

Semiclassical theory of trapped fermionic dipoles

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We investigate the properties of a degenerate dilute gas of neutral fermionic particles in a harmonic trap that interact via dipole-dipole forces. We employ the semiclassical Thomas-Fermi method and discuss the Dirac correction to the interaction energy. A nearly analytic as well as an exact numerical minimization of the Thomas-Fermi-Dirac energy functional are performed in order to obtain the density distribution. We determine the stability of the system as a function of the interaction strength, the particle number, and the trap geometry. We find that there are interaction strengths and particle numbers for which the gas cannot be trapped stably in a spherically symmetric trap, but both prolate and oblate traps will work successfully.

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I. INTRODUCTION

The experimental achievement of quantum degeneracy in a dilute trapped gas of cold fermionic atoms [1] has stimulated theoretical interest in the properties of this fundamental system. Attention has focused on the critical temperature [2] and the detection [3] of Cooper pairing as well as on the properties of mixtures of various fermionic and bosonic species [4,5]. Another important problem concerns the interactions between ultracold fermions [5–7]. Owing to the exclusion principle, spin-polarized Fermi atoms do not interact via *s*-wave collisions, whereas they dominate in the low-energy regime for bosons and have pronounced effects on the static and dynamical properties of cold boson gases [8]. Hence, in the absence of low-energy collisions other types of forces come into play.

A good candidate is a dipole-dipole interaction between atoms or molecules, not analyzed so far in the context of cold trapped fermions. Some atoms possess permanent magnetic dipole moments of considerable magnitude (chromium, for instance, has $\mu = 6\mu_B$). It was also proposed to induce electric dipoles in atoms [9,10]. Huge permanent electric dipole moments occur naturally in diatomic polar molecules [11]. The behavior of atomic bosonic dipoles in traps has been investigated in [12,13], which addressed the question of instabilities in the system caused by an attractive component of dipolar interactions. The conclusion drawn was that a large enough positive scattering length (providing repulsive interactions) can stabilize a system of bosonic dipoles. For strong (e.g. molecular) dipoles, when supposedly the scattering length can be neglected, it is the trap geometry that plays a crucial role [10]—the system is stable provided the trap assures the domination of repulsive interactions in the gas.

A full quantum mechanical description of the system of many interacting fermions is of course very complex. But the lessons of semiclassical atomic physics can be applied, in particular the Thomas-Fermi approach [14,15] and its refinements (see, e.g., [16]). Its success in describing static properties of atoms is well known, and we note that, in recent years, these methods were also used successfully for study-

ing dynamical processes of atoms and molecules in super-strong light pulses [17].

Our paper is organized as follows: in Sec. II, the Thomas-Fermi model is revisited with an eye on the dipole-dipole forces. The Dirac correction to the interaction energy term is discussed and scaling properties derived with the help of the virial theorem. In Sec. III the results of approximate (nearly analytic) and numerical minimizations of the Thomas-Fermi-Dirac energy functional are presented. Unexpectedly, we find that the system may withstand larger dipolar forces both for very flat and for highly elongated traps.

II. THOMAS-FERMI MODEL

A. General considerations

Let us begin by recalling some basic things, mainly collected from Refs. [16,18,19], with due attention to the changed situation: here fully spin-polarized fermions—there fermions with no net spin. The spatial one-particle density is denoted by $n(\vec{r})$ and is normalized to the total particle number N ,

$$N = \int (d\vec{r})n(\vec{r}), \quad (1)$$

where $(d\vec{r}) \equiv dx dy dz$ denotes the volume element. The spatial one-particle density matrix $n^{(1)}(\vec{r}'; \vec{r}'')$ and the one-particle Wigner function $\nu(\vec{r}, \vec{p})$ are related by

$$n^{(1)}(\vec{r}'; \vec{r}'') = \int \frac{(d\vec{p})}{(2\pi\hbar)^3} \nu\left(\frac{1}{2}(\vec{r}' + \vec{r}''), \vec{p}\right) e^{i\vec{p} \cdot (\vec{r}' - \vec{r}'')/\hbar}. \quad (2)$$

The spatial and momental one-particle densities are obtained by integrating $\nu(\vec{r}, \vec{p})$ over the other variable,

$$n(\vec{r}) = n^{(1)}(\vec{r}; \vec{r}) = \int \frac{(d\vec{p})}{(2\pi\hbar)^3} \nu(\vec{r}, \vec{p}),$$

$$\rho(\vec{p}) = \int \frac{(d\vec{r})}{(2\pi\hbar)^3} \nu(\vec{r}, \vec{p}). \quad (3)$$

They are needed for the calculation of the kinetic energy,

$$E_{\text{kin}} = \int (d\vec{p}) \frac{\vec{p}^2}{2M} \rho(\vec{p}), \quad (4)$$

and the external potential energy (of the harmonic trap),

$$E_{\text{trap}} = \int (d\vec{r}) \frac{1}{2} M \omega^2 [x^2 + y^2 + (\beta z)^2] n(\vec{r}), \quad (5)$$

where M is the mass of the atom species considered, ω is the (transverse) trap frequency, and β is the aspect ratio of the cylindrical trap. The trap is spherically symmetric for $\beta = 1$; for $\beta < 1$, the equipotential surfaces are prolate (“cigar shaped”) ellipsoids; for $\beta > 1$, they are oblate (“lenticular shaped”) ellipsoids.

