Universal manipulation of a single qubit

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We find the optimal universal way of manipulating a single qubit, $|\psi(\vartheta,\varphi)\rangle$, such that $(\vartheta,\varphi)\rightarrow(\vartheta-\alpha,\varphi-\beta)$. Such optimal transformations fall into two classes. For $0\leqslant\alpha\leqslant\pi/2$, the optimal map is the identity and the fidelity varies monotonically from 1 (for $\alpha=0$) to $\frac{1}{2}$ (for $\alpha=\pi/2$). For $\pi/2\leqslant\alpha\leqslant\pi$, the optimal map is the universal-NOT gate and the fidelity varies monotonically from $\frac{1}{2}$ (for $\alpha=\pi/2$) to $\frac{2}{3}$ (for $\alpha=\pi$). The fidelity $\frac{2}{3}$ is equal to the fidelity of measurement. It is therefore rather surprising that for some values of α the fidelity is lower than $\frac{2}{3}$. For instance, a universal square root of NOT operation is more difficult to approximate than the universal NOT gate itself.

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A unit of classical information is a bit, i.e., 0 or 1. Quantum information consists of qubits that are a superposition of the states $|0\rangle$ and $|1\rangle$. Classical and quantum information differs in many ways. While classical information can be copied perfectly, the same is not true with qubits [1–3]. Another feature that distinguishes classical and quantum information is a measurement. Unlike classical information, an unknown single qubit cannot be measured to give complete information about the qubit. In order to get the maximum information about an unknown qubit, we measure the qubit along any chosen basis $\{|\phi\rangle,|\phi^{\perp}\rangle\}$. If the result is $|\phi\rangle$, then we guess the unknown qubit to be $|\phi\rangle$, and if the result is $|\phi^{\perp}\rangle$, then we guess $|\phi^{\perp}\rangle$. Averaging over all possible $|\phi\rangle$'s (assuming a uniform distribution over the Bloch sphere), the fidelity is equal to $\frac{2}{3}$. We cannot achieve a higher fidelity by using generalized measurements and hence $\frac{2}{3}$ is the optimal measurement fidelity of an unknown qubit.

A qubit in Bloch vector notation is

$$|\psi(\vartheta,\varphi)\rangle\equiv\begin{pmatrix}\cos(\vartheta/2)\\ e^{-i\varphi}\sin(\vartheta/2)\end{pmatrix}. \tag{1}$$

The most general linear transformation on (ϑ,φ) is

$$(\vartheta,\varphi)\rightarrow(\vartheta-\alpha,\varphi-\beta) \tag{2}$$

with $0\leqslant\alpha\leqslant\pi$ and $0\leqslant\beta\leqslant 2\pi$. If $\vartheta=0,\pi$, then φ is undefined. For definiteness, we will take $\varphi=0$ in such cases. Since, when taking averages over the Bloch sphere, these anomalous cases are of measure zero, we need not pay any special attention to them. This general transformation can be composed from two transformations. First

$$(\vartheta,\varphi)\rightarrow(\vartheta-\alpha,\varphi) \tag{3}$$

and then $\varphi\rightarrow\varphi-\beta$. The transformation on φ can be achieved perfectly by a unitary operation and so is of less interest to us. However, the transformation on ϑ cannot be achieved unitarily. To find the fidelity of general linear transformations of the form (2), it suffices to consider only the nonunitary part (3). We are interested only in universal transformations. These are those transformations for which the fidelity is independent of ϑ and φ of the input state. Furthermore, we will assume that the input distribution is uniform over the Bloch sphere. Since the area element $\sin\vartheta d\vartheta d\varphi$ is not preserved in form (except when $\vartheta=0,\pi$), the output will not be uniform. In taking averages, we intergrate over a uniform distribution of the input variables that corresponds to integrating over a nonuniform distribution of the output variables.

Changing bits 0 to 1 and 1 to 0 is a NOT gate in the classical information case. In the quantum case, changing $|\psi\rangle=a|0\rangle+b|1\rangle$ to $|\psi^{\perp}\rangle=b^{*}|0\rangle-a^{*}|1\rangle$ requires antiunitary transformation, which is not allowed in quantum mechanics. In [4–6], it was shown that universal-NOT (U-NOT) operation can be achieved with $\frac{2}{3}$ fidelity for a single input. This fidelity is the same as the measurement fidelity. They showed that the U-NOT operation is no better than measuring a qubit first and then preparing an orthogonal state. In Bloch vector notation, the U-NOT gate corresponds to transforming $|\psi(\vartheta,\varphi)\rangle$ to $|\psi(\vartheta-\pi,\varphi)\rangle$. This is a special case of the transformation (3) with $\alpha=\pi$. Now consider the general case in which we transform $|\psi(\vartheta,\varphi)\rangle$ to $|\psi(\vartheta-\alpha,\varphi)\rangle$. Naively, it may seem that the fidelity should be at least $\frac{2}{3}$, since one could measure a qubit with $\frac{2}{3}$ fidelity and prepare a state in an appropriate direction. We will show in this paper that this is not so.

