{7/2}}{2}\tau_{\rm cl}\left[\frac{g_{5/2}(z_0)g_{3/2}(z_0)}{I_{22}^{\eta}(z_0)+n_{c0}\Lambda_0^3I_{12}^{\eta}(z_0)}\right].\end{aligned}\tag{B5}$$

The relaxation times in Eq. (134) are defined by

$$\begin{aligned}\tau_{\eta,22}\equiv-\left[\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)^2f_0(1+f_0)\right]\\ \times\Bigg\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^3\right)\\ \times\hat{L}_{22}\left[u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^3\right]\Bigg\}^{-1}\end{aligned}$$

$$
\begin{aligned}
&= \frac{5\pi^{9/2}\hbar^3}{16m(k_BT_0)^2a^2}\left[\frac{g_{5/2}(z_0)}{I_{22}^{\eta}(z_0)}\right] \\
&= \frac{5\sqrt{2}\pi^{7/2}}{2}\tau_{\mathrm{cl}}\left[\frac{g_{5/2}(z_0)g_{3/2}(z_0)}{I_{22}^{\eta}(z_0)}\right],
\end{aligned}
\tag{B6}
$$

and

$$
\begin{aligned}
\tau_{\eta,12} \equiv & -\left[\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)^2 f_0(1+f_0)\right] \\
& \times\left\{\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right)\right. \\
& \left.\times\hat{L}_{12}\left[u_\mu u_\nu-\frac{1}{3}\delta_{\mu\nu}u^2\right]\right\}^{-1} \\
& = \frac{5\pi^{9/2}\hbar^3}{16m(k_BT_0)^2a^2}\left[\frac{g_{5/2}(z_0)}{n_{c0}\Lambda_0^3 I_{12}^{\eta}(z_0)}\right] \\
& = \frac{5\sqrt{2}\pi^{7/2}}{2}\tau_{\mathrm{cl}}\left[\frac{g_{5/2}(z_0)g_{3/2}(z_0)}{n_{c0}\Lambda_0^3 I_{12}^{\eta}(z_0)}\right].
\end{aligned}
\tag{B7}
$$

We note that in a nondegenerate gas, these expressions simplify and we find $\tau_\kappa=\tau_{\kappa,22}=\frac{15}{8}\tau_{\mathrm{cl}}(n_0(\mathbf{r}))$ and $\tau_\eta=\tau_{\eta,22}=\frac{5}{4}\tau_{\mathrm{cl}}(n_0(\mathbf{r}))$. The latter expression for τ_η agrees with the result for the shear viscosity given in Eq. (11) by Kavoulakis *et al.* [27].

The physical meaning of these new relaxation times becomes clear when we compare our Chapman–Enskog analysis in the text with a simple relaxation time approximation [20]. In the relaxation time approximation, one simply replaces the collision term in Eq. (1) with $-[f-f^{(0)}]/\tau_{\mathrm{rel}}$, where τ_{rel} is a phenomenological relaxation time characterizing how fast the system relaxes to local equilibrium. With this approximation, the solutions of the linearized equations in Eqs. (55a) and (55b) for the functions $A(u)$ and $B(u)$ are found to be simply given by

$$
A(u)=-\tau_{\mathrm{rel}}\left[\frac{mu^2}{2k_BT}-\frac{5g_{5/2}(z)}{2g_{3/2}(z)}\right],\quad B(u)=-\tau_{\mathrm{rel}}\frac{m}{k_BT}.
\tag{B8}
$$

In contrast, our Chapman–Enskog solution for $A(u)$ is given by Eq. (108) with the coefficient A given by Eq. (109), while one has $B(u)=B$ with the constant B given by Eq. (128). In terms of the relaxation times τ_κ and τ_η defined above, we find Eqs. (108) and (128) can be written as

$$
A(u)=-\tau_\kappa\left[\frac{mu^2}{2k_BT}-\frac{5g_{5/2}(z)}{2g_{3/2}(z)}\right],\quad B(u)=-\tau_\eta\frac{m}{k_BT}.
\tag{B9}
$$

Comparing Eq. (B9) with Eq. (B8), we see that both τ_κ and τ_η can be identified with the relaxation time τ_{rel}. That is, in the simple relaxation time approximation, κ and η are still given by the formulas (125) and (133), but with $\tau_\kappa=\tau_\eta=\tau_{\mathrm{rel}}$. We also note that Eq. (7) of Ref. [27] gives a general expression for various collisional relaxation times above T_{BEC}, which is given by a formula analogous to Eqs. (B1) and (B5).

It is these effective relaxation times that determine whether one is in the hydrodynamic two-fluid domain, $\omega\tau_\kappa\lesssim 1$, $\omega\tau_\eta\lesssim 1$ (see discussion in Sec. VI). They should be used in place of the classical collision time τ_{cl} in Eq. (B2).