For the dipole-dipole interaction energy, E_{dd} , we need (the diagonal part of) the two-particle density matrix $n^{(2)}(\vec{r}'_1, \vec{r}'_2; \vec{r}''_1, \vec{r}''_2)$,

$$E_{\text{dd}} = \frac{1}{2} \int (d\vec{r}') (d\vec{r}'') V_{\text{dd}}(\vec{r}' - \vec{r}'') n^{(2)}(\vec{r}', \vec{r}''; \vec{r}', \vec{r}''), \quad (6)$$

where

$$V_{\text{dd}}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{\vec{\mu}^2}{r^3} - 3 \frac{(\vec{\mu} \cdot \vec{r})^2}{r^5} - \frac{8\pi}{3} \vec{\mu}^2 \delta(\vec{r}) \right]. \quad (7)$$

We note that the contact term, proportional to $\delta(\vec{r})$, is required by the condition that the magnetic field made by the point dipole be divergence-free [20]. An alternative way of presenting V_{dd} is

$$V_{\text{dd}}(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\mu} \cdot \left[-\vec{\nabla} \frac{1}{r} - 4\pi \vec{1} \delta(\vec{r}) \right] \cdot \vec{\mu}, \quad (8)$$

which is a particularly convenient starting point for evaluating the Fourier transform

$$\int (d\vec{r}) e^{i\vec{k} \cdot \vec{r}} V_{\text{dd}}(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\mu} \cdot \left[4\pi \frac{\vec{k}\vec{k}}{k^2} - 4\pi \vec{1} \right] \cdot \vec{\mu}. \quad (9)$$

The vanishing divergence just mentioned is here immediately recognized, inasmuch as $\vec{k} \cdot [\dots] = 0$.

B. Thomas-Fermi-Dirac functionals

The semiclassical approximation now employed—in the spirit of what the TFD trio (Thomas [14], Fermi [15], and Dirac [21]) did, although in a technically different manner—is twofold: We approximate $n^{(2)}$ by products of $n^{(1)}$ factors (Dirac),

$$n^{(2)}(\vec{r}'_1, \vec{r}'_2; \vec{r}''_1, \vec{r}''_2) = n^{(1)}(\vec{r}'_1; \vec{r}'_1) n^{(1)}(\vec{r}'_2; \vec{r}'_2) - n^{(1)}(\vec{r}'_1; \vec{r}''_2) n^{(1)}(\vec{r}'_2; \vec{r}'_1), \quad (10)$$

and $n^{(1)}$ by a brutally simple Wigner function (Thomas and Fermi),

$$\nu(\vec{r}, \vec{p}) = \eta(\{\hbar^2 [6\pi^2 n(\vec{r})]^{2/3} - \vec{p}^2\}), \quad (11)$$

where $\eta()$ is Heaviside’s unit step function. This gives the density functional of the kinetic energy as

$$E_{\text{kin}}[n] = \int (d\vec{r}) \frac{\hbar^2}{M} \frac{1}{20\pi^2} [6\pi^2 n(\vec{r})]^{5/3}, \quad (12)$$

and the density functional for the potential energy in the trap is E_{trap} of Eq. (5).

The dipole-dipole interaction energy consists of two parts, corresponding to the two summands in Eq. (10),

$$E_{\text{dd}}[n] = E_{\text{dd}}^{(\text{dir})}[n] + E_{\text{dd}}^{(\text{ex})}[n] \quad (13)$$

with the direct term

$$E_{\text{dd}}^{(\text{dir})}[n] = \frac{1}{2} \int (d\vec{r}') (d\vec{r}'') n(\vec{r}') V_{\text{dd}}(\vec{r}' - \vec{r}'') n(\vec{r}'') \quad (14)$$

and the exchange term

$$E_{\text{dd}}^{(\text{ex})}[n] = -\frac{1}{2} \int (d\vec{r}') (d\vec{r}'') n^{(1)}(\vec{r}'; \vec{r}'') n^{(1)}(\vec{r}''; \vec{r}') \times V_{\text{dd}}(\vec{r}' - \vec{r}'') = -\frac{1}{2} \int (d\vec{r}) \int (d\vec{s}) V_{\text{dd}}(\vec{s}) \times n^{(1)}(\vec{r} + \frac{1}{2}\vec{s}; \vec{r} - \frac{1}{2}\vec{s}) n^{(1)}(\vec{r} - \frac{1}{2}\vec{s}; \vec{r} + \frac{1}{2}\vec{s}). \quad (15)$$

