Correlated errors in quantum-error corrections

Won Young Hwang,* Doyeol (David) Ahn,† and Sung Woo Hwang‡
Institute of Quantum Information Processing and Systems, University of Seoul, 90 Jeonnong, Tongdaemoon, Seoul 130-743, Korea
(Received 31 January 2000; published 11 January 2001)

We show that errors are not generated correlatedly provided that quantum bits do not directly interact with (or couple to) each other. Generally, this no-qubit-interaction condition is assumed except for the case where two-qubit gate operation is being performed. In particular, the no-qubit-interaction condition is satisfied in the collective decoherence models. Thus, errors are not correlated in the collective decoherence. Consequently, we can say that current quantum error correcting codes that correct single-qubit errors will work in most cases including the collective decoherence.

DOI: 10.1103/PhysRevA.63.022303 PACS number(s): 03.67.Lx, 03.65.Ta, 89.20.Ff

Information processing with quantum bits (qubits), e.g., quantum computing and quantum cryptography is a technique that will solve some classically intractable problems [1–5]. However, in order to make quantum computing practical, quantum error correcting codes (QECCs) [6–13] are indispensable [14]. With QECC's, we can correct errors on qubits induced by interactions of qubits with the environment.

However, there exists no QECC that can correct all errors. That is, only some subsets of all possible errors can be corrected with QECC's. So, the strategy is to choose certain subclasses of errors that constitute dominant parts as to-be-corrected ones, while other classes of errors that constitute negligible parts as not-to-be-corrected ones. Generally, single-qubit errors, where only one qubit has undergone interaction with the environment or arbitrary unitary operation, are assumed to be the most common ones. More precisely, it is assumed that *the probability of* k (integer $k \geqslant 0$) *errors are of order* ϵ^k, which is much smaller than ϵ the probability of a single error if ϵ is small enough and $k \geqslant 2$ [14]. This is the independence condition. However, it should be noted that the independence condition is distinguished from the independent decoherence, where each qubits interact with their own environments, which do not interact with one another.[1] Although the independence of qubit-environment interaction ensures the independence condition, the converse is not guaranteed. The purpose of this paper is to show that even if qubits do not interact independently with environments, the generated errors satisfy the independence condition to the second order, provided that quantum bits do not directly interact with (or couple to) each other. Generally, this no-qubits-interaction condition is assumed except for the case where two-qubit gate operation is being performed. In particular, the no-qubits-interaction condition is satisfied in the collective decoherence models [15–17]. Thus, we can say that correlated errors are not generated in most cases including the collective decoherence. Therefore, current QECC's [6–9] which correct single-qubit-errors work in most cases including the collective decoherence. Recently Knill *et al.* have shown that there exist some QECC's that can correct errors due to general interaction [12]. So, there exist some QECC's, which correct errors due to collective interaction. However, their results do not directly mean that QECC's correcting single-qubit-errors work in collective decoherence.

First, let us consider complete independent decoherence, where qubits interact with their own environments, which do not interact with one another. This has been addressed and worked out thoroughly in Refs. [10] and [11]. We will consider this in Hamiltonian formulations. Let us consider the following total Hamiltonian.

