PHYSICAL REVIEW A, VOLUME 63, 032101

Electroproduction of relativistic positronium

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(Received 4 January 2000; published 5 February 2001)

Electroproduction of relativistic positronium may be of importance for obtaining beams of polarized or unpolarized positronium. We give cross sections for singlet and triplet positronium production in high-energy collisions of electrons on atoms, correct to all orders in the atomic number *Z*, including a complete description of atomic screening, using the Thomas-Fermi-Molière atomic model. The results are analogous to the photoproduction cross section results recently published [Phys. Rev. A **60**, 1883 (1999)]. However, since in electroproduction the positronum is produced as a spectrum with spectral distributions varying considerably according to the spin of the positronium, singlet or triplet, the resulting cross sections are strongly dependent on screening, more so than in photoproduction. We show that with initially longitudinaly polarized electrons, polarization may be transferred to the positronium, so that beams of sizably polarized positronium may be obtained.

DOI: 10.1103/PhysRevA.63.032101 PACS number(s): 12.20.Ds, 14.60.Cd, 13.40.−f

I. INTRODUCTION

After a number of theoretical papers in various approximations on photoproduction [1,2,4] and electroproduction [3,4] of relativistic positronium also combined with the crystal enhancement effect [5] and with only one experimental paper on positronium production in $\pi^0 \to 2\gamma$ decay [6], serious attempts at proposing experimental production of positronium beams at the REFER facility of Hiroshima University have now been reported [7,8]. Descriptions of experimental setups and energy conditions versus positronium production rates are discussed in Ref. [8]. Important figures are the electron and photon beams estimated at 10^{13} electrons/sec and 10^{10} photons/sec, respectively, which show that careful theoretical discussions on cross sections and angular distributions are needed for choosing the optimal experimental conditions for positronium beam production. In addition it is probably of importance in the future to have information on the possibility of obtaining polarized beams of triplet positronium. The present paper addresses these questions for electroproduction of positronium, following up the recent paper on photoproduction [2].

Since the discovery of positronium production in π^0 decay $\pi^0 \to \gamma + \mathrm{Ps}$ [6] in 1984, and the first theoretical calculations of the production process [1] in 1986, it has been realized that the production of beams of relativistic positronium could be of importance for positronium research. This is discussed in several of the quoted papers, see, for instance, Pivovarov *et al.* [8]. Briefly it can be said that with this technique positronium research is on the level of particle physics in general, with positronium particle beams in vacuum interacting with photon or electron beams for study of energy levels or linewidths. This may then replace the present methods for study of positronium, where positronium is produced in gases or liquids with large background effects from neighboring atoms. This prevents to a large degree the possibility of obtaining the desired accuracy which is needed for future study of positronium, the most pure QED system.

The paper is organized in the following way. In Sec. II the singlet Born approximation cross section is derived which leads to the high-energy exact cross section in Sec. III with the angular distribution in Sec. IV. The singlet positronium spectrum is obtained in Sec. V where accurate formulas including also the important case of arbitrary screening are given. In Sec. VI approximate simplified formulas for low-energy positronium for the singlet positronium cross section are given. High-energy triplet positronium electroproduction is discussed in Sec. VII, with specifications to one-virtual-photon production $eZ \to eZ\gamma \to eZP_{S_t}$ in Sec. VIII and two-virtual-photon production $eZ \to eZ\gamma\gamma \to eZP_{S_t}$ in Sec. IX, always including all valid powers of the atomic number *Z*. With initially polarized electrons it is possible to produce polarized triplet positronium, and we give in Sec. X formulas for positronium polarization for one-virtual-photon production which is high, and for two-virtual-photon production which gives almost no positronium polarization since the positronium energy is rather low in this process.

II. THE SINGLET CROSS SECTION TO LOWEST ORDER IN *Z*

The positronium creation in electron-atom collisions is described by

$$e + Z \to e + Z + \mathrm{Ps}$$

corresponding to the momentum-energy balance

$$p_1 = p_2 + q + p_{\mathrm{Ps}}, \tag{1}$$

where the initial and final electron four momenta are p_1 and p_2, respectively, q the momentum transfer to the atom, and p_{Ps} the positronium four momentum. The cross section is obtained from the pair production cross section for equal energies and momenta of the electron and positron, p_- and p_+, respectively, and appropriate change in phase space from two to one final state particle and by multiplying with inverse squared positronium normalization constant $N_{\mathrm{Ps}} = \alpha^3 m_e^3 / 8\pi n^3$ for the nth positronium energy state. The singlet cross section is given by

$$d^5\sigma_s=\frac{\alpha^7Z^2}{m_e^2n^3}\frac{|\langle p_{\mathrm{Ps}},q,p_2|M|p_1\rangle|^2}{\vec{q}^4k^4[(k-p_+)^2-m_e^2]^2}d^3p_{\mathrm{Ps}}d\Omega_2 \quad (2)$$

with the matrix element

$$|\langle p_{\mathrm{Ps}},q,p_2|M|p_1\rangle|^2=p_{\mathrm{Ps}}^2k^2\sin^2\Theta\left\{2\sin^2\varphi\left[E_1^2-E_1E_{\mathrm{Ps}}+\frac{m_e^2\vec{k}^2}{k^2}+\frac{k^2}{4}\right]+\frac{1}{2}\vec{k}^2\right\}. \quad (3)$$

We have here summed over final spin states and averaged over the spin of the initial state since we do not here consider observations of positronium polarization effects. In Eqs. (2) and (3) k is the spacelike four momentum of the virtual photon, $k=(E_{\mathrm{Ps}},\vec{k})$ with the relations to observable particles

$$k=p_1-p_2=p_{\mathrm{Ps}}+q. \quad (4)$$

Written out, the cross section is

$$d^4\sigma_s=\frac{1}{2\pi}\frac{\alpha^7Z^2}{m_e^2n^3}\times\frac{\left[2\sin^2\varphi\left(E_1^2-E_{\mathrm{Ps}}E_1+\frac{m_e^2\vec{k}^2}{k^2}+\frac{\vec{k}^2}{4}\right)+\frac{1}{2}\vec{k}^2\right]p_{\mathrm{Ps}}^2\sin^2\Theta\, d^3p_{\mathrm{Ps}}dk^2}{(k^2-2E_{\mathrm{Ps}}^2+m_{\mathrm{Ps}}^2+2|\vec{k}||\vec{p}_{\mathrm{Ps}}|\cos\Theta)^2(k^2-E_{\mathrm{Ps}}^2+|\vec{k}||\vec{p}_{\mathrm{Ps}}|\cos\Theta)^2k^2}, \quad (5)$$

where Θ and φ are the polar and azimuthal angles of $\vec{p}_{\mathrm{Ps}}$, respectively, with $\vec{k}$ along the z axis and the scattering $(\vec{p}_1,\vec{p}_2)$ plane in the x-z plane. We have for convenience for later integrations introduced the invariant k^2 as an integration variable by

$$d\cos\alpha=\frac{1}{2|\vec{p}_1||\vec{p}_2|}dk^2,$$

with α the initial electron scattering angle.