Now we note that

$$n^{(1)}(\vec{r} + \frac{1}{2}\vec{s}; \vec{r} - \frac{1}{2}\vec{s}) n^{(1)}(\vec{r} - \frac{1}{2}\vec{s}; \vec{r} + \frac{1}{2}\vec{s}) = \int \frac{(d\vec{p}') (d\vec{p}'')}{(2\pi\hbar)^6} \nu(\vec{r}, \vec{p}') \nu(\vec{r}, \vec{p}'') e^{i(\vec{p}' - \vec{p}'') \cdot \vec{s}/\hbar} \quad (16)$$

depends only on the length $s = |\vec{s}|$ of vector \vec{s} , not on its direction \vec{s}/s , because—in the TF approximation (11)—the product $\nu(\vec{r}, \vec{p}') \nu(\vec{r}, \vec{p}'')$ involves only $p' = |\vec{p}'|$ and $p'' = |\vec{p}''|$. As a consequence, it is permissible to replace, in Eq. (15), $V_{\text{dd}}(\vec{s})$ by its average over the solid angle associated with \vec{s} ,

$$V_{\text{dd}}(\vec{s}) \rightarrow \frac{\mu_0}{4\pi} \left[-\frac{8\pi}{3} \vec{\mu}^2 \delta(\vec{s}) \right]. \quad (17)$$

We thus arrive at

$$\begin{aligned}
E_{\text{dd}}^{(\text{ex})}[n] &= \frac{1}{2} \int (d\vec{r}) [n(\vec{r})]^2 \frac{\mu_0}{4\pi} \frac{8\pi}{3} \vec{\mu}^2 \\
&= \frac{1}{2} \int (d\vec{r})(d\vec{r}') n(\vec{r}) \frac{\mu_0}{4\pi} \left[\frac{8\pi}{3} \vec{\mu}^2 \delta(\vec{r}-\vec{r}') \right] n(\vec{r}'), \quad (18)
\end{aligned}$$

and accordingly

$$E_{\text{dd}}[n] = \frac{1}{2} \int (d\vec{r})(d\vec{r}') n(\vec{r}) \bar{V}_{\text{dd}}(\vec{r}-\vec{r}') n(\vec{r}') \quad (19)$$

with

$$\bar{V}_{\text{dd}}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{\vec{\mu}^2}{r^3} - 3 \frac{(\vec{\mu} \cdot \vec{r})^2}{r^5} \right], \quad (20)$$

which is $V_{\text{dd}}(\vec{r})$ of Eq. (7) with the contact term removed. In Eqs. (8) and (9) this corresponds to multiplying the unit dyadic $\vec{\Gamma}$ by $\frac{1}{3}$ [13].

The rotational symmetry of the trap potential in Eq. (5) distinguishes the z axis, and we take for granted that this is also the direction of spin polarization,

$$\vec{\mu} = \mu \vec{e}_z. \quad (21)$$

Then

$$\bar{V}_{\text{dd}}(\vec{r}) = \frac{\mu_0}{4\pi} \mu^2 \frac{1-3(z/r)^2}{r^3}, \quad (22)$$

and the whole system is invariant under rotations around the z axis.

The total TFD energy functional is then given by the sum of the kinetic energy (12), the potential energy in the trap (5), and the dipole-dipole interaction energy (19),

$$\begin{aligned}
E^{(\text{TFD})}[n] &= \int (d\vec{r}) \left[\frac{\hbar^2}{M} \frac{1}{20\pi^2} [6\pi^2 n(\vec{r})]^{5/3} + \frac{1}{2} M \omega^2 r^2 n(\vec{r}) \right] \\
&+ \frac{1}{2} \int (d\vec{r})(d\vec{r}') n(\vec{r}) \bar{V}_{\text{dd}}(\vec{r}-\vec{r}') n(\vec{r}'). \quad (23)
\end{aligned}$$

The density that minimizes $E^{(\text{TFD})}$ under the constraint (1) obeys the nonlinear integral equation

$$\begin{aligned}
\frac{\hbar^2}{2M} [6\pi^2 n(\vec{r})]^{2/3} + \frac{1}{2} M \omega^2 [x^2 + y^2 + (\beta z)^2] \\
+ \int (d\vec{r}') \bar{V}_{\text{dd}}(\vec{r}-\vec{r}') n(\vec{r}') = \frac{1}{2} M \omega^2 R^2, \quad (24)
\end{aligned}$$

where $\frac{1}{2} M \omega^2 R^2$ is a convenient way of writing the Lagrange multiplier for the constraint.

