Decoherence and entanglement in two-mode squeezed vacuum states

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I investigate the decoherence of two-mode squeezed vacuum states by analyzing the relative entropy of entanglement. I consider two sources of decoherence: (i) the phase damping and (ii) the amplitude damping due to the coupling to the thermal environment. In particular, I give the exact value of the relative entropy of entanglement for the phase damping model. For the amplitude damping model, I give an upper bound for the relative entropy of entanglement, which turns out to be a good approximation for the entanglement measure in the usual experimental situations.

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Quantum entanglement is an essential ingredient in quantum communication and computation [1]. Therefore, it is of great importance to quantify the entanglement to assess the efficacy of quantum information processing. Recently, unconditional quantum teleportation of an unknown coherent state has been realized experimentally by exploiting a two-mode squeezed vacuum state as an entanglement resource [2], shortly after the theoretical proposal of Ref. [3]. The two-mode squeezed vacuum state shared between two parties—Alice and Bob—is formally generated from the vacuum state $|\text{vac}\rangle$ by the unitary transformation $U(r) = \exp[-r(a_1^\dagger a_2^\dagger - a_1 a_2)]$, where $r(\geqslant 0)$ is called a squeezing parameter. The indices 1 and 2 refer to the optical modes of Alice and Bob, respectively. The two-mode squeezed vacuum state $|\Psi\rangle = U(r)|\text{vac}\rangle$ is written in the Fock basis as $|\Psi\rangle = (\cosh r)^{-1}\Sigma_{n=0}^{\infty}\tanh^n r|n,n\rangle$, where $|n,n\rangle \equiv |n\rangle_1 \otimes |n\rangle_2$. Since $|\Psi\rangle$ is a pure state, its amount of entanglement is uniquely quantified by the von Neumann entropy of the reduced state of Alice or Bob:

$$S(|\Psi\rangle) = \cosh^2 r \log_2(\cosh^2 r) - \sinh^2 r \log_2(\sinh^2 r). \quad (1)$$

In real experimental situations, due to coupling to the environment, the entangled state inevitably loses its purity; it becomes mixed. This phenomenon—*decoherence*—is the most dangerous obstacle for all entanglement manipulations. Several protocols for entanglement enhancement or purification in continuous variable systems have been proposed [4–8] and the decoherence of continuous variable states has been studied [9–11]. However, the quantification of entanglement of mixed entangled states in continuous variable systems is still not well understood.

In this paper, I investigate the decoherence of two-mode squeezed vacuum states by analyzing the relative entropy of entanglement [12]. I consider two sources of decoherence separately: (i) *the phase damping* and (ii) *the amplitude damping* due to the coupling to the thermal environment. The damping is assumed to affect each mode of the state independently with the same coupling parameters. In particular, I show that the exact calculation of the relative entropy of entanglement can be performed for the phase damping model. For the amplitude damping model, I give an upper bound for the relative entropy of entanglement, which turns out to be a good approximation for the entanglement measure in the usual experimental situations.

The relative entropy of entanglement of pure states reduces to the von Neumann entropy of the reduced state of either subsystem. For a mixed state ρ it is defined as $E_R(\rho) = \min_{\sigma\in\mathcal{D}} S(\rho||\sigma)$, where $S(\rho||\sigma) = \text{Tr}[\rho(\log_2\rho - \log_2\sigma)]$ is the quantum relative entropy. The minimum is taken over $\mathcal{D}$, the set of all disentangled states. It is usually difficult to calculate the relative entropy of entanglement for mixed states, except for some specific states. Recently, the following theorem on the relative entropy of entanglement has been proved [13,14] and it turns out to be quite suitable for the present analysis.

Theorem 1. For a bipartite quantum state described by a density matrix of the form

$$\rho = \sum_{n_1,n_2} a_{n_1,n_2} |\phi_{n_1}, \psi_{n_1}\rangle\langle\phi_{n_2}, \psi_{n_2}|, \quad (2)$$

the relative entropy of entanglement is given by

$$E_R(\rho) = -\sum_n a_{n,n}\log_2 a_{n,n} + \text{Tr}(\rho \log_2\rho), \quad (3)$$

and the disentangled state ρ^* that minimizes the quantum relative entropy $S(\rho||\rho^*)$ is

$$\rho^* = \sum_n a_{n,n} |\phi_n, \psi_n\rangle\langle\phi_n, \psi_n|. \quad (4)$$

Here, $|\phi_n\rangle$ and $|\psi_n\rangle$ are orthonormal states of each subsystem.

