Lyapunov exponents for the differences between quantum and classical dynamics

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The moments and correlations of the (classical or quantum) position and momentum variables satisfy a hierarchy of coupled equations, which have been studied and solved numerically for the Hénon-Heiles model. It is found, for chaotic states of the model, that the second moments of the classical and quantum variables grow exponentially at a rate governed by the classical Lyapunov exponent. The differences between quantum and classical variables also grow exponentially, but with a larger exponent. The behavior of this quantum-classical difference exponent is studied in this paper.

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In the study of classical and quantum chaos it is necessary to understand the quantitative differences between quantum and classical dynamics. One way to study these differences is by way of the moments of the position and momentum variables. The moments and correlation functions of these (quantum or classical) variables satisfy a hierarchy of equations, which may be truncated and solved numerically, provided the widths of the probability distributions remain small enough. By choosing the initial quantum and classical probabilities to be equal, one can study [1] how the differences between the two theories grow in time, how they depend on the parameters of the initial state, and how they scale with $\hbar$.

The model studied in [1] was that of Hénon and Heiles [2], whose Hamiltonian has two degrees of freedom coupled by a cubic potential,

$$H=\tfrac{1}{2}(p_1^2+p_2^2)+\tfrac{1}{2}(q_1^2+q_2^2)+q_1^2q_2-\tfrac{1}{3}q_2^3. \tag{1}$$

The probability distributions were chosen to be Gaussian parametrized by their centroids and variances, with the sum of the variances of the two coordinates,

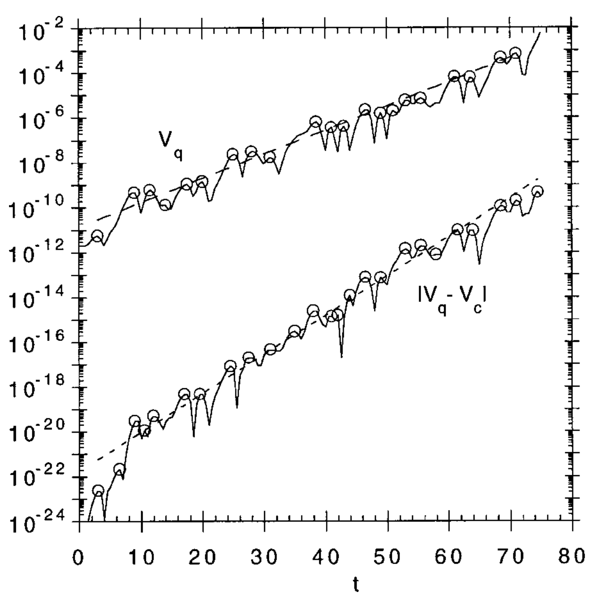

FIG. 1. Quantum variance and the difference between quantum and classical variances. Parameters for the initial chaotic state are $E=0.16$, $q_1=0$, $q_2=-0.20$, $p_1=0.48\,442$, $p_2=0.20$, $\hbar=2\times10^{-12}$.

$$V=\langle(q_1-\langle q_1\rangle)^2\rangle+\langle(q_2-\langle q_2\rangle)^2\rangle, \tag{2}$$

being a convenient parameter. The average brackets $\langle\cdots\rangle$ are to be interpreted as quantum or classical averages, according to the context.

Figure 1 shows the time dependence of the quantum variance V_q and the difference between the quantum and classical variances, $|V_q-V_c|$, for a classically chaotic state. The classical variance V_c would be indistinguishable from V_q on the scale of the graph. It is well known that in a chaotic state the separation between two initially close trajectories will oscillate within an exponentially growing envelope, proportional to $e^{\lambda t}$, with λ being the largest Lyapunov exponent [3]. Thus the variance, being the mean-squared separation of the ensemble, should grow as $e^{2\lambda t}$.

More interesting was the discovery [1] that the small difference between the quantum and classical variances also grows exponentially, but with a larger exponent than that of the variances themselves. The difference between the centroids of the quantum and classical probability distributions also grows exponentially with the same exponent as $|V_q-V_c|$. This exponent, which we shall call λ_{qc}, was found to be independent of the magnitude of $\hbar$. The Lyapunov exponents are well established as useful parameters for characterizing the nature of classical chaotic dynamics, so we may expect that the quantum-classical difference exponent should come to play a similarly useful role in the theory of quantum chaos. This paper presents the first detailed study of the quantum-classical difference exponent in a particular model.

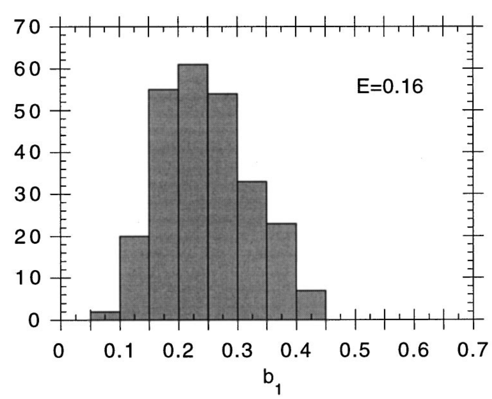

FIG. 2. Histogram of the exponent b_1 for the growth of the quantum variance.

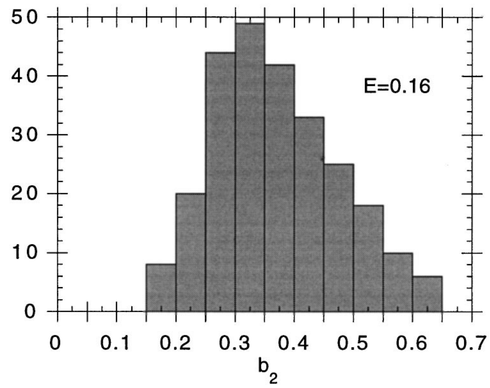

FIG. 3. Histogram of the quantum-classical difference exponent b_2, whose average is λ_{qc}.