Let us take an example where $\alpha=3\pi/4$. Therefore, for a given unknown state $|\psi\rangle$, we want to prepare a state as close as possible to $|\psi'\rangle=|\psi(\vartheta-3\pi/4,\varphi)\rangle$. We choose a random state

$$|\phi(\mu,\nu)\rangle\equiv\begin{pmatrix}\cos(\mu/2)\\ e^{-i\nu}\sin(\mu/2)\end{pmatrix} \tag{4}$$

and measure $|\psi(\vartheta,\varphi)\rangle$ on the basis of $\{|\phi\rangle,|\phi^{\perp}\rangle\}$. If we get $|\phi\rangle$, we prepare $|\phi'\rangle\equiv|\phi(\mu-3\pi/4,\nu)$, and if we get $|\phi^{\perp}\rangle$, then $|\phi'^{\perp}\rangle$ is prepared. As a density matrix, the state we prepare by this method is

$$\rho^{(1)}=|\langle\psi|\phi\rangle|^{2}|\phi'\rangle\langle\phi'|+|\langle\psi|\phi^{\perp}\rangle|^{2}|\phi'^{\perp}\rangle\langle\phi'^{\perp}|. \tag{5}$$

We take the average of $\rho^{(1)}$ over uniform distributions of $|\phi\rangle$ on the Bloch sphere to obtain $\overline{\rho^{(1)}}$ and then the fidelity is given by

$$F=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\langle\psi'|\overline{\rho^{(1)}}|\psi'\rangle\sin\vartheta d\vartheta d\varphi=0.5833\ldots. \tag{6}$$

This value is lower than $\frac{2}{3}$. Can we do any better? If we prepare $|\phi^{\perp}\rangle$ when the result is $|\phi\rangle$ and prepare $|\phi\rangle$ for $|\phi^{\perp}\rangle$, i.e.,

$$\rho^{(2)}=|\langle\psi|\phi\rangle|^2|\phi^{\perp}\rangle\langle\phi^{\perp}|+|\langle\psi|\phi^{\perp}\rangle|^2|\phi\rangle\langle\phi|, \tag{7}$$

then

$$F=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\langle\psi'|\overline{\rho^{(2)}}|\psi'\rangle\sin\vartheta d\vartheta d\varphi=0.6178\ldots. \tag{8}$$

This fidelity is still lower than $\frac{2}{3}$ but higher than the value in Eq. (6). This rather surprising result is due to the different phase angles of $|\psi\rangle$ and $|\phi\rangle$. If $|\langle\psi|\phi\rangle|^2=\frac{2}{3}$, then the rotation $|\psi(\vartheta,\varphi)\rangle\rightarrow|\psi(\vartheta-\pi,\varphi)\rangle$ and $|\phi(\mu,\nu)\rangle\rightarrow|\phi(\mu-\pi,\nu)\rangle$ yields the same fidelity $|\langle\psi(\vartheta-\pi,\varphi)|\phi(\mu-\pi,\nu)\rangle|^2=\frac{2}{3}$. However, if the rotation is over some other angle $\alpha\neq\pi$ or 0, then $|\langle\psi(\vartheta-\alpha,\varphi)|\phi(\mu-\alpha,\nu)\rangle|^2$ may not be $\frac{2}{3}$ because φ and ν are not necessarily the same. If the phase angles φ and ν are the same, then for any α, $|\langle\psi(\vartheta-\alpha,\varphi)|\phi(\mu-\alpha,\varphi)\rangle|^2=\frac{2}{3}$. For $\pi/2\leqslant\alpha\leqslant\pi$ in $|\psi(\vartheta-\alpha,\varphi)\rangle$, $\overline{\rho^{(2)}}$ yields the fidelity

$$F^{(2)}=\tfrac{1}{12}[6+\cos(\pi-\alpha)+\cos(\pi+\alpha)]. \tag{9}$$

For $0\leqslant\alpha\leqslant\pi/2$, we consider the usual measurement density matrix,