C. Virial theorems

Scaling transformations of the form

$$n(\vec{r}) \rightarrow \lambda^{3+\alpha} n(\lambda \vec{r}), \quad N \rightarrow \lambda^\alpha N \quad (25)$$

are consistent with the constraint (1). They affect the various pieces of $E^{(\text{TFD})}$ in accordance with

$$\begin{aligned}
E_{\text{kin}} &\rightarrow \lambda^{2+(5/3)\alpha} E_{\text{kin}}, \\
E_{\text{trap}} &\rightarrow \lambda^{-2+\alpha} E_{\text{trap}}, \\
E_{\text{dd}} &\rightarrow \lambda^{3+2\alpha} E_{\text{dd}}, \quad (26)
\end{aligned}$$

so that ($E \equiv E^{(\text{TFD})}$)

$$\begin{aligned}
E = E_{\text{kin}} + E_{\text{trap}} + E_{\text{dd}} &\rightarrow \lambda^{2+(5/3)\alpha} E_{\text{kin}} + \lambda^{-2+\alpha} E_{\text{trap}} \\
&+ \lambda^{3+2\alpha} E_{\text{dd}}. \quad (27)
\end{aligned}$$

In the infinitesimal vicinity of $\lambda = 1$, all first-order changes of E originate in the explicit change of N , $\delta N = \delta \lambda \alpha N$, and therefore

$$\alpha N \frac{\partial E}{\partial N} = (2 + \frac{5}{3}\alpha) E_{\text{kin}} + (-2 + \alpha) E_{\text{trap}} + (3 + 2\alpha) E_{\text{dd}} \quad (28)$$

must hold irrespective of the value of parameter α . In view of the linear α dependence, we get two independent statements,

$$\begin{aligned}
\alpha = 0: \quad 2E_{\text{kin}} - 2E_{\text{trap}} + 3E_{\text{dd}} &= 0, \\
\alpha = -\frac{3}{2}: \quad E_{\text{kin}} + 7E_{\text{trap}} &= 3N \frac{\partial E}{\partial N}. \quad (29)
\end{aligned}$$

They enable us to express E_{kin} , E_{trap} , and E_{dd} in terms of E and $N \partial E / \partial N$,

$$\begin{aligned}
E_{\text{kin}} &= \frac{21}{2} E - \frac{15}{2} N \frac{\partial E}{\partial N}, \\
E_{\text{trap}} &= -\frac{3}{2} E + \frac{3}{2} N \frac{\partial E}{\partial N}, \\
E_{\text{dd}} &= -8E + 6N \frac{\partial E}{\partial N}. \quad (30)
\end{aligned}$$

In conjunction with ($\mu = |\vec{\mu}|$)

$$\mu \frac{\partial}{\partial \mu} E = 2E_{\text{dd}},$$

$$\omega \frac{\partial}{\partial \omega} E = 2E_{\text{trap}},$$

$$M \frac{\partial}{\partial M} E = E_{\text{trap}} - E_{\text{kin}}, \quad (31)$$

they imply that the ground-state energy $E(M, \omega, \mu, N)$ is of the form (we leave the β dependence implicit)

$$E(M, \omega, \mu, N) = \hbar \omega N^{4/3} e(N^{1/6} \varepsilon) \quad (32)$$

with

$$\varepsilon = (\omega M^3 / \hbar^5)^{1/2} \frac{\mu_0}{4\pi} \mu^2. \quad (33)$$

Clearly, the dimensionless number ε measures the relative strength of the dipole-dipole interaction. The universal function $e(\cdot)$ is to be found numerically (for each β value of interest). For $\varepsilon = 0$, we can solve Eq. (24) immediately,

$$n(\vec{r}) = \frac{1}{6\pi^2} (M\omega/\hbar)^3 [R^2 - x^2 - y^2 - (\beta z)^2]^{3/2}, \quad (34)$$

where, by convention, $[\dots]^{3/2} = 0$ for negative arguments, and

$$R = (48\beta N)^{1/6} (M\omega/\hbar)^{-1/2} \quad (35)$$

as a consequence of Eq. (1), and so we find

$$e_0 \equiv e(0) = \frac{3}{4} (6\beta)^{1/3} = 1.363\beta^{1/3}. \quad (36)$$

For the spherically symmetric case $\beta = 1$, E_{dd} vanishes in first-order perturbation theory, so that

$$\beta = 1: \quad e(N^{1/6} \varepsilon) = 2^{-5/3} 3^{4/3} + e_2 N^{1/3} \varepsilon^2 + O(N^{1/2} \varepsilon^3) \quad (37)$$

with $e_2 < 0$.

D. Dimensionless variables

These scaling laws invite the use of correspondingly chosen dimensionless variables, such as

$$\vec{x} = \vec{r}/a \quad \text{with} \quad a = N^{1/6} \sqrt{\frac{\hbar}{M\omega}} \quad (38)$$

for the position and

$$g(\vec{x}) = \frac{a^3}{N} n(a\vec{x}) \quad \text{or} \quad n(\vec{r}) = \frac{N}{a^3} g(\vec{r}/a) \quad (39)$$

for the density. The constraint (1) then appears as

$$\int (d\vec{x}) g(\vec{x}) = 1, \quad (40)$$

and the TFD energy acquires the form

$$\begin{aligned} \frac{E^{(\text{TFD})}}{\hbar \omega N^{4/3}} [g] &= \frac{3}{10} (6\pi^2)^{2/3} \int (d\vec{x}) [g(\vec{x})]^{5/3} \\ &+ \frac{1}{2} \int (d\vec{x}) (x_1^2 + x_2^2 + \beta^2 x_3^2) g(\vec{x}) \\ &+ \frac{1}{2} N^{1/6} \varepsilon \int (d\vec{x}) (d\vec{x}') g(\vec{x}) \frac{1 - 3 \cos^2 \theta}{|\vec{x} - \vec{x}'|^3} g(\vec{x}'), \end{aligned} \quad (41)$$

where θ is the angle between the polarization direction (the x_3 direction) and the relative position vector $\vec{x} - \vec{x}'$,

$$(\vec{x} - \vec{x}') \cdot \vec{\mu} = |\vec{x} - \vec{x}'| \mu \cos \theta = (x_3 - x'_3) \mu. \quad (42)$$