The quantities V_q and $|V_q - V_c|$ are highly oscillatory, but their upper envelopes are approximately exponential, being proportional to $e^{b_1 t}$ and $e^{b_2 t}$, respectively. Therefore we find the local maxima of the curves (the circles in Fig. 1), and estimate the exponents b_1 and b_2 from the slopes of the fitted straight lines on the semilogarithmic plot. Since b_2 is greater than b_1, it would appear that the two curves in Fig. 1 would eventually cross. But the extrapolated straight lines cross for variances greater than the size of the system, which is 0(1), so the curves do not actually cross. The exponents b_1 and b_2 are analogous to the *local* or *finite-time* Lyapunov exponents, which may be defined as

$$\lambda(x_0, T) = \lim_{\epsilon \to 0} T^{-1} \ln[d(\epsilon, T)/\epsilon], \tag{3}$$

where $d(\epsilon, 0) = \epsilon$ is the initial separation between two nearby trajectories that start within ϵ of the point x_0 in phase space, and $d(\epsilon, T)$ is their separation at the later time T. The usual Lyapunov exponent is given by $\lambda = \lim_{T\to\infty} \lambda(x_0, T)$ and is independent of the starting point x_0 on the trajectory. For finite T there is a distribution of values of $\lambda(x_0, T)$ corresponding to the different starting points x_0 within the region that is densely explored by the trajectory. This distribution may contain useful information beyond that of the usual Lyapunov exponent [4].

Figures 2 and 3 show, respectively, the distributions of the exponents b_1 and b_2 at the energy $E = 0.16$, for which the accessible region of phase space contains a large chaotic zone. That zone was sampled by examining the Poincaré section defined by the condition $q_1 = 0$ with $p_1 > 0$, and selecting initial conditions within the chaotic zone on a square grid in the (q_2, p_2) plane of cell size 0.05×0.05. Although the ranges of b_1 and b_2 overlap considerably, the quantum-classical difference exponent b_2 is always larger than b_1, as is shown by the distribution of $b_2 - b_1$ in Fig. 4. Apparently, there must be a significant correlation between b_1 and b_2, which is confirmed by the linear regression plot, Fig. 5. This close relation between b_1 and b_2 is surprising, since b_1 is essentially a classical parameter, whereas b_2 is purely quantum mechanical in origin. There are significant residual deviations from the regression line, but they form no apparent pattern.

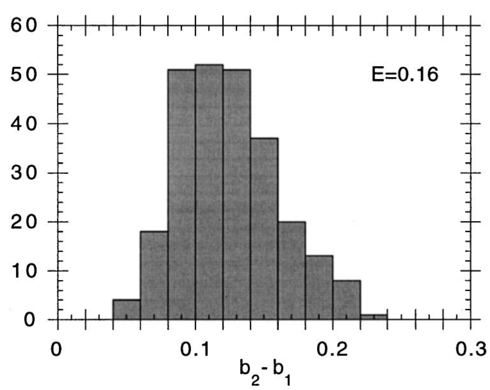

FIG. 4. Histogram of $b_2 - b_1$.

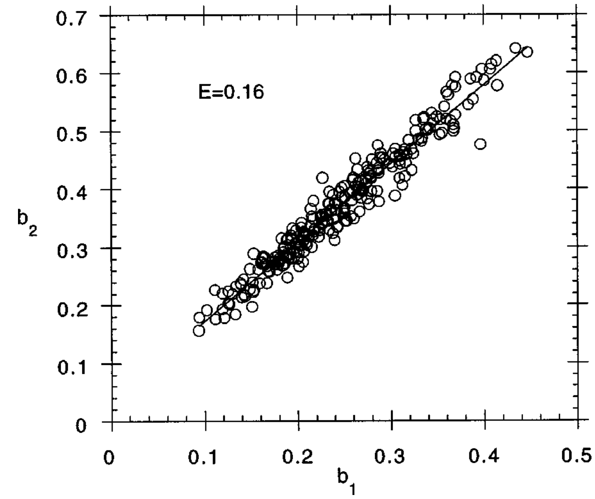

FIG. 5. Linear regression of the exponent b_2 versus b_1.

The shapes of the histograms do not change much as the energy is varied, but all of the parameters depend weakly on energy. Table I summarizes the results for two different energies. The average of b_1 over the entire chaotic zone is compared with 2λ, where λ is the classical (infinite time) Lyapunov exponent, computed by the algorithm of Wolf *et al.* [5]. The agreement is good for $E = 0.16$, but less good for the lower energy. This happens because at lower energies the chaotic zone is smaller and has some very thin parts, hence the chosen grid in the (q_2, p_2) phase plane may provide a less representative sample of the chaotic zone. The average of b_2 over the chaotic zone, denoted λ_{qc}, is distinctly larger than 2λ.

TABLE I. Lyapunov-like exponents for the Hénon-Heiles model at two energies.

	$E = 0.13$	$E = 0.16$
2λ	0.113	0.222
$(b_1)_{\rm av}$	0.145	0.246
$\lambda_{qc} = (b_2)_{\rm av}$	0.221	0.371
Regression line	$b_2 = 0.017 + 1.412 b_1$	$b_2 = 0.037 + 1.356 b_1$

The quantum-classical difference exponent λ_{qc} provides a good measure of the rate at which quantum and classical dynamics depart from each other in chaotic systems. It is being studied for other systems, and preliminary results support the generalizations made above. It will be interesting to determine whether the behavior of λ_{qc} can distinguish different kinds of chaotic systems.

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