First, I consider the phase damping model. The density matrix obeys the following master equation in the interaction picture:

$$\frac{d}{dt}\rho(t) = (\mathcal{L}_1 + \mathcal{L}_2)\rho(t), \quad (5)$$

with

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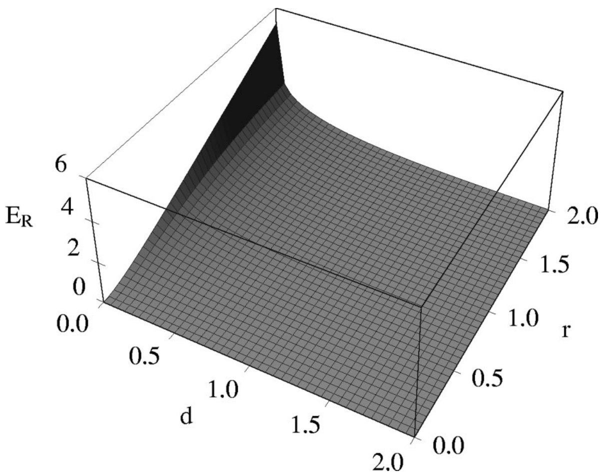

FIG. 1. The relative entropy of entanglement E_R of the state of Eq. (7) as a function of the squeezing parameter r and the degree of damping $d(=\gamma t)$. The amount of E_R is shown in units of bits.

$$\mathcal{L}_i\rho=\frac{\gamma}{2}[2a_i^\dagger a_i\rho a_i^\dagger a_i-(a_i^\dagger a_i)^2\rho-\rho(a_i^\dagger a_i)^2]. \tag{6}$$

The solution of Eq. (5) with the initial condition $\rho(t=0)=|\Psi\rangle\langle\Psi|$ is calculated as

$$\rho(t)=\frac{1}{\cosh^2 r}\sum_{n_1,n_2=0}^{\infty}(\tanh r)^{n_1+n_2}\exp(-\gamma t|n_1-n_2|^2)|n_1,n_1\rangle\langle n_2,n_2|. \tag{7}$$

It should be noted that the density matrix of Eq. (7) takes the form of Eq. (2) in Theorem 1. Consequently, it is possible to compute numerically the relative entropy of entanglement of the state of Eq. (7). Figure 1 shows the relative entropy of entanglement E_R thus computed as a function of the squeezing parameter r and the degree of damping $d\equiv\gamma t$. In numerical computations, the truncated photon number has been taken to be $\max(n_1)=\max(n_2)=100$, the value of which is sufficiently large for numerical convergence. It is seen that with an increasing amount of squeezing the amount of entanglement decreases rapidly with the damping. For large values of d, E_R is vanishingly small but remains finite; it is still larger than the conceivable numerical errors. It is not clear from the present numerical analysis whether the state is always entangled for finite γt.

Next, I consider the amplitude damping model. The master equation for the density matrix is the same as Eq. (5), but

$$\mathcal{L}_i\rho=\frac{\gamma}{2}(1+\overline{n})(2a_i\rho a_i^\dagger-a_i^\dagger a_i\rho-\rho a_i^\dagger a_i)+\frac{\gamma}{2}\overline{n}(2a_i^\dagger\rho a_i-a_i a_i^\dagger\rho-\rho a_i a_i^\dagger), \tag{8}$$

where $\overline{n}$ is the average photon number of the thermal environment. To solve the master equation, I first represent the two-mode squeezed vacuum state as a continuous superposition of two-mode coherent states [9]:

$$|\Psi\rangle=\int d^2\alpha G(\alpha,r)|\alpha,\alpha^*\rangle, \tag{9}$$

where $G(\alpha,r)=\exp[-(\coth r-1)|\alpha|^2]/(\pi\sinh r)$. According to the treatment of Ref. [15], the solution for the density matrix with the initial condition $\rho(t=0)=|\Psi\rangle\langle\Psi|$ is calculated as

$$\rho(t)=\int d^2\alpha\int d^2\beta G(\alpha,r)G(\beta,r)\sigma_1(\alpha,\beta)\sigma_2(\alpha^*,\beta^*), \tag{10}$$

where

$$\sigma_i(\alpha,\beta)=\frac{\langle\beta|\alpha\rangle}{n(t)+1}\exp(-\xi_1\xi_2^*)D_i(\xi_1)\exp(\kappa a_i^\dagger a_i)D_i^\dagger(\xi_2). \tag{11}$$