The minimum of the scaled TFD energy is $e(N^{1/6} \varepsilon)$ of Eq. (32) (with its implicit β dependence); it is obtained for the $g(\vec{x})$ that obeys the dimensionless analog of Eq. (24),

$$\begin{aligned} \frac{1}{2} [6\pi^2 g(\vec{x})]^{2/3} + \frac{1}{2} (x_1^2 + x_2^2 + \beta^2 x_3^2) \\ + N^{1/6} \varepsilon \int (d\vec{x}') \frac{1 - 3 \cos^2 \theta}{|\vec{x} - \vec{x}'|^3} g(\vec{x}') = \frac{1}{2} X^2, \end{aligned} \quad (43)$$

where the value of $X = R/a$ is determined by the constraint (40).

III. RESULTS

Owing to its intrinsic semiclassical approximations, one expects a few-percent deviation of the TFD energy from the true ground-state energy. It is, therefore, not really necessary to solve the nonlinear integral equation (43). A reasonable variational estimate, in conjunction with a few full-blown numerical solutions for comparison, will do.

A. Gaussian variational ansatz

The Gaussian ansatz

$$g(\vec{x}) = (2\pi)^{-3/2} \kappa^3 \gamma \exp[-\frac{1}{2} \kappa^2 (x_1^2 + x_2^2 + \gamma^2 x_3^2)] \quad (44)$$

is convenient. In addition to its aspect ratio γ , the shape parameter, it contains the scale parameter κ , so that the virial theorems of Sec. II C will be obeyed for the optimal choice of κ . For this scaled density, the scaled kinetic energy is given by

$$\frac{E_{\text{kin}}}{\hbar \omega N^{4/3}} = 2^{-4/3} 3^{19/6} 5^{-5/2} \pi^{1/3} \kappa^2 \gamma^{2/3}, \quad (45)$$

and the scaled value of E_{trap} is

$$\frac{E_{\text{trap}}}{\hbar \omega N^{4/3}} = \frac{1}{\kappa^2} \left(1 + \frac{\beta^2}{2\gamma^2} \right). \quad (46)$$

They exhibit the anticipated dependence on the scale parameter κ , and so does the dipole-dipole interaction energy,

$$\frac{E_{\text{dd}}}{\hbar \omega N^{4/3}} = \frac{\kappa^3}{4\sqrt{\pi}} N^{1/6} \varepsilon \gamma (\gamma^2 - 1) \int_0^1 d\zeta \frac{\zeta^2 - \zeta^4}{1 - \zeta^2 + \gamma^2 \zeta^2}. \quad (47)$$

This integral can, of course, be evaluated in terms of elementary functions, but as it stands we see immediately that the integrand, and thus the integral, is positive for $0 < \gamma < \infty$, so that

$E_{\text{dd}} > 0$ for $\gamma > 1$ (oblate Gaussian),

$E_{\text{dd}} = 0$ for $\gamma = 1$ (spherical Gaussian),

$E_{\text{dd}} < 0$ for $\gamma < 1$ (prolate Gaussian), (48)

consistent with the expectation that an oblate density has a larger magnetic interaction energy than a prolate density. More explicitly, then, we have

$$\frac{E_{\text{dd}}}{\hbar \omega N^{4/3}} = \frac{\kappa^3}{4\sqrt{\pi}} N^{1/6} \varepsilon \gamma \begin{cases} \frac{1 - \vartheta \cot \vartheta}{\sin^2 \vartheta} - \frac{1}{3} & \text{for } \gamma = \frac{1}{\cos \vartheta} > 1 \\ \frac{\vartheta \coth \vartheta - 1}{\sinh^2 \vartheta} - \frac{1}{3} & \text{for } \gamma = \frac{1}{\cosh \vartheta} < 1. \end{cases} \quad (49)$$

The Gaussian density (44) cannot mimic the $\varepsilon = 0$ solution (34) very well. But nevertheless, the resulting estimate of the $\varepsilon = 0$ energy,

$$\text{Gaussian: } e_0 = 2^{-1/6} 3^{25/12} 5^{-5/4} \pi^{1/6} \beta^{1/3} = 1.42 \beta^{1/3} \quad (50)$$

is only 4.4% in excess of the correct value (36). This accuracy is sufficient for our purposes; and, in any case, an error of a few percent is a small price for the enormous simplification that the Gaussian ansatz brings about.