In Eq. (4) E_{Ps} is the positronium energy

$$E_{\mathrm{Ps}}=E_1-E_2, \quad (6)$$

the momentum transfer is given by

$$q^2=(k-p_{\mathrm{Ps}})^2=k^2-2E_{\mathrm{Ps}}^2+m_{\mathrm{Ps}}^2+2|\vec{k}||\vec{p}_{\mathrm{Ps}}|\cos\Theta, \quad (7)$$

and the positron propagator denominator by

$$(k-p_+)^2-m_e^2=k^2-E_{\mathrm{Ps}}^2+|\vec{k}||\vec{p}_{\mathrm{Ps}}|\cos\Theta. \quad (8)$$

III. HIGH-ENERGY SINGLET POSITRONIUM ELECTROPRODUCTION

The high-energy pair production process is given by the Born approximation cross section for small values of the momentum transfer $\vec{q}$ [9]. This is physically reasonable since small values of $\vec{q}$ means distant collisions which in turn means collisions in a weak Coulomb field. Moreover, for these small values of $\vec{q}$ atomic screening, which amounts to including the screening function $1-F(q)$, is important. For the atomic screening function $F(q)$ we shall use the Thomas-Fermi-Molière model [9]. As in photoproduction of positronium [2], the cross section for electroproduction is given by the Born approximation for small values of $\vec{q}$, $|\vec{q}|^2\ll m_{\mathrm{Ps}}^2$, and for ‘‘large’’ values of $\vec{q}$, $|\vec{q}|^2\sim m_{\mathrm{Ps}}^2$ the cross section is obtained by multiplying the Born cross section by the factor $[V(x)/V(1)]^2$, where $V(x)$ is the hypergeometric function [9,10] $F(ia,-ia;1,x)$ and $V(1)=\sinh\pi a/\pi a$. Here $x=1-q^2\xi^2$ with $\xi=(1+u^2)^{-1}$, $u=p_\perp/m_{\mathrm{Ps}}$ as given in Ref. [2].

The cross section for high energies and small positronium emission angles for electroproduction of singlet positronium in the field of an atom of charge number Z including screening and Coulomb interactions to all orders is then from Eq. (5) given by

$$d^4\sigma_s=\frac{1}{2\pi}\frac{\alpha^7Z^2[1-F(q)]^2}{m_e^2n^3}\frac{p_\perp^3dp_\perp m_{\mathrm{Ps}}^4\left\{2\sin^2\varphi\left(1-\frac{E_{\mathrm{Ps}}}{E_1}+\frac{m_e^2\vec{k}^2}{k^2E_1^2}+\frac{k^2}{4E_1^2}\right)+\frac{1}{2}\frac{\vec{k}^2}{E_1^2}\right\}}{[p_\perp^2+m_{\mathrm{Ps}}^2-k^2]^2\left[p_\perp^2+\frac{(m_{\mathrm{Ps}}^2-k^2)^2}{4E_{\mathrm{Ps}}^2}\right]^2}[V(x)/V(1)]^2\frac{dE_{\mathrm{Ps}}}{E_{\mathrm{Ps}}}\frac{dk^2}{k^2}\frac{d\varphi}{2\pi}, \quad (9)$$

where $p_\perp = p_{\mathrm{Ps}}\Theta$ is the component of the positronium momentum perpendicular to $\vec{k}$.

IV. ANGULAR DEPENDENCE

The azimuthal dependence of the cross section relative to the scattering $(\vec{p}_1 - \vec{p}_2 - \vec{k})$ plane is such that while the coefficient of $\sin^2\varphi$ in Eqs. (5) and (9) is positive, positronium is preferently emitted in a plane perpendicular to the scattering plane with the φ dependence

$$d\sigma/d\varphi \sim 2\sin^2\varphi\left[E_1^2 - E_1 E_{\mathrm{Ps}} + \frac{m_e^2\vec{k}^2}{k^2} + \frac{k^2}{4}\right] + \frac{1}{2}\vec{k}^2. \tag{10}$$

Note that the φ dependence is similar to the dependence for photoproduction of positronium [2], in that case the photon polarization plane plays the role of the reference plane.

For high energies and small angles Θ the angular dependence is governed by the narrow region $\Theta^2 \sim (m_{\mathrm{Ps}}^2 - k^2)/p_{\mathrm{Ps}}^2$ dictated by the q^2 dependence Eq. (7), similar to the case of photoproduction dicussed by the author in Ref. [1],

$$d\sigma/d\Theta \sim \Theta^3\left(\Theta^2 + \frac{(m_{\mathrm{Ps}}^2 - k^2)^2}{4E_{\mathrm{Ps}}^4}\right)^{-2} \tag{11}$$

followed by an angular tail due to the positron propagator Eq. (7),

$$d\sigma/d\Theta \sim \left[\Theta\left(\Theta^2 + \frac{m_{\mathrm{Ps}}^2 - k^2}{p_{\mathrm{Ps}}^2}\right)\right]^{-2}. \tag{12}$$

These properties are characteristic of high-energy pair production and bremsstrahlung processes. These two regions also divide between strong influence of screening for small values of q^2, while screening effects are absent in the angular tail.

V. HIGH-ENERGY SMALL ANGLE SINGLET POSITRONIUM PRODUCTION: POSITRONIUM SPECTRUM

The differential cross section Eq. (9) is integrated over φ, summed over all s states, and the effect of screening [10] is included in a simplified Thomas-Fermi-Molière model

$$\frac{1 - F(q)}{q^2} = \frac{1}{\beta^2 + q^2}, \tag{13}$$

with $\beta = (Z^{1/3}/242)m_e$, an approximation for the full Molière model [11]. Equation (9) then becomes

$$d^3\sigma_s = \frac{\alpha^7 Z^2}{m_e^2}\zeta(3)\frac{p_\perp^3 dp_\perp m_{\mathrm{Ps}}^4\left[1 - \frac{E_{\mathrm{Ps}}}{E_1} + \frac{1}{2}\left(\frac{E_{\mathrm{Ps}}}{E_1}\right)^2 + \frac{m_e^2}{k^2}\frac{E_{\mathrm{Ps}}^2}{E_1^2} - \frac{k^2}{4E_1^2} - \frac{m_e^2}{E_1^2}\right]}{[p_\perp^2 + m_{\mathrm{Ps}}^2 - k^2]^2\left[p_\perp^2 + \beta^2 + \frac{(m_{\mathrm{Ps}}^2 - k^2)^2}{4E_{\mathrm{Ps}}^2}\right]^2} \times [V(x)/V(1)]\frac{dE_{\mathrm{Ps}}}{E_{\mathrm{Ps}}}\frac{dk^2}{k^2}, \tag{14}$$

where $\zeta(p) = \Sigma_1^\infty(1/n^p)$ is the Riemann ζ function, with $\zeta(3) = 1.20205$.