APPENDIX C

Hohenberg and Martin [23] worked out the dispersion relation of the hydrodynamic modes in a uniform Bose superfluid using the Landau–Khalatnikov two-fluid equations. The frequencies of the first and second sound modes are given by

$$
\omega^2=u_i^2k^2-iD_ik^2\omega,
\tag{C1}
$$

where the sound velocities u_i are determined by the coupled equations

$$
u_1^2+u_2^2=\frac{T\rho_s\bar{s}^2}{\rho_n\bar{c}_v}+\left.\frac{\partial P}{\partial\bar{s}}\right|_{\bar{s}},
\tag{C2}
$$

$$
u_1^2u_2^2=\frac{T\rho_s\bar{s}^2}{\rho_n\bar{c}_v}\left.\frac{\partial P}{\partial\rho}\right|_T.
\tag{C3}
$$

Here $\bar{s}\equiv s/mn$ is the entropy per unit mass and $\bar{c}_v$ is the specific heat per unit mass. The damping coefficients D_i are determined by the coupled equations

$$
D_1+D_2=\frac{4\eta}{3\rho_n}+\frac{\zeta_2}{\rho_n}-\frac{\rho_s}{\rho_n}(\zeta_1+\zeta_4)+\frac{\zeta_3\rho_s}{\rho_n}\rho+\frac{\kappa}{\rho\bar{c}_v},
\tag{C4}
$$

$$
u_1^2D_1+u_2^2D_2=\frac{\partial P}{\partial\rho}+\frac{\zeta_2+4\eta/3}{\rho}\frac{\rho_s}{\rho_n}\left(\frac{T\bar{s}^2}{\bar{c}_v}-\frac{2T\bar{s}}{\rho\bar{c}_v}\left.\frac{\partial P}{\partial T}\right|_\rho\right.
$$

$$
\left.+\left.\frac{\partial P}{\partial\rho}\right|_{\bar{s}}\right)+\zeta_3\frac{\rho_s}{\rho_n}\left.\frac{\partial P}{\partial\rho}\right|_{\bar{s}}\rho
\tag{C5}
$$

$$
-(\zeta_1+\zeta_4)\frac{\rho_s}{\rho_n}\left(-\frac{T\bar{s}}{\bar{c}_v\rho}\left.\frac{\partial P}{\partial T}\right|_\rho+\left.\frac{\partial P}{\partial\rho}\right|_{\bar{s}}\right).
\tag{C6}
$$

We note that the above general expressions are valid for both liquid ^{4}He and Bose gases. For the liquid ^{4}He these formulas can be simplified by using $\bar{c}_v\approx\bar{c}_p$. One has

$$
u_1^2\approx\frac{\partial P}{\partial\rho},
\tag{C7}
$$

$$
u_2^2\approx-\frac{\rho_s}{\rho_n}\frac{\partial T}{\partial(1/\bar{s})},
\tag{C8}
$$

$$D_1 \approx \frac{\zeta_2 + 4\eta/3}{\rho}, \tag{C9}$$

$$D_2 \approx \frac{1}{\rho}\left\{ \frac{\kappa}{T(\partial \bar{s}/\partial T)} + \frac{\rho_s}{\rho_n}\left[\zeta_3 \rho^2 - \rho(\zeta_1 + \zeta_4) + \zeta_2 + \frac{4}{3}\eta \right] \right\}. \tag{C10}$$

However, these are not valid for a dilute Bose gas. Calculating the various thermodynamic derivatives with the Hartree–Fock single-particle spectrum used in this paper [28], the sound velocities and damping coefficients for a uniform Bose gas are given by

$$u_1^2 \approx \frac{5\tilde{P}_0}{3m\tilde{n}_0} + \frac{2g\tilde{n}_0}{m}\left(1 - \frac{2n_{c0}^2}{9\tilde{n}_0^2}\sigma_H \right), \tag{C11}$$

$$u_2^2 \approx \frac{gn_{c0}}{m}(1 - \sigma_H), \tag{C12}$$

$$D_1 \approx \frac{4\eta}{3\rho_n} + \frac{\zeta_2}{\rho_n} - \frac{\rho_s}{\rho_n}(\zeta_1 + \zeta_4) + \frac{\zeta_3 \rho_s^2}{\rho_n} + \frac{4}{9}\frac{\kappa T_0}{\rho_n u_1^2} \times \left(1 + \frac{2n_{c0}}{3\tilde{n}_0}\sigma_1 \sigma_H \right)^2, \tag{C13}$$

$$D_2 \approx \rho_s \zeta_3 + \frac{4}{9}\frac{\rho_s}{\rho_n^2 u_2^2}(\sigma_1 \sigma_H)^2 \kappa T_0. \tag{C14}$$

Here σ_H is defined in Eq. (25), while $\rho_n = m\tilde{n}_0$ and $\rho_s = mn_{c0}$.

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