Dipolar interactions are partially attractive and partially repulsive, depending on the configuration of the dipoles. One should keep in mind the simple situation of two dipoles in the plane perpendicular to their polarization, which repel each other, as opposed to the situation of two attracting dipolar particles placed along the direction of their polariza-

tion. Extending this picture to a cloud of trapped dipoles one would expect that attraction dominates in prolate traps, and repulsion in oblate traps (provided the dipoles are polarized along the trap axis as is the case here). In the case of predominant attraction one may surmise that instabilities occur. By varying the two system parameters ($N^{1/6} \varepsilon$ and β) we have investigated the issue of stability, see Fig. 1.

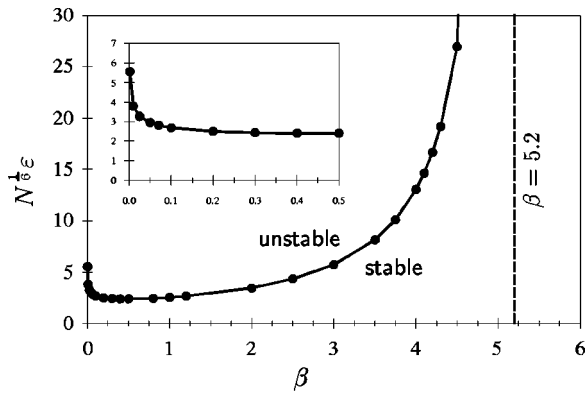


FIG. 1. Stability diagram. The dots, connected by a solid line to guide the eye, define the border between system parameters (β , aspect ratio of the trap; $N^{1/6} \varepsilon$, effective interaction strength) for which $E^{(\text{TFD})}$ is bounded from below (stable) or not bounded (unstable). The inset shows an enlargement of the region of pronouncedly prolate traps ($\beta < 0.5$). For oblate traps with $\beta = 5.2$ or larger, the system is always stable, irrespective of the value of $N^{1/6} \varepsilon$. This figure presents results obtained in the Gaussian approximation, and so do all the other figures.

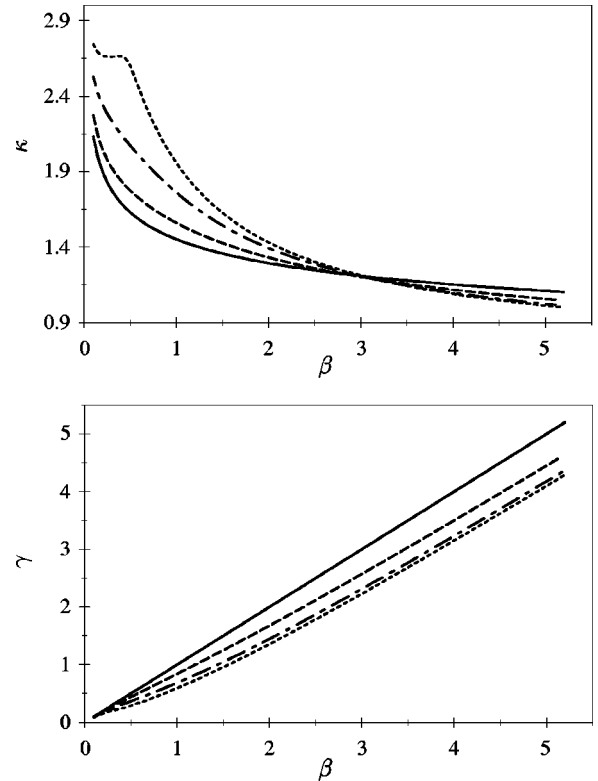


FIG. 2. Dependence of cloud parameters κ (cloud size; top) and γ (cloud shape; bottom) on β , the aspect ratio of the trap. The lines refer to different values of the interaction strength: $N^{1/6} \varepsilon = 0$ (solid lines), $N^{1/6} \varepsilon = 1$ (dashed lines), $N^{1/6} \varepsilon = 2$ (dash-dotted lines), and $N^{1/6} \varepsilon = 2.4$ (dotted lines).

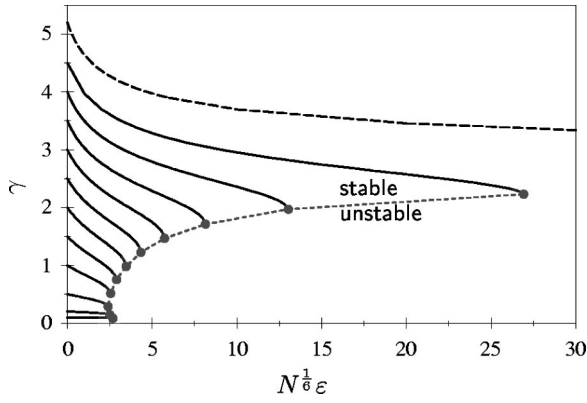


FIG. 3. Aspect ratio γ of the cloud as a function of the interaction strength $N^{1/6}\epsilon$, for various values of the aspect ratio β of the trap. Note that $\gamma = \beta$ for $N^{1/6}\epsilon = 0$. The dashed line is for $\beta = 5.2$, for which the cloud is stable for all values of $N^{1/6}\epsilon$.