In Eq. (11), $n(t)=\overline{n}(1-e^{-\gamma t})$, $D_i(\xi_j)=\exp(\xi_j a_i^\dagger-\xi_j^* a_i)$, and $\kappa=\ln\{n(t)/[n(t)+1]\}$. Furthermore, ξ_1 and ξ_2 are given by $\xi_1=e^{-\gamma t/2}\{[n(t)+1]\alpha+n(t)\beta\}/[2n(t)+1]$ and $\xi_2=e^{-\gamma t/2}\{n(t)\alpha+[n(t)+1]\beta\}/[2n(t)+1]$. By noting the completeness of coherent states, it is straightforward to show that

$$\begin{aligned}{}_i\langle m_1|D_i(\xi_1)\exp(\kappa a_i^\dagger a_i)D_i^\dagger(\xi_2)|m_2\rangle_i&=\frac{1}{\pi^2}\sqrt{\frac{1}{m_1!m_2!}}\int d^2\gamma_1\int d^2\gamma_2\gamma_1^{m_1}(\gamma_2^*)^{m_2}\\&\times\exp\Big[-\frac{1}{2}(|\gamma_1|^2+|\gamma_2|^2+|\gamma_1-\xi_1|^2+|\gamma_2-\xi_2|^2)\\&+(\gamma_1-\xi_1)^*(\gamma_2-\xi_2)e^{\kappa}+\sqrt{-1}\mathrm{Im}(\xi_1\gamma_1^*)\\&-\sqrt{-1}\mathrm{Im}(\xi_2\gamma_2^*)\Big].\end{aligned} \tag{12}$$

Using Eq. (12), we obtain

$$\begin{aligned}\langle n_1,n_3|\rho(t)|n_2,n_4\rangle&=\frac{R}{\pi^4\sinh^4 r}\sqrt{\frac{1}{n_1!n_2!n_3!n_4!}}\\&\times\int d^2\gamma_1\int d^2\gamma_2\int d^2\gamma_3\\&\times\int d^2\gamma_4\gamma_1^{n_1}(\gamma_2^*)^{n_2}\gamma_3^{n3}(\gamma_4^*)^{n_4}\\&\times\exp[-|\gamma_1|^2-|\gamma_2|^2-|\gamma_3|^2\\&-|\gamma_4|^2]\exp[P(\gamma_1^*\gamma_3^*+\gamma_2\gamma_4)\\&+Q(\gamma_1^*\gamma_2+\gamma_3^*\gamma_4)],\end{aligned} \tag{13}$$

where

$$P=R\,e^{-\gamma t}\coth r, \tag{14}$$

$$Q=\frac{1}{n(t)+1}\{n(t)+R\,e^{-\gamma t}[n(t)+1-e^{-\gamma t}]\}, \tag{15}$$

and

$$R=\{\coth^2 r[n(t)+1]^2-[n(t)+1-e^{-\gamma t}]^2\}^{-1}. \quad (16)$$

The integral of Eq. (13) can be evaluated by expanding $\exp[P(\gamma_1^*\gamma_3^*+\gamma_2\gamma_4)+Q(\gamma_1^*\gamma_2+\gamma_3^*\gamma_4)]$ with respect to $\gamma_1^*\gamma_3^*$, $\gamma_2\gamma_4$, $\gamma_1^*\gamma_2$, and $\gamma_3^*\gamma_4$. Finally we obtain

$$\rho(t)=\sum_{n_1,n_2=0}^{\infty} c_{n_1,n_2}^{(0)}|n_1,n_1\rangle\langle n_2,n_2| +\sum_{k=1}^{\infty}\sum_{n_1,n_2=0}^{\infty} c_{n_1,n_2}^{(k)}|n_1,n_1+k\rangle\langle n_2,n_2+k| +\sum_{k=1}^{\infty}\sum_{n_1,n_2=0}^{\infty} c_{n_1,n_2}^{(k)}|n_1+k,n_1\rangle \times\langle n_2+k,n_2|, \quad (17)$$

where

$$c_{n_1,n_2}^{(k)}=\frac{\sqrt{n_1!n_2!(n_1+k)!(n_2+k)!}}{\sinh^2 r}P^{n_1+n_2}Q^kR \times\sum_{l=0}^{\min(n_1,n_2)}\frac{1}{l!(l+k)!(n_1-l)!(n_2-l)!}\left(\frac{Q}{P}\right)^{2l}. \quad (18)$$

To estimate the relative entropy of entanglement of the state $\rho(t)$ of Eq. (17), I write $\rho(t)$ as a convex combination of new density matrices $\rho_0(t)$ and $\rho_k^{(\pm)}(t)$ $(k=1,2,\dots)$:

$$\rho(t)=p_0\rho_0(t)+\sum_{k=1}^{\infty}p_k^{(+)}\rho_k^{(+)}(t)+\sum_{k=1}^{\infty}p_k^{(-)}\rho_k^{(-)}(t). \quad (19)$$