Integrations over $p_\perp$ and the virtual spacelike photon, Eq. (4), are straightforward in principle and the resultant positronium spectrum can be written in the form

$$d\sigma_s = \frac{\alpha^7 Z^2}{m_e^2}\zeta(3)\int_{\eta_{\min}}^\infty \frac{d\eta}{\eta(1+\eta)^2}[\mathcal{F}(\eta, E_{\mathrm{Ps}}, \beta) - 1 - f_s(Z)] \times\left[1 - \frac{E_{\mathrm{Ps}}}{E_1} + \frac{1}{2}\left(\frac{E_{\mathrm{Ps}}}{E_1}\right)^2 - \frac{1}{4\eta}\left(\frac{E_{\mathrm{Ps}}}{E_1}\right)^2\right]\frac{dE_{\mathrm{Ps}}}{E_{\mathrm{Ps}}} \tag{15}$$

with $\eta = -k^2/m_{\mathrm{Ps}}^2$, and $\eta_{\min} = (-k^2/m_{\mathrm{Ps}}^2)_{\min} = E_{\mathrm{Ps}}^2/(4E_1E_2)$. Here $\mathcal{F} - 1 - f_s(Z)$ is the result of the $p_\perp$ integration, analogous to photoproduction [2], where the Coulomb correction function for singlet positronium production is given by

$$f_s(Z) = \frac{1}{4}\left[2(2\ln 2 - 1) - \ln y_1 - \int_0^{1-y_1}\frac{dx}{\sqrt{x}}\frac{1+x}{1-x}[V(x)/V(1)]^2\right], \quad y_1 \ll 1 \tag{16}$$

and tabulated in Table I.

The function $\mathcal{F}(\eta, E_{\mathrm{Ps}}, \beta)$ is given for Thomas-Fermi-Molière screening, Eq. (13), by

$$\mathcal{F}(\eta, E_{\mathrm{Ps}}, \beta) = -\frac{1}{2}\ln\left[\left(\frac{m_{\mathrm{Ps}}}{2E_{\mathrm{Ps}}}\right)^2(1+\eta) + \left(\frac{\beta}{m_e}\right)^2(1+\eta)^{-1}\right]. \tag{17}$$

It is of some importance to note the comparison with the case of photoproduction [2] where the function

TABLE I. The Coulomb functions $f(Z)$, $f_s(Z)$, and $f_t(Z)$ used in the text.

	Z	$f(Z)$	$f_s(Z)$	$f_t(Z)$
C	6	0.0023	0.0057	0.0047
Al	13	0.0107	0.0265	0.0218
Fe	26	0.0420	0.100	0.0828
Kr	36	0.0784	0.185	0.1504
Sn	50	0.144	0.325	0.2632
Pt	78	0.303	0.655	0.4952
Pb	82	0.332	0.705	0.538
U	92	0.395	0.815	0.595

$$-\frac{1}{2}\ln\left[\left(\frac{m_{\mathrm{Ps}}}{2E_1}\right)^2+\left(\frac{\beta}{m_e}\right)^2\right] \tag{18}$$

occurs for Thomas-Fermi-Molière screening, with $E_1=\omega$, the initial particle energy. This shows that the effect of the screening on photoproduction of positronium is determined by the initial particle energy since $E_{\mathrm{Ps}}=\omega$, the photon energy. The experimental setup determines whether screening is absent, partial or complete. For electroproduction on the other hand, the effect of screening varies over the positronium spectrum—it is in general stronger in the upper part of the spectrum while for the lower part where E_{Ps}/E_1 may become small, the factor $(E_{\mathrm{Ps}}/E_1)^2(1+\eta)^{-2}$ may reduce the effect of the screening considerably, which is seen by writing Eq. (17) in the form

$$\mathcal{F}(\eta,E_{\mathrm{Ps}},\beta)=-\frac{1}{2}\ln\left\{\left(\frac{E_1}{E_{\mathrm{Ps}}}\right)^2(1+\eta)\times\left[\left(\frac{m_{\mathrm{Ps}}}{2E_1}\right)^2+\left(\frac{\beta}{m_e}\right)^2\left(\frac{E_{\mathrm{Ps}}}{E_1}\right)^2(1+\eta)^{-2}\right]\right\}.$$

As a consequence we feel that a satisfactory theory of positronium production can be given only if the complete effect of screening as in Eq. (17) is taken into account. We rewrite Eq. (15) in the form

$$d\sigma_s=\frac{\alpha^7Z^2}{m_e^2}\zeta(3)[W_1(x,\beta)+W_2(x)]\frac{dx}{x}, \tag{19}$$

where we, as we did earlier [3], have introduced

$$x=E_{\mathrm{Ps}}/E_1$$

and where the η integrals are

$$W_1(x,\beta)=(1-x)\int_{\eta_m}^{\infty}\frac{d\eta}{\eta(1+\eta)^2}\times\mathcal{F}(\eta,E_{\mathrm{Ps}},\beta)\left(1+2\eta_m-\frac{\eta_m}{\eta}\right), \tag{20}$$

$$W_2(x)=-(1-x)[1+f_s(Z)]\int_{\eta_m}^{\infty}\frac{d\eta}{\eta(1+\eta)^2}\times\left(1+2\eta_m-\frac{\eta_m}{\eta}\right). \tag{21}$$

We have here introduced η_m defined above, $\eta_m=(x^2/4)(1-x)^{-1}$, as a convenient variable which simplifies the equations. The results of the calculations of the integrals in Eqs. (20) and (21) are obtained in Appendix A. From Eqs. (A3) and (A6) the positronium spectrum for arbitrary screening can be obtained as

$$\begin{aligned}d\sigma_s={}&\frac{\alpha^7Z^2}{m_e^2}\zeta(3)\frac{1}{2}(1-x)\Big\{[(1+2\eta_m)I_0(\eta_m)-\eta_mI_1(\eta_m)]\\&\times\left[\ln\left(\frac{2E_{\mathrm{Ps}}}{m_{\mathrm{Ps}}}\right)^2-2-2f_s(Z)\right]+\eta_mF(\eta_m,a)\\&-(1+4\eta_m)G(\eta_m,a)+(1+3\eta_m)H(\eta_m,a)\Big\}\\&\times\frac{dx}{x}\quad\text{(arbitrary screening)},\end{aligned} \tag{22}$$

where the functions I_0, I_1, F, G, and H are given in Appendix A, Eqs. (A4), and $a=(\beta/m_e)^2/(m_{\mathrm{Ps}}/2E_{\mathrm{Ps}})^2$.

For the case of no screening, i.e., for positronium energies for which the screening parameter is very small

$$\left(\frac{m_{\mathrm{Ps}}}{2E_{\mathrm{Ps}}}\right)^2\gg(Z^{1/3}/242)^2, \tag{23}$$

so that a is negligible compared to one, the cross section is obtained from Eq. (22) by neglecting a. Written out, the cross section for no screening is

$$\begin{aligned}d\sigma_s={}&\frac{\alpha^7Z^2}{m_e^2}\zeta(3)\frac{1}{2}(1-x)\left\{\left[(1+4\eta_m)\ln\frac{1+\eta_m}{\eta_m}-2\frac{1+2\eta_m}{1+\eta_m}\right]\right.\\&\times\left[\ln\left(\frac{2E_{\mathrm{Ps}}}{m_{\mathrm{Ps}}}\right)^2-2-2f_s(Z)\right]+\eta_m\ln\frac{1+\eta_m}{\eta_m}\\&+\left(1+\frac{\eta_m}{1+\eta_m}\right)\ln(1+\eta_m)+\frac{1+3\eta_m}{1+\eta_m}-(1+4\eta_m)\\&\left.\times\left[\frac{\pi^2}{6}-L_2\left(\frac{\eta_m}{1+\eta_m}\right)\right]\right\}\frac{dx}{x}\quad\text{(no screening)}.\end{aligned} \tag{24}$$