$$\rho^{(3)}=|\langle\psi|\phi\rangle|^2|\phi\rangle\langle\phi|+|\langle\psi|\phi^{\perp}\rangle|^2|\phi^{\perp}\rangle\langle\phi^{\perp}|, \tag{10}$$

and the fidelity is given as

$$F^{(3)}=\tfrac{1}{2}+\tfrac{1}{6}\cos\alpha. \tag{11}$$

For $\pi/2\leqslant\alpha\leqslant\pi$, we will show that $F^{(2)}$ in Eq. (9) is indeed the optimal fidelity. $|\psi(\vartheta-\alpha,\varphi)\rangle$ for $0\leqslant\alpha\leqslant\pi/2$ can be obtained with better fidelity than $F^{(3)}$ in Eq. (11). We expect this since for $\alpha=0$, the identity operation gives $|\psi\rangle$ with fidelity 1.

By considering the most general type of transformation on a single qubit, we will find the one that maximizes the fidelity for the transformation (3). We will follow the method of Bužek *et al.* [5]. The most general operation available to us is to perform unitary evolution on the single qubit and some ancilla prepared in a known state $|Q\rangle$ (this is taken to be normalized). This gives

$$\begin{aligned}|0\rangle|Q\rangle&\rightarrow|1\rangle|A\rangle+|0\rangle|B\rangle,\\ |1\rangle|Q\rangle&\rightarrow|0\rangle|\tilde{A}\rangle+|1\rangle|\tilde{B}\rangle,\end{aligned} \tag{12}$$

where $|A\rangle,|\tilde{A}\rangle,|B\rangle,|\tilde{B}\rangle$ may not be normalized. From the normalization and the orthogonality of (12), $|A|^2+|B|^2=|\tilde{A}|^2+|\tilde{B}|^2=1$ and $\langle A|\tilde{B}\rangle+\langle B|\tilde{A}\rangle=0$. We let $|\psi\rangle$ transform under (12), trace over the ancilla, and obtain a density matrix $\rho^{(\mathrm{out})}$ for our qubit. The fidelity is then given by $F=\langle\psi'|\rho^{(\mathrm{out})}|\psi'\rangle$, where $|\psi'\rangle=|\psi(\vartheta-\alpha,\varphi)\rangle$. When expressed explicitly in terms of ϑ, φ, $|A\rangle$, $|B\rangle$, $|\tilde{A}\rangle$, and $|\tilde{B}\rangle$, this expression for the fidelity has 48 terms. We can compare coefficients of those terms dependent on $e^{\pm i\varphi}$ and $e^{\pm 2i\varphi}$. These coefficients must vanish in order for the transformation in (12) to be independent of φ (which we require for universality), $\langle A|\tilde{A}\rangle=\langle A|B\rangle=\langle\tilde{B}|\tilde{A}\rangle=\langle\tilde{A}|B\rangle=\langle\tilde{B}|A\rangle=0$. Of those terms remaining, some have a dependence on ϑ. These terms must also vanish (by universality). This leaves only two terms giving us an expression for the fidelity:

$$\begin{aligned}F&=\sin^2\frac{\alpha}{2}|\tilde{A}|^2+\cos^2\frac{\alpha}{2}|\tilde{B}|^2\\ &=\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)|\tilde{B}|^2+\sin^2\frac{\alpha}{2}.\end{aligned} \tag{13}$$

By comparing coefficients of functions of ϑ of those terms having a ϑ dependence and setting them to zero, we obtain the following conditions:

$$2|\tilde{A}|^2-2|\tilde{B}|^2+\langle\tilde{B}|B\rangle+\langle B|\tilde{B}\rangle=0, \tag{14}$$

$$2|B|^2-2|A|^2-\langle\tilde{B}|B\rangle-\langle B|\tilde{B}\rangle=0, \tag{15}$$

$$|A|^2+|\tilde{A}|^2-|B|^2-|\tilde{B}|^2+\langle B|\tilde{B}\rangle+\langle\tilde{B}|B\rangle=0, \tag{16}$$

$$\begin{aligned}&\left(\cos^2\frac{\alpha}{2}-2\sin^2\frac{\alpha}{2}\right)|\tilde{A}|^2+\left(\sin^2\frac{\alpha}{2}-2\cos^2\frac{\alpha}{2}\right)|\tilde{B}|^2\\ &\quad+\sin^2\frac{\alpha}{2}|B|^2+\cos^2\frac{\alpha}{2}|A|^2+\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)\langle\tilde{B}|B\rangle\\ &\quad+\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)\langle B|\tilde{B}\rangle=0.\end{aligned} \tag{17}$$