The first, second, and third terms of the right-hand side of Eq. (19) correspond to the first, second, and third terms of the right-hand side of Eq. (17), respectively. By convexity of the relative entropy of entanglement [12], we have that

$$E_R(\rho(t))\leqslant p_0E_R(\rho_0(t))+\sum_{k=1}^{\infty}p_k^{(+)}E_R(\rho_k^{(+)}(t)) +\sum_{k=1}^{\infty}p_k^{(-)}E_R(\rho_k^{(-)}(t)). \quad (20)$$

The density matrices $\rho_0(t)$ and $\rho_k^{(\pm)}(t)$ $(k=1,2,\dots)$ have the form of Eq. (2) in Theorem 1, so the right-hand side of Eq. (20) can be calculated exactly and it yields an upper bound for the relative entropy of entanglement of $\rho(t)$. At $\gamma t=0$, this upper bound E_R^* gives the exact value of the relative entropy of entanglement of Eq. (1). Figures 2 and 3 show E_R^* thus computed as a function of the squeezing parameter r and the degree of damping $d(\equiv\gamma t)$. In Fig. 2 (3), $\overline{n}$, the averaged number of photons in the thermal environment, has been taken to be 0.01 (0.1). The dotted line on the data surface in each figure indicates the separability-inseparability borderline, which is given by the necessary and sufficient separability criterion for the two-mode squeezed state in the thermal environment [16,17]; $\gamma t=\ln[1+(1-e^{-2r})/(2\overline{n})]$. Within the inside region (above the dotted line) the state is entangled, while within the outside region (below the dotted line) it is separable and the relative entropy of entanglement E_R vanishes. It is seen that the values of the upper bound E_R^* on these borderlines for $r\lesssim 1$ are already negligibly small. Although E_R^* is merely an upper bound for E_R, it may be considered as an approximation for E_R. Since $r\lesssim 1$ and $\overline{n}\ll 1$ in the usual experimental situations, this approximation is a fairly good one in the sense

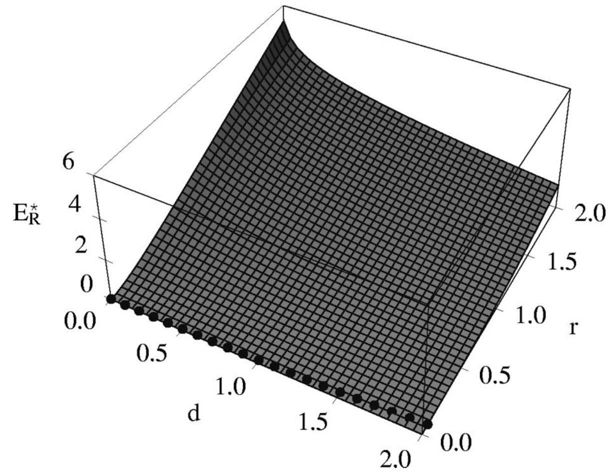

FIG. 2. The upper bound E_R^* for the relative entropy of entanglement of the state of Eq. (17) as a function of the squeezing parameter r and the degree of damping $d(=\gamma t)$. The averaged number of photons in the thermal environment $\overline{n}=0.01$. The dotted line is the separability-inseparability borderline. The amount of E_R^* is shown in units of bits.

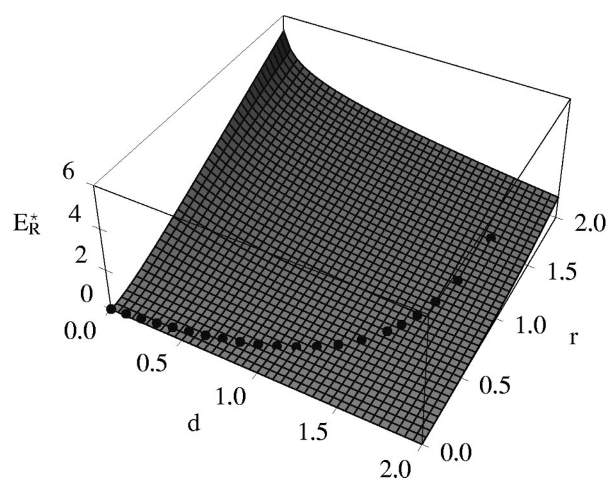

FIG. 3. The same as Fig. 2, but $\overline{n}=0.1$.

that the upper bound $E_R^* \approx 0$ for the separable region.

In summary, the decoherence of two-mode squeezed vacuum states has been investigated by analyzing the relative entropy of entanglement for (i) the phase damping model and (ii) the amplitude damping model. For the phase damping model, a method for the exact numerical computation of the relative entropy of entanglement has been established. For the amplitude damping model, a good approximation for the relative entropy of entanglement in the usual experimental situations has been introduced.

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