This result was obtained by Gevorkyan *et al.* [4] in a different notation. For the case of complete screening, $(Z^{1/3}/242)^2\gg(m_{\mathrm{Ps}}/2E_{\mathrm{Ps}})^2$ the spectrum is obtained from Eq. (24) by replacing $2E_{\mathrm{Ps}}/m_{\mathrm{Ps}}$ by $242/Z^{1/3}$ and changing the sign of the last four terms. These changes are obvious from Eq. (A1)

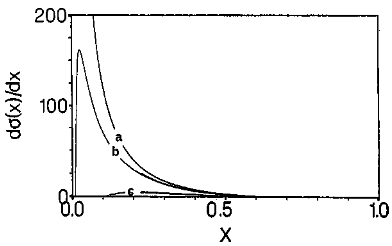

FIG. 1. Singlet electroproduction positronium spectra in units of $\zeta(3)Z^2\alpha^7/m_e^2$ for an aluminum target. Curves (a)–(c) are for E_1 equal to 1 GeV, 100 MeV and 10 MeV, respectively.

$$d\sigma_s=\frac{\alpha^7Z^2}{m_e^2}\zeta(3)\frac{1}{2}(1-x)\Bigg\{\left[(1+4\eta_m)\ln\frac{1+\eta_m}{\eta_m}-2\frac{1+2\eta_m}{1+\eta_m}\right]\times\left[\ln\left(\frac{242}{Z^{1/3}}\right)^2-2-2f_s(Z)\right]-\eta_m\ln\frac{1+\eta_m}{\eta_m}-\left(1+\frac{\eta_m}{1+\eta_m}\right)\ln(1+\eta_m)-\frac{1+3\eta_m}{1+\eta_m}+(1+4\eta_m)\times\left[\frac{\pi^2}{6}-L_2\left(\frac{\eta_m}{1+\eta_m}\right)\right]\Bigg\}\frac{dx}{x}\quad(\text{complete screening}). \tag{25}$$

VI. APPROXIMATE FORMULAS FOR THE SINGLET POSITRONIUM SPECTRUM FOR ARBITRARY SCREENING

The spectrum of Eq. (22) gives an exact description of the positronium spectrum for high initial energies for any element. Figure 1 for Al for some energies shows that as pointed out in this paper, it is necessary to use formulations for arbitrary screening also for very high energies: Even for such high energies as 100 MeV and 1 GeV, screening and energy dependence has to be taken into account in order to obtain a correct description of the process. In fact complete screening is strictly speaking never realized, in particular since with increasing energies the spectrum gets more peaked towards lower positronium energies where non-screening effects are most important. To study these effects we derive the spectrum in the limit of small values of x, which is the most important part of the spectrum.

The exact spectrum Eq. (22), becomes for small values of x, including first order in x (remember that η_m is of second order in x in this region)

$$d\sigma_s=\frac{\alpha^7Z^2}{m_e^2}\zeta(3)\frac{1}{2}(1-x)\left\{[I_0(0)-\eta_mI_1(\eta_m)]\left[\ln\left(\frac{2E_{\mathrm{Ps}}}{m_{\mathrm{Ps}}}\right)^2-2-2f_s(Z)\right]+\eta_mF(\eta_m,a)-G(0,a)+H(0,a)\right\}\frac{dx}{x},$$

where from Eqs. (A4), it is found that

$$I_o(0)=-\ln\eta_m-1,\qquad \eta_mI_1(\eta_m)=1,$$

TABLE II. The $g(a)$ functions, Eq. (27), related to arbitrary screening for singlet positronium electroproduction.

a	0.0	0.2	0.4	0.6	0.8	1.0	∞
$g(a)$	0.64	0.50	0.40	0.28	0.21	0.17	-0.64

$$\eta_mF(\eta_m,a)=\ln(1+a),$$

$$G(0,a)=-\ln\eta_m\ln(1+a)+\frac{\pi^2}{6}+\frac{1}{2}L_2(-a)+\frac{1}{4}\ln^2(1+a)-\gamma_0^2,$$

$$H(0,a)=\ln(1+a)+\frac{2}{\sqrt{a}}\gamma_0-1,$$

where $\tan\gamma_0=\sqrt{a}$. This gives the result

$$d\sigma_s=\frac{\alpha^7Z^2}{m_e^2}\zeta(3)\frac{1}{2}(1-x)\Bigg(-\left(2\ln\frac{2-x}{x}-2\right)\times\left\{\ln\left[\left(\frac{m_{\mathrm{Ps}}}{2E_{\mathrm{Ps}}}\right)^2\left(\frac{\beta}{m_e}\right)^2\right]+2+2f_s(Z)\right\}-g(a)\Bigg)\frac{dx}{x}\quad(\text{arbitrary screening}) \tag{26}$$

with

$$g(a)=\frac{\pi^2}{6}+1+\frac{1}{2}L_2(-a)-\frac{2}{\sqrt{a}}\gamma_0+\frac{1}{2}\ln^2(1+a)-\gamma_0, \tag{27}$$

which is given in Table II.

We have in Eq. (27) used

$$-\ln\eta_m=\ln\left(\frac{4(1-x)}{x^2}\right)=2\ln\left(\frac{2\sqrt{1-x}}{x}\right)=2\ln\frac{2-x}{x}$$

to first order in x. For completeness we give the results for no screening and complete screening. We find for $a=0$, no screening, $g(0)=\pi^2/6-1$,

$$d\sigma_s=\frac{\alpha^7Z^2}{m_e^2}\zeta(3)(1-x)\times\left\{\left(2\ln\frac{2-x}{x}-2\right)\left[\ln\frac{2E_{\mathrm{Ps}}}{m_{\mathrm{Ps}}}-1-f_s(Z)\right]-\frac{\pi^2-6}{12}\right\}\frac{dx}{x}, \tag{28}$$

$a=\infty$, complete screening, $g(a)=-\pi^2/6+1$,

$$d\sigma_s = \frac{\alpha^7 Z^2}{m_e^2}\zeta(3)(1-x) \times \left\{ \left(2\ln\frac{2-x}{x} - 2\right)\left[\ln\frac{242}{Z^{1/3}} - 1 - f_s(Z)\right] - \frac{\pi^2-6}{12}\right\}\frac{dx}{x}. \tag{29}$$

Except for our extra factor $(1-x)$ and the factor $(2-x)/x$ in the logarithm instead of $2/x$, which are necessary in order that our results shall be correct to order x, these particular results, Eqs. (28),(29), were previously obtained by Gevorkyan *et al.* [4]. Values for $g(a)$ are given in Table II, and values for $d\sigma_s/dx$ for some energies and Al target are given in Fig. 1.

VII. HIGH-ENERGY TRIPLET POSITRONIUM ELECTROPRODUCTION

The Born approximation triplet positronium electroproduction cross section was obtained in Ref. [3]. The production process is closely related to the bremsstrahlung process, except that the virtual bremsstrahlung photon is converted to a positronium particle with mass m_{Ps}. This has an important effect on the spectrum of the produced particle: While the bremsstrahlung spectrum is strongly peaked at the low-energy end, the positronium spectrum, or any final state ''axion'' or other massive particle spectrum [12], lacks this effect: the spectrum is for high energies strongly concentrated near the upper end of the spectrum.