From this stability diagram one concludes that, for oblate traps ($\beta > 1$), the bigger the trap aspect ratio, the bigger values of the dipole parameter $N^{1/6}\epsilon$ can be stabilized. In fact, we found numerically that fermions form stable configurations for $\beta > 5.2$ irrespective of the strength of their dipole interaction (an analogous effect was observed for dipolar bosons by Santos *et al.* [10]). On the other hand, one might naively expect an exactly opposite effect in prolate traps ($\beta < 1$) where attractive interactions should dominate. This is not quite true, indeed for moderate trap aspect ratios ($1 > \beta > 0.5$) the critical value of the dipole parameter is smaller, but in traps that are very soft ($\beta < 0.5$) in the z direction of rotational symmetry, we observe an increase of the critical value of $N^{1/6}\epsilon$ (see the inset in Fig. 1). This can be understood with the help of the following argument. We note that the dipole-dipole energy term vanishes for a uniform density distribution. As the trap is made softer in the polarization direction, the shape of the cloud along the soft axis becomes more and more uniform, contributing to the interaction energy to a lesser extent (this argument was also used to interpret our earlier results for bosons interacting via contact and dipole-dipole forces, see [13]).

The dependence of γ and κ on β , shown in Fig. 2, is consistent with this argument. We see that γ decreases with decreasing β whereas κ increases. Accordingly, the cloud gets stretched along the z axis of symmetry, and the diameter of the circular cross section in the x, y plane ($\propto \kappa^{-1}$) is reduced. At the center of the cloud, we thus have a relatively large volume of (almost) constant density, and the inhomogeneous parts of the cloud are relatively far apart. Taken together, these geometric features lead to a rather small dipole-dipole interaction energy.

Whereas the stability of the system considered can be well understood, the spatial behavior of the fermionic cloud, especially near the collapse, seems to be much less intuitive. As we approach the critical parameter values, the aspect ratio γ of the cloud decreases and the cloud becomes elongated in the attractive z direction. This type of behavior is general in the sense that it does not depend on the trap aspect ratio, see Fig. 3. It is only for traps with $\beta > 5.2$ that γ reaches an

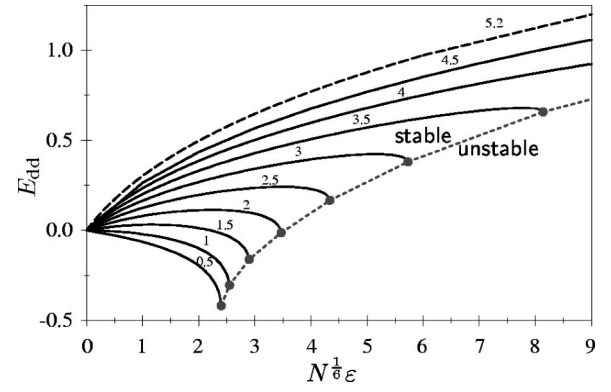


FIG. 4. Dipole-dipole interaction energy E_{dd} as a function of the interaction strength $N^{1/6}\epsilon$. The solid lines are for trap aspect ratios $\beta = 0.5, 1, 1.5, \dots, 4.5$; the dashed line is for $\beta = 5.2$.

asymptotic value (dependent on β) for extremely large values of $N^{1/6}\epsilon$.

The dependence of the dipolar energy E_{dd} on the dipole parameter $N^{1/6}\epsilon$ and the trap geometry is also of interest as it is the quantity responsible for the (in)stability of the system, see Fig. 4. For all prolate traps, E_{dd} remains negative approaching some critical value at the collapse point. For $\beta < 5.2$, the dipolar energy can be positive for moderate dipole parameters, but if their values are large enough, E_{dd} turns negative indicating the collapse. For $\beta > 5.2$, E_{dd} is always positive and increases as a function of $N^{1/6}\epsilon$.

Finally, we take a look at the total TFD energy, see Fig. 5. For $\beta < 5.2$, the system becomes unstable at the critical value of $N^{1/6}\epsilon$. Consistent with Fig. 1, we observe that larger values of $N^{1/6}\epsilon$ are supported for $\beta = 0.1$ and $\beta = 1$ than for $\beta = 0.5$.

Let us now discuss the dipole parameter $N^{1/6}\epsilon$ and give some typical values for it. Owing to the N dependence, one can locate the experimental system in various regions of the stability diagram of Fig. 1 not only by choosing (or inducing, as proposed for bosons [9,10]) a specific value of μ , but