From Eqs. (14) and (15) and the normalization condition of the transformations in (12),

$$|A|^2=|\tilde{A}|^2,\quad |B|^2=|\tilde{B}|^2, \tag{18}$$

which then implies Eq. (17) is equal to Eq. (15). From Eq. (16), with $\eta=\mathrm{Re}(\langle B|\tilde{B}\rangle)/|\tilde{B}|^2$ (therefore $|\eta|\leqslant1$),

$$|\tilde{B}|^2=\frac{1}{2-\eta}. \tag{19}$$

For $\pi/2\leqslant\alpha\leqslant\pi$, $|\tilde{B}|^2$ needs to be minimum to give a maximum fidelity in Eq. (13). Therefore, with $\eta=-1$, the fidelity is

$$F=\tfrac{1}{3}\cos^2\frac{\alpha}{2}+\tfrac{2}{3}\sin^2\frac{\alpha}{2}, \tag{20}$$

which is the same as Eq. (9). Therefore, for $\pi/2\leqslant\alpha\leqslant\pi$, the

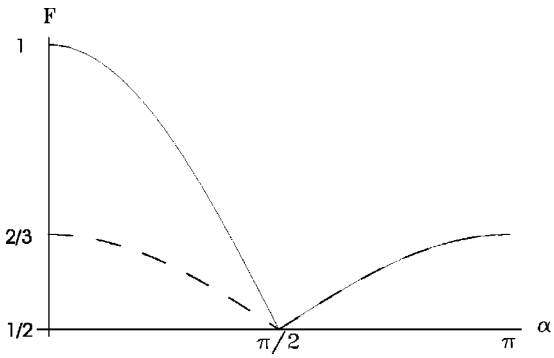

FIG. 1. A graph of F versus α is shown. For $0 \leqslant \alpha \leqslant \pi/2$, the upper curve corresponds to the optimal quantum scheme and the lower curve represents the measurement scheme. For $\pi/2 \leqslant \alpha \leqslant \pi$, both measurement and optimal quantum schemes yield the identical results.

measurement-based preparation as in $\rho^{(2)}$ is indeed optimal. The transformation satisfying Eqs. (14)–(18) and (20) is the same as the U-NOT transformation of Bužek *et al.* in [4,5]. The fidelity in Eq. (20) has the highest value of $\frac{2}{3}$ when $\alpha = \pi$ (U-NOT gate) and the lowest $\frac{1}{2}$ when $\alpha = \pi/2$ [U-SQRT(NOT) gate]. The graph for α and F is shown in Fig. 1, where the measurement [i.e., with $\rho^{(2)}$ in Eq. (7)] and the quantum schemes yield the identical result. For $0 \leqslant \alpha \leqslant \pi/2$, $|\widetilde{B}|^2$ needs to be maximum to have the maximum fidelity in Eq. (13). Therefore, with the choice of $\eta = 1$,

$$F = \cos^2 \frac{\alpha}{2}. \tag{21}$$

This transformation is simply a trivial identity map and it has a maximum fidelity of 1 when $\alpha = 0$ and a minimum of $\frac{1}{2}$ when $\alpha = \pi/2$. A graph of α and F for $0 \leqslant \alpha \leqslant \pi/2$ is shown in Fig. 1. In this case, the quantum scheme has a higher fidelity than the measurement scheme.

It follows that for a general transformation linear in the spherical coordinates, namely $(\vartheta, \varphi) \rightarrow (\vartheta - \alpha, \varphi - \beta)$, the procedures that optimize fidelity fall into two distinct classes. (i) For $0 \leqslant \alpha \leqslant \pi/2$, the optimal procedure is the identity map that performs better than a measurement-based scheme. In this range, the maximum fidelity (equal to 1) is achieved, not surprisingly, when $\alpha = 0$. (ii) For $\pi/2 \leqslant \alpha \leqslant \pi$, the U-NOT transformation is optimal. This procedure performs only as well as a measurement-based scheme. In this range, the maximum fidelity (equal to $\frac{2}{3}$) is achieved, perhaps a little surprisingly, only for the case $\alpha = \pi$, which, if $\beta = 0$, corresponds to a universal NOT operation. Since φ can be varied linearly by a unitary transformation, β can take any value in either of these two classes. Therefore, our result shows that the U-SQRT(NOT) operation is harder to approximate than the U-NOT gate. In fact, U-SQRT(NOT) is the most difficult transformation yielding a fidelity of $\frac{1}{2}$.

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