This is a consequence of the momentum transfer, $\vec{q}$, to the nucleus, which occurs as a factor $1/q$ [4] in the bremsstrahlung and positronium production cross sections, due to the Coulomb field. Atomic screening modifies this factor, but the singularity for $q=0$ still remains. It is now easy to see the reason for the differences in the spectra. For bremsstrahlung the momentum transfer

$$\vec{q} = \vec{p}_1 - \vec{p}_2 - \vec{k}$$

has a minimum

$$q_{\min} = \frac{\omega m_e^2}{2E_1E_2} \tag{30}$$

as can be easily seen. Here E_1 and E_2 are the initial and final energies of the electron, respectively, and ω the photon energy. This shows that the emission of the zero mass photon gives a strong peak at the lower end of the spectrum. The electroproduction of positronium does not have this effect, the minimum value of the momentum transfer is never zero, as can be seen from the formula

$$q_{\min} = (E_{\mathrm{Ps}}^2 m_e^2 + E_1E_2 m_{\mathrm{Ps}}^2)/2E_1E_2E_{\mathrm{Ps}} = \frac{E_{\mathrm{Ps}} m_e^2}{2E_1E_2} + \frac{m_{\mathrm{Ps}}^2}{2E_{\mathrm{Ps}}} \tag{31}$$

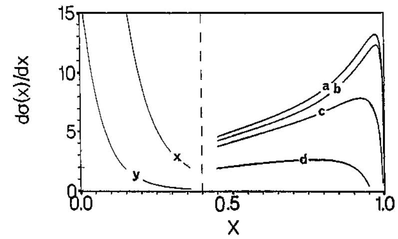

FIG. 2. Triplet electroproduction positronium spectra in units of $\zeta(3)Z^2\alpha^7/m_e^2$ for $eZ\to eZ\gamma\to eZ\,\mathrm{Ps}_t$ [thick lines, curves (a)–(d)] and $eZ\to eZ\gamma\gamma\to eZ\,\mathrm{Ps}_t$ (thin lines, curves x and y). Curves (a) and (b) refer to $E_1=1$ GeV for Al and Sn, respectively, and curves (c) and (d) for Al targets and $E_1=100$ and 10 MeV, respectively. Curves x and y are for Sn and Al. As described in the text, interference is not taken into account.

which can be easily derived or can be found in Ref. [3], Eq. (10). This very simple discussion shows that the emission of a nonzero mass particle in a bremsstrahlung-type process will completely change the particle spectrum from mostly relatively low energy photons in the bremsstrahlung process to relatively high-energy particles in the massive particle process.

In the previous work [3], only Born approximation cross sections were calculated. Beyond the Born approximation there is the Coulomb correction to the Born approximation singlet positronium production cross section which is obtainable from the bremsstrahlung cross section to all orders in Z, given in Ref. [9], in much the same way as the singlet positronium production cross section was obtained [2]. We denote this bremsstrahlung related process by $eZ\to eZ\gamma \to eZ\mathrm{Ps}_t$ and $d\sigma(e\gamma\to e\mathrm{Ps})$.

In addition there is the triplet positronium electroproduction cross section which does not exist in Born approximation, analogous to the photoproduction triplet cross section [2]—and which was calculated by Gevorkyan *et al.* [4]. We denote this two-photon related process by $eZ\to eZ\gamma\gamma \to eZ\,\mathrm{Ps}_t$ and $d\sigma(e\gamma\gamma\to e\,\mathrm{Ps})$.

One should bear in mind that these two production processes are coherent and show interference effects—except in the Born approximation, which is a somewhat unusual situation. We shall, however, not take into account interference effects, which is an acceptable approximation since the bremsstrahlung related process $eZ\to eZ\gamma\to eZ$ is confined to high-energy positronium as discussed above, while the two-photon related process $eZ\to eZ\gamma\gamma\to eZ\,\mathrm{Ps}_t$ is like the corresponding photoproduction process [2], confined to the lower part of the spectrum. Specifically this is shown in Fig. 2.

VIII. THE $eZ\to eZ\gamma\to eZ\mathrm{Ps}_t$ PROCESS

The triplet electroproduction cross section by the bremsstrahlunglike process is derived in Appendix B with the result for no screening, with $x=E_{\mathrm{Ps}}/E_1$

$$d\sigma_t^{\text{no scr}}(e\gamma\to e\,\text{Ps})=\frac{1}{4}Z^2\alpha^7\zeta(3)/m_e^2\left[1+\left(\frac{x}{2-x}\right)^4\right]\times\left\{2\ln\left[\frac{2(1-x)E_1}{(2-x)m_e}\right]-1-2f(Z)\right\}x\,dx, \tag{32}$$

where we have used

$$q_{\min}=(2-x)^2/[2x(1-x)]m_e^2/E_1, \tag{33}$$

and $f(Z)$ is the bremsstrahlung Coulomb correction function given in Table I.

For arbitrary screening, with the screening parameter $\Lambda=m_eZ^{1/3}/111$, the cross section is from Ref. [3] and Appendix B

$$\begin{aligned}d\sigma_t(e\gamma\to eP_s)=\frac{1}{4}Z^2\alpha^7\zeta(3)/m_e^2\Bigg\{&\left[1+\left(\frac{x}{2-x}\right)^4\right]\\&\times\Bigg[2\ln\left(\frac{2(1-x)}{2-x}\frac{E_1}{m_e}\right)+1\\&-2\frac{q_{\min}}{\Lambda}\arctan\frac{\Lambda}{q_{\min}}-\ln\left(1+\frac{\Lambda^2}{q_{\min}^2}\right)\\&-2f(Z)\Bigg]+4\frac{x^2(1-x)}{(2-x)^4}\Bigg[\frac{1}{3}\\&+3\frac{q_{\min}^2}{\Lambda^2}\ln\left(1+\frac{\Lambda^2}{q_{\min}^2}\right)-2\frac{q_{\min}}{\Lambda}\arctan\frac{\Lambda}{q_{\min}}\\&-4\frac{q_{\min}^2}{\Lambda^2}\left(1-\frac{q_{\min}}{\Lambda}\arctan\frac{\Lambda}{q_{\min}}\right)\Bigg]\Bigg\}x\,dx,\end{aligned} \tag{34}$$

with

$$\Lambda/q_{\min}=2x(1-x)/(2-x)^2(E_1/m_e)(Z^{1/3}/111). \tag{35}$$

As discussed in Sec. V for singlet positronium the effect of screening depends on $x=E_{\text{Ps}}/E_1$, not only on the initial energy E_1. In fact, for x close to the minimum value $x_{\min}=m_{\text{Ps}}/E_1$ as well as close to the maximum value $x_{\max}=1-m_e/E_1$ screening is absent: for

$$x_{\min}=m_{\text{Ps}}/E_1,\quad \Lambda/q_{\min}=Z^{1/3}/222$$

and

$$x_{\max}=1-m_e/E_1,\quad \Lambda/q_{\min}=Z^{1/3}/55.5,$$

and $\Lambda/q_{\min}$ is much smaller than 1, i.e., the effect of screening is essentially absent for any element, and the cross section is given by $d\sigma_t^{\text{no scr.}}(e\gamma\to e\,\text{Ps})$. In fact for all values of x the cross section to second order in $\Lambda/q_{\min}$ can be written

$$\begin{aligned}d\sigma_t(e\gamma\to e\text{Ps})&=d\sigma_t^{\text{no scr.}}(e\gamma\to e\,\text{Ps})\\&+\frac{1}{4}Z^2\alpha^7\zeta(3)/m_e^2\left(-\frac{1}{3}\right)\\&\times\left[1+\frac{1}{5}\frac{x^2}{(2-x)^4}(5x^2-2x+2)\right](\Lambda/q_{\min})^2x\,dx.\end{aligned} \tag{36}$$

Since the maximum value of $(\Lambda/q_{\min})^2$ which occurs for $x=2/3$ is

$$(\Lambda/q_{\min})^2_{\max}=0.0625\left(\frac{E_1}{m_e}\frac{Z^{1/3}}{111}\right)^2,$$

the formula for the cross section is to a fair degree of accuracy given by $d\sigma_t^{\text{no scr.}}(e\gamma\to e\,\text{Ps})$ Eq. (32) for energies far above the conventional ''no screening'' limit $(E_1/m_e)(Z^{1/3}/111)\ll 1$.