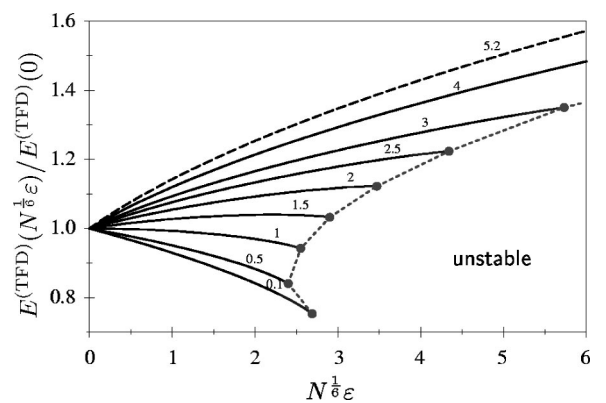


FIG. 5. Normalized TFD energy as a function of the interaction strength for various trap shapes. The universal function $e(N^{1/6}\epsilon)$ of Eq. (32), normalized to its initial value $e_0 = e(0)$, is shown for trap aspect ratios $\beta = 0.1, 0.5, 1, 1.5, \dots, 4$ (solid lines) and $\beta = 5.2$ (dashed line). Note the horizontal slope of the $\beta = 1$ line (spherical trap), as required by Eq. (37).

also, to some extent, by varying N . One could also exploit the ω dependence of $\varepsilon \propto \sqrt{\omega}$, which is particularly relevant for optical traps with tight confinement [22].

For the parameters of the fermionic chromium isotope, $\omega = 300$ Hz and $N = 10^6$, one obtains $N^{1/6}\varepsilon = 0.012$, which is a very small value. Therefore, the conclusion for atoms possessing even relatively large magnetic dipole moments is that irrespective of their number and the trap frequency they will always remain stable against collapse. The following amusing analogy offers a good reason for this observation [6]. Let us compare the characteristic sizes of the noninteracting Fermi gas and the Bose-condensed atomic gas interacting via a repulsive contact potential. In the Thomas-Fermi approximation, the appropriate quantities, in units of $\sqrt{\hbar/(M\omega)}$, read $R_{\text{Fermi}} = (48N)^{1/6}$ for fermions and $R_{\text{Bose}} = (15Na_0)^{1/5}$ for bosons, where a_0 is the scattering length. By equating the two sizes one can calculate the effective, N dependent, scattering length due to the exclusion-principle-induced repulsion: $a_0 \approx 1.68N^{-1/6}$. For typical numbers of atoms, $N = 10^3 - 10^6$, this a_0 is huge on the scale set by typical scattering lengths for bosonic atoms ($a_0 \approx 10^{-3}$). Once we realize how strong is the repulsion that originates in the Fermi statistics, we understand that small atomic magnetic dipoles can hardly have a noticeable effect on the behavior of dipolar fermionic gases. However, for polar molecules the situation is different. For a trap frequency of $\omega = 300$ Hz and $N = 10^6$ molecules of mass $m \approx 100$ a.m.u. (the typical mass of an alkaline dimer) and a typical electric dipole moment of 1 Debye unit, one reaches $N^{1/6}\varepsilon \approx 11.5$, which may very well put the system into the unstable regime, see Fig. 1. In this situation, a sufficiently large β value will stabilize the system.

In order to assess the quality of our variational results we have computed exact numerical solutions of Eq. (43). This equation was solved for the density distribution $g(\vec{x})$ self-consistently starting from the known analytical result (34) for a noninteracting ($\varepsilon = 0$) Fermi gas in a trap [6] and slowly increasing the dipole parameter $N^{1/6}\varepsilon$. For each value of $N^{1/6}\varepsilon$ the solution was iterated until convergence was reached. Then, the value of the dipole parameter was slightly increased. In order to compute the dipole (integral) term we note that it has the form of a convolution. Thus, it can be conveniently evaluated in the Fourier space where it is a simple product of the Fourier transforms of the density (computed numerically with the aid of an FFT) and the interaction potential, the latter being known analytically [13]:

$$\int (d\vec{x}) e^{i\vec{q}\cdot\vec{x}} \frac{1 - 3 \cos^2 \theta}{|\vec{x}|^3} = -\frac{4\pi}{3} (1 - 3 \cos^2 \alpha), \quad (51)$$

where α is the angle between the Fourier variable \vec{q} and the z direction. In order to assure that the integral term is evaluated accurately we used a Gaussian distribution for comparison and chose the grid parameters accordingly.

Our numerical calculations, performed in three dimensions, were quite demanding so we limited their use to a check of the main features of the stability diagram in Fig. 1. The solutions obtained satisfy the virial relations (29) very well, and total energies obtained numerically are always below the corresponding values from the variational analysis. The critical value of the dipole parameter for a spherical trap is $N^{1/6}\varepsilon = 2.29$ as compared to 2.55 obtained by the variational calculation. We also confirmed the effect of increase of the critical interaction strength that we found, in the Gaussian approximation, for prolate traps: for $\beta = 0.07$ the critical value of $N^{1/6}\varepsilon$ is 2.75 (> 1.96) as compared to 2.81 (> 2.55) obtained analytically in the Gaussian approximation.

Now two remarks about possible extensions of our work are in order. First, the results presented in this paper describe the situation of a dipolar fermionic gas at the temperature $T = 0$. An interesting subject of study would be extension of our theory to finite temperatures. Second, there exists a parallel approach in the Thomas-Fermi model, namely the one in the momentum space [18,19]. As many experiments with cold gases yield their momental characteristics, investigation of this alternative approach also presents an attractive theoretical task.

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