$$\begin{aligned}\mathbf{H}_T = \Big[& \mathbf{H}_1 \otimes \mathbf{I}_2 \otimes \cdots \otimes \mathbf{I}_n \otimes \mathbf{I}_1^E \otimes \mathbf{I}_2^E \otimes \cdots \otimes \mathbf{I}_n^E + \mathbf{I}_1 \otimes \mathbf{I}_2 \otimes \cdots \otimes \mathbf{I}_n \otimes \mathbf{H}_1^E \otimes \mathbf{I}_2^E \otimes \cdots \otimes \mathbf{I}_n^E + \sum_j \mathbf{Q}_1^j \otimes \mathbf{I}_2 \otimes \cdots \otimes \mathbf{I}_n \otimes \mathbf{E}_1^j \otimes \mathbf{I}_2^E \otimes \cdots \\ & \otimes \mathbf{I}_n^E \Big] + \Big[\mathbf{I}_1 \otimes \mathbf{H}_2 \otimes \cdots \otimes \mathbf{I}_n \otimes \mathbf{I}_1^E \otimes \mathbf{I}_2^E \otimes \cdots \otimes \mathbf{I}_n^E + \mathbf{I}_1 \otimes \mathbf{I}_2 \otimes \cdots \otimes \mathbf{I}_n \otimes \mathbf{I}_1^E \otimes \mathbf{H}_2^E \otimes \cdots \otimes \mathbf{I}_n^E + \sum_j \mathbf{I}_1 \otimes \mathbf{Q}_2^j \otimes \cdots \otimes \mathbf{I}_n \otimes \mathbf{I}_1^E \otimes \mathbf{E}_2^j \\ & \otimes \cdots \otimes \mathbf{I}_n^E \Big] + \cdots + \Big[\mathbf{I}_1 \otimes \mathbf{I}_2 \otimes \cdots \otimes \mathbf{H}_n \otimes \mathbf{I}_1^E \otimes \mathbf{I}_2^E \otimes \cdots \otimes \mathbf{I}_n^E + \mathbf{I}_1 \otimes \mathbf{I}_2 \otimes \cdots \otimes \mathbf{I}_n \otimes \mathbf{I}_1^E \otimes \mathbf{I}_2^E \otimes \cdots \otimes \mathbf{H}_n^E + \sum_j \mathbf{I}_1 \otimes \mathbf{I}_2 \otimes \cdots \otimes \mathbf{Q}_n^j \\ & \otimes \mathbf{I}_1^E \otimes \mathbf{I}_2^E \otimes \cdots \otimes \mathbf{E}_n^j \Big]. \end{aligned} \tag{1}$$

*Email address: wyhwang@iquips.uos.ac.kr
†Also with Department of Electrical Engineering, University of Seoul, Seoul 130-743, Korea, Email address: dahn@uoscc.uos.ac.kr
‡Permanent address: Department of Electronics Engineering, Korea University, 5-1 Anam, Sungbook-ku, Seoul 136-701, Korea.

[1]Correlated decoherence should also be distinguished from collective decoherence. The former is the one that does not satisfy the independence condition while the latter is the one where qubits interact with environments collectively.

Here, $\mathbf{H}_\alpha$ and $\mathbf{H}_\alpha^E$ are the free Hamiltonian of αth qubit and αth environment, respectively, ($\alpha=1,2,\ldots,n$ and n is the number of qubits and integer $j\geqslant 1$) and $\mathbf{I}$ is the identity operator. $\mathbf{Q}_\alpha^j$ is an operator that acts on αth qubit and $\mathbf{E}_\alpha^j$ is an operator that acts on αth environment. It is clear that a set of terms in a parentheses commute with those in other parentheses in Eq. (1). Since $\exp(\Sigma_i A_i)=\Pi_i \exp(A_i)$ when $[A_i,A_j]=0$ for each i,j ($[A,B]=AB-BA$), the total unitary time evolution operator $U(t)=\exp(-i\mathbf{H}_T t)$ decomposes into n factors. Thus each qubit-environment system evolves separately by their own unitary operators, for example, the first qubit- environment system by $U_1(t)=\exp(-i[\mathbf{H}_1\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{I}_1^E\otimes\mathbf{I}_2^E\otimes\cdots\otimes\mathbf{I}_n^E+\mathbf{I}_1\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{H}_1^E\otimes\mathbf{I}_2^E\otimes\cdots\otimes\mathbf{I}_n^E+\Sigma_j\mathbf{Q}_1^j\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{E}_1^j\otimes\mathbf{I}_2^E\otimes\cdots\otimes\mathbf{I}_n^E]t)$. Each qubit-environment's evolution can be decomposed [6,14] as, for example,

$$U_1(t)|\psi\rangle_1|e\rangle_1=\sum_{k=0}^{3}(\sigma^k\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n)|\psi\rangle_1|e_k\rangle_1$$
$$\equiv\sum_{k=0}^{3}\sigma_1^k|\psi\rangle_1|e_k\rangle_1. \tag{2}$$