For strong screening, $\Lambda/q_{\min}\gg 1$, the cross section to first order in $q_{\min}/\Lambda$ is given by

$$\begin{aligned}d\sigma_t^{\text{st scr}}(e\gamma\to e\,\text{Ps})=d\sigma_t^{\text{compl scr}}(e\gamma\to e\,\text{Ps})\\-\frac{1}{4}Z^2\alpha^7[\zeta(3)/m_e^2]4\pi\\\times\frac{x^2(1-x)}{(2-x)^2}\frac{q_{\min}}{\Lambda}x\,dx,\end{aligned} \tag{37}$$

where for complete screening

$$\begin{aligned}d\sigma_t^{\text{compl scr}}(e\gamma\to e\,\text{Ps})=\frac{1}{4}Z^2\alpha^7\zeta(3)/m_e^2\Bigg\{&\left[1+\left(\frac{x}{2-x}\right)^4\right]\\&\times\left[2\ln\left(\frac{2-x}{x\Lambda}\right)+1-2f(Z)\right]\\&+\frac{4x^2(1-x)}{3(2-x)^2}\Bigg\}x\,dx.\end{aligned} \tag{38}$$

When we use Eq. (35) for $\Lambda/q_{\min}$, Eq. (37) can be written

$$\begin{aligned}d\sigma_t^{\text{st scr}}(e\gamma\to e\,\text{Ps})=\frac{1}{4}Z^2\alpha^7\zeta(3)/m_e\times\Bigg\{&\left[1+\left(\frac{x}{2-x}\right)^2\right]\\&\times\left[2\ln\left(\frac{2-x}{x\Lambda}\right)+1-2f(Z)\right]\\&+\frac{4x^2(1-x)}{3(2-x)^2}-\pi x\frac{m_e222}{E_1Z^{1/3}}\Bigg\}x\,dx\end{aligned} \tag{39}$$

so the requirement for complete screening is

$$\frac{E_1}{xm_e} \gg \pi \frac{222}{Z^{1/3}}. \tag{40}$$

Curves of $d\sigma_t(e\gamma \to e\ \mathrm{Ps})$ are given in Fig. 2, for some energies and elements.

IX. THE $eZ \to eZ\gamma\gamma \to eZ\mathrm{Ps}_t$ PROCESS

We shall calculate the cross section for this process, including positronium polarization effects. From the improved Weizsäcker-Williams method [14] we obtain

$$d^3\sigma_t(e\gamma\gamma \to e\ \mathrm{Ps}) = \frac{d\omega}{\omega} N(\omega) f(P_1,\xi) d^2\sigma_t(\gamma \to \mathrm{Ps}) \tag{41}$$

with the number of virtual photons emitted from the initial electron

$$N(\omega) = \frac{\alpha}{2\pi}\left[\frac{E_1^2+E_2^2}{E_1^2}\left(\ln\frac{s_0}{|k_-^2|}-1\right)-2\frac{E_2}{E_1}\right] \tag{42}$$

and [2]

$$d^2\sigma_t(\gamma \to \mathrm{Ps}) = 4\pi \frac{Z^4\alpha^8}{m_e^2 n^3} u\,du\,\xi^4(2\xi-1)^2(1+\vec{s}\cdot\hat{p}_{\mathrm{Ps}}P_{\mathrm{ph}}) \times\left[\frac{W(x)}{V(1)}\right]^2\frac{d\varphi}{2\pi} \tag{43}$$

with $W(x) = F(1+ia,1-ia,2;x)$ and $u = p_{\mathrm{Ps}}^{\perp}/(2E_1)$, is the positronium photoproduction cross section. The factor $f(P_1,\xi)$ is the polarization dependent factor describing the transfer of the initial electron polarization, P_1, to the virtual photon, which can be shown to be

$$f(P_1,\xi) = 1 + \frac{\omega}{E_1+E_2} P_1 P_{\mathrm{ph}}. \tag{44}$$

In these formulas E_1 and E_2 are the energies of the initial and final electron, respectively, $\omega = E_{\mathrm{Ps}}$ the virtual photon energy which is equal to the positronium energy E_{Ps} since no energy is transferred to the nucleus, P_{ph} the unit virtual photon polarization, and $\vec{s}$ the unit positronium polarization. The invariant s_0 describes the fusion of the virtual photon k and the Coulomb photon k_0

$$s_0 = (k+k_0)^2 = m_{\mathrm{Ps}}^2$$

and k_-^2 is the minimum value of the virtual photon squared $-k_-^2 = (p_1-p_2)_-^2 = -E_{\mathrm{Ps}}^2 m_e^2/E_1E_2$.

The cross section is summed over energy levels n, $\Sigma(1/n^3) = \zeta(3)$, integrated over u, which is converted to an integral over x, as in Ref. [2], given in Table I,

$$f_t(Z) = \frac{1}{2}(\alpha Z)^2 \int_0^1 dx\,\sqrt{x}(1+x)[W(x)/V(1)]^2, \tag{45}$$

and averaged over the unobservable virtual photon polarization

$$\frac{1}{2}\sum_{P_{\mathrm{ph}}=-1}^{+1}\left(1+\frac{\omega}{E_1+E_2}P_1P_{\mathrm{ph}}\right)(1+\vec{s}\cdot\hat{p}_{\mathrm{Ps}}P_{\mathrm{ph}}) = 1 + \frac{E_{\mathrm{Ps}}}{E_1+E_2}\vec{s}\cdot\hat{p}_{\mathrm{Ps}}P_1. \tag{46}$$

This gives the positronium spectrum and polarization dependence

$$d\sigma_t(e\gamma\gamma \to e\ \mathrm{Ps}) = \frac{1}{2}Z^2\alpha^7\zeta(3)/m_e^2 f_t(Z)\left(1+\frac{x}{2-x}\vec{s}\cdot\hat{p}_{\mathrm{Ps}}P_1\right) \times(1-x)\left\{(1+2\eta_m)\ln\left[\frac{(2-x)^2-x^2}{x^2}\right] -2(1+\eta_m)\right\}\frac{dx}{x}, \tag{47}$$

where we have introduced as in Sec. V

$$\eta_m = x^2/4(1-x). \tag{48}$$

Neglecting terms of order x^2, which is a good approximation for the cross section, we obtain

$$d\sigma_t(e\gamma\gamma \to e\ \mathrm{Ps}) = \frac{1}{2}Z^2\alpha^7\zeta(3)/m_e^2 f_t(Z)\left(1+\frac{x}{2-x}\vec{s}\cdot\hat{p}_{\mathrm{Ps}}P_1\right) \times(1-x)\left[2\ln\frac{2-x}{x}-2\right]\frac{dx}{x}. \tag{49}$$

We return to polarization effects in Sec. X. Summing over positronium polarizations, $\vec{s}\cdot\hat{p}_{\mathrm{Ps}} = \pm 1$, we obtain the spectrum

$$d\sigma_t(e\gamma\gamma \to e\ \mathrm{Ps}) = Z^2\alpha^7[\zeta(3)/m_e^2] \times(1-x)\left[2\ln\frac{2-x}{x}-2\right]\frac{dx}{x}. \tag{50}$$

This spectrum agrees with the result of Gevorkyan *et al.* [4] in the same approximation [see their equations (3.24) and (3.26)]. Curves of $d\sigma_t(e\gamma\gamma \to e\ \mathrm{Ps})$ are given for some elements in Fig. 2.

X. POSITRONIUM POLARIZATION

With an initial longitudinally polarized electron, part of the electron polarization P_e is transferred to the positronium particles.

A. $eZ \to eZ\gamma \to eZ\mathrm{Ps}_t$

For this process the information on the positronium polarization is obtained from calculation of the Born approximation cross section including a polarized initial electron. The calculation is similar to the cross section calculation [3] with the result

$$d\sigma_t(e\gamma\to e\,\mathrm{Ps,pol}) = \frac{1}{16}Z^2\alpha^2\zeta(3)\frac{E_{\mathrm{Ps}}dE_{\mathrm{Ps}}}{E_1^3E_2q_{\min}^2}\times\left\{\frac{2(E_1^2+E_2^2)q_{\min}}{E_{\mathrm{Ps}}}-\frac{2m_e^2+m_{\mathrm{Ps}}^2}{3}+\left[\frac{2(E_1^2+E_2^2)q_{\min}}{E_{\mathrm{Ps}}}-\frac{2m_e^2+m_{\mathrm{Ps}}^2}{3}\frac{E_{\mathrm{Ps}}}{E_1}\right]\times\vec{\zeta}_e\cdot\hat{p}_{\mathrm{Ps}}\vec{\zeta}_{\mathrm{Ps}}\cdot\hat{p}_{\mathrm{Ps}}\right\}\left[\ln\frac{2E_1E_2}{E_{\mathrm{Ps}}q_{\min}}-1\right]x\,dx. \tag{51}$$

Again this result can be checked against the bremsstrahlung result [9] by replacing the positronium by the zero mass photon, and $\vec{\zeta}_{\mathrm{Ps}}$ by $i\vec{e}\times\vec{e}$ [4]. In Eq. (51) ζ_e and ζ_{Ps} are the electron and positronium polarization unit vectors, respectively.

By the same procedure as in Appendix B, the cross section to all orders in Z is obtained, now also containing polarization effects

$$d\sigma_t(e\gamma\to e\,\mathrm{Ps,pol}) = \frac{1}{8}Z^2\alpha^7\zeta(3)/m_e^2\frac{1}{(2-x)^4}\times\{[(2-x)^4+x^4]+2x[(2-x)^3-2x^2(1-x)]\vec{\zeta}_e\cdot\hat{p}_{\mathrm{Ps}}\vec{\zeta}_{\mathrm{Ps}}\cdot\hat{p}_{\mathrm{Ps}}\}\times\left[2\ln\left[\frac{2(1-x)E_1}{(2-x)m_e}\right]-1-2f(Z)\right]x\,dx. \tag{52}$$

The polarization of positronium is then given by

$$P_{\mathrm{Ps}} = \frac{2x[(2-x)^3-2x^2(1-x)]}{(2-x)^4+x^4}P_1\cos\vartheta, \tag{53}$$

which is shown in Fig. 3 for $P_1=1$ and $\vartheta=0$. Here P_1 is the actual electron polarization and ϑ is the very small angle (of order m_e/E_{Ps}) between $\vec{p}_1$ and $\vec{p}_{\mathrm{Ps}}$. It is seen that for complete transfer of energy to the positron, $x=1$, also the polarization transfer is complete, which is a common feature in quantum electrodynamics. Again, as expected, no polarization is transferred if x is very small, actually of order m_e/E_1. We have here only discussed polarization for no screening. For screening the polarization can be obtained by using Sec. VIII.

It is to be noticed that since the polarization is high for the upper part of the spectrum where the cross section is large, the positronium polarization from the $eZ\to eZ\gamma\to eZP_t$ process is rather high. The average positronium polarization is

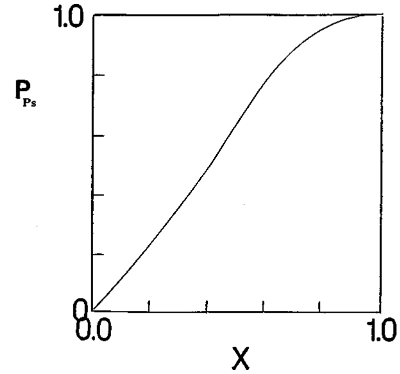

FIG. 3. Positronium polarization in the $eZ\to eZ\gamma\to eZ\,\mathrm{Ps}_t$ process, Eq. (53) for complete electron polarization, $P_1=1$ and $\vartheta=0$.

$$\langle P_{\mathrm{Ps}}\rangle = \int P_{\mathrm{Ps}}(x)\cdot d\sigma_t(x)\Big/\int d\sigma_t \approx \frac{2}{3}P_1. \tag{54}$$

B. $eZ\to eZ\gamma\gamma\to eZ\,\mathrm{Ps}_t$

In Eq. (49) it was shown that the positronium polarization dependence of the cross section for this process is

$$d\sigma_t(e\gamma\gamma\to e\,\mathrm{Ps}) \sim \left(1+\frac{x}{2-x}\vec{s}\cdot\hat{p}_{\mathrm{Ps}}P_1\right)\frac{dx}{x}. \tag{55}$$

This gives the positronium polarization

$$P_{\mathrm{Ps}} = \frac{d\sigma_t(\vec{s}\cdot\vec{p}_{\mathrm{Ps}}=1)-d\sigma_t(\vec{s}\cdot\vec{p}_{\mathrm{Ps}}=-1)}{d\sigma_t(\vec{s}\cdot\vec{p}_{\mathrm{Ps}}=1)+d\sigma_t(\vec{s}\cdot\vec{p}_{\mathrm{Ps}}=-1)} = \frac{x}{2-x}P_1. \tag{56}$$

Again as above, the polarization transfer is complete, $P_{\mathrm{Ps}}=P_1$, when the energy transfer is complete, while essentially no polarization is transferred, $P_{\mathrm{Ps}}\sim 0$, when the energy transfer is essentially absent, $x\sim m_{\mathrm{Ps}}/E_1$. The overall effect of the polarization is in this case small since the spectrum is confined to small values of x where the polarization is small.

APPENDIX A

We write $\mathcal{F}(\eta,E_{\mathrm{Ps}},\beta)$ in the form

$$\mathcal{F}(\eta,E_{\mathrm{Ps}},\beta) = -\frac{1}{2}\left\{\ln\left(\frac{m_{\mathrm{Ps}}}{2E_{\mathrm{Ps}}}\right)^2-\ln(1+\eta)+\ln[(1+\eta)^2+a]\right\}, \tag{A1}$$

where a is the ratio

$$a = \left(\frac{\beta}{m_e}\right)^2\Big/\left(\frac{m_{\mathrm{Ps}}}{2E_{\mathrm{Ps}}}\right)^2 \tag{A2}$$

i.e., the effect of ‘‘screening’’ to ‘‘no screening.’’

The result Eq. (20) can be written in the form

$$W_1(x,\beta)=-\frac{1}{2}(1-x)\Bigg\{[(1+2\eta_m)I_0(\eta_m)-\eta_m I_1(\eta_m)]\ln\left(\frac{m_{\rm Ps}}{2E_{\rm Ps}}\right)^2-\eta_m F(\eta_m,a)+(1+4\eta_m)G(\eta_m,a)-(1+3\eta_m)H(\eta_m,a)\Bigg\}, \tag{A3}$$

where

$$I_0(\eta_m)=\ln\frac{1+\eta_m}{\eta_m}-\frac{1}{1+\eta_m},$$

$$I_1(\eta_m)=-2\ln\frac{1+\eta_m}{\eta_m}+\frac{1}{1+\eta_m}+\frac{1}{\eta_m},$$

$$F(\eta_m,a)=\frac{1}{1+a}\{-2\ln\eta_m+\ln[(1+\eta_m)^2+a]\}-2\sqrt{a}\gamma+\frac{1}{\eta_m}\ln[(1+\eta_m)^2+a]-\left[\frac{1}{\eta_m}\ln(1+\eta_m)+\ln\frac{1+\eta_m}{\eta_m}\right],$$

$$G(\eta_m,a)=\ln^2(1+\eta_m)-\ln^2\eta_m+\frac{1}{2}L_2\left(\frac{-a}{(1+\eta_m)^2}\right)-2\,\mathrm{Re}L_2\left(\frac{-1+i\sqrt{a}}{\eta_m}\right)-\left[\frac{\pi^2}{6}-L_2\left(\frac{\eta_m}{1+\eta_m}\right)\right],$$

$$H(\eta_m,a)=\frac{1}{1+\eta_m}\ln[(1+\eta_m)^2+a]+\frac{2}{\sqrt{a}}\gamma-\frac{1}{1+\eta_m}[\ln(1+\eta_m)+1]. \tag{A4}$$

Here $\tan\gamma=\sqrt{a}/(1+\eta_m)$ and $L_2(Z)$ the Euler dilogarithm

$$L_2(Z)=-\int_0^Z\frac{\ln(1+x)}{x}dx. \tag{A5}$$

Directly from this result follows

$$W_2(x)=-(1-x)[1+f_s(Z)][(1+2\eta_m)I_0(\eta_m)-\eta_m I_1(\eta_m)]. \tag{A6}$$

It should be noted that the construction is such that F, G, and H for $a=0$ are equal to the last parenthesis in each formula with opposite sign. This follows from the definition, Eq. (A1).

APPENDIX B

The cross section for the electroproduction of positronium by the bremsstrahlunglike process is obtained from the corresponding photon bremsstrahlung process [9]. The procedure is analogous to the singlet positronium photoproduction. The cross section consists of the Born approximation part [3] which contains all screening effects and added to this the Coulomb effects of higher orders in Z:

$$d\sigma_{\rm brems}(\omega)=d\sigma^{\rm Born}_{\rm brems}(\omega)+2Z^2\frac{\alpha^2}{m_e^2}\frac{1}{E_1^2}\frac{d\omega}{\omega}\left[E_1^2+E_2^2-\frac{2}{3}E_1E_2\right]\left[\int_0^{1-q^2_{\rm min}}\frac{x}{1-x}R\,dx+\ln q^2_{\rm min}+1\right], \tag{B1}$$

where $q_{\rm min}$ is given in Eq. (30) and [Ref. [9], Eq. (3.9)]

$$R=[V^2(x)+(Z\alpha)^2(1-x)^2W^2(x)]/V^2(1). \tag{B2}$$

As shown by Davies, Bethe, and Maximon [13] the integral in Eq. (B1) can be solved in closed form

$$\int_0^{1-q^2_{\rm min}}\frac{x}{1-x}R\,dx=-\ln q^2_{\rm min}-1-2f(Z) \tag{B3}$$

with the final result, here given for no screening

$$d\sigma_{\rm brems}=2Z^2\frac{\alpha^2}{m_e^2}\frac{1}{E_1^2}\frac{d\omega}{\omega}\left[E_1^2+E_2^2-\frac{2}{3}E_1E_2\right]\times\left(2\ln\frac{2E_1E_2}{\omega m_e}-1-2f(Z)\right), \tag{B4}$$

where

$$f(Z)=(Z\alpha)^2\sum_{n=0}^{\infty}\frac{1}{n[n+(Z\alpha)^2]}. \tag{B5}$$

For triplet positronium production the Born approximation cross section is [3]

$$d\sigma_t^{\rm Born}(\gamma\to Ps)=\frac{1}{8}Z^2\alpha^7\zeta(3)\frac{dE_{\rm Ps}}{E_1^2E_2q^2_{\rm min}}\left[2(E_1^2+E_2^2)q_{\rm min}-\frac{1}{3}(2m_e^2+m^2_{\rm Ps})E_{\rm Ps}\right]\left(\ln\frac{2E_1E_2}{E_{\rm Ps}q_{\rm min}}-1\right) \tag{B6}$$

for no screening. Here $q_{\rm min}$ for triplet production is given by Eq. (31). This cross section for arbitrary screening is also given in Ref. [3]. We shall return to this complete cross section in the text. As for photon bremsstrahlung we sort out the Born approximation

$$d\sigma_t(\gamma\to\text{Ps}) = d\sigma_t^{\text{Born}}(\gamma\to\text{Ps}) + \frac{1}{8}Z^2\alpha^7\zeta(3)\frac{dE_{\text{Ps}}}{E_1^2E_2q_{\min}^2}\times\left[2(E_1^2+E_2^2)q_{\min} - \frac{1}{3}(2m_e^2+m_{\text{Ps}}^2)E_{\text{Ps}}\right]\times\left[\int_1^{1-A}\frac{x}{1-x}R\,dx - \ln A + 1\right], \tag{B7}$$

where screening effects are contained in the Born approximation term and where

$$A = \frac{2E_1E_2}{E_{\text{Ps}}q_{\min}}. \tag{B8}$$

R is the same function of x as in Eq. (B2). The result of the integration can be written down from (B3) with A replacing $q_{\min}^2$. The cross section is then given here for no screening

$$d\sigma_t(\gamma\to\text{Ps}) = \frac{1}{8}Z^2\alpha^7\zeta(3)\frac{dE_{\text{Ps}}}{E_1^2E_2q_{\min}^2}\times\left[2(E_1^2+E_2^2)q_{\min} - \frac{1}{3}(2m_e^2+m_{\text{Ps}}^2)E_{\text{Ps}}\right]\times\left(\ln\frac{2E_1E_2}{E_{\text{Ps}}q_{\min}} - 1 - 2f(Z)\right). \tag{B9}$$

It should be noted as a check on the result that Eqs. (B7)–(B9) agree with the corresponding photon bremsstrahlung results for $m_{\text{Ps}}=0$, which gives $A=(2E_1E_2/E_{\text{Ps}}m_e)^2$.

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