Here, $|\psi\rangle_\alpha$ and $|e\rangle_\alpha$ denotes αth qubits and αth environment state, respectively. $\sigma^0=\mathbf{I},\sigma^1=\sigma^x,\sigma^2=-i\sigma^y$, and $\sigma^3=\sigma^z$, $\mathbf{I}$ is the identity operator, and $\sigma^x,\sigma^y,\sigma^z$ are the Pauli operators. σ_α^k denotes σ^k acting on αth qubit leaving others intact. $|e_k\rangle$ are not normalized and not necessarily orthogonal [6,9]. However, in general the norm of the terms with $\sigma_1^1,\sigma_1^2,\sigma_1^3$ in Eq. (2) are of the first order of time t while that with σ_1^0 is of the zeroth order. This property is required to ensure the quantum Zeno effect [18–20]. Therefore,

$$\sum_{k=0}^{3}\sigma_1^k|\psi\rangle_1|e\rangle_1=c_1^0t^0\sigma_1^0|\psi\rangle_1|\overline{e_0}\rangle_1+c_1^kt\sum_{k=1}^{3}\sigma_1^k|\psi\rangle_1|\overline{e_k}\rangle_1, \tag{3}$$

where $|\overline{e_k}\rangle$ is the normalized state of $|e_k\rangle$ and c_1^k's are some constants. The same relation is satisfied for other α's. As noted above, the total qubits-environments system can be expressed as direct products of each qubit-environment system, each of which satisfy an equation similar to Eq. (3). Then, we can see by inspection that *terms with k errors are of order* t^k in general (Note that the total state is in a form similar to $[1+t]^n$). So we can say that the independence of qubits-environments interactions ensure the independence condition.

Next, let us consider incomplete independent decoherence, where qubits interact with different environments that are still interacting with one another. In this case total states do not decompose into factors in general and thus the above method cannot be used to derive the independence condition. On the other hand, one may guess that collective decoherence generates correlated errors. However, there is no reason why the collective interaction of qubits with the environment necessarily induces correlated errors. However, in both models, qubits do not couple to each other or they satisfy the no-qubits-interaction condition. Then correlated errors are not generated, as we show in the following. Therefore, we can say that both incomplete-independent and collective decoherence do not generate correlated errors. Now, we state the no-qubits-interaction condition more precisely: in each term of the qubit-environment interaction Hamiltonian $\mathbf{H}_I$, only one qubit-operator is a nonidentity. That is,

$$\mathbf{H}_I=\sum_j \mathbf{Q}_1^j\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{E}_1^j+\sum_j \mathbf{I}_1\otimes\mathbf{Q}_2^j\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{E}_2^j$$
$$+\cdots+\sum_j \mathbf{I}_1\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{Q}_n^j\otimes\mathbf{E}_n^j. \tag{4}$$

The total Hamiltonian is the following:

$$\mathbf{H}_T=\mathbf{H}_1\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{I}_E+\mathbf{I}_1\otimes\mathbf{H}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{I}_E+\cdots$$
$$+\mathbf{I}_1\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{H}_n\otimes\mathbf{I}_E+\mathbf{I}_1\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{H}_E+\mathbf{H}_I$$
$$\equiv\mathbf{H}_0+\mathbf{H}_I. \tag{5}$$

Here we adopt the interaction picture [21], where $|\psi\rangle_I$ (the state vector in the interaction picture) $=\exp(it\mathbf{H}_0)|\psi\rangle_S$ (the state vector in Schrodinger picture). The time evolution of $|\psi\rangle_I$ is determined by the Schrodinger-like equation

$$i\frac{\partial|\psi\rangle_I}{\partial t}=V(t)|\psi\rangle_I, \tag{6}$$

where

$$V(t)\equiv\exp(it\mathbf{H}_0)\mathbf{H}_I\exp(-it\mathbf{H}_0). \tag{7}$$

Since $V(t)$ is time dependent, the time evolution operator $U_I(t)$ for $|\psi\rangle_I$ is given by the Dyson series [21].

$$U_I(t)=1+\sum_{m=1}^{\infty}(-i)^m\int_0^t dt_1\int_0^{t_1}dt_2\cdots$$
$$\times\int_0^{t_{m-1}}dt_mV(t_1)V(t_2)\cdots V(t_m). \tag{8}$$

From Eqs. (4) and (7),

$$V(t)=\exp(it\mathbf{H}_0)\left[\sum_j \mathbf{Q}_1^j\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{E}_1^j\right]\exp(-it\mathbf{H}_0)$$
$$+\exp(it\mathbf{H}_0)\left[\sum_j \mathbf{I}_1\otimes\mathbf{Q}_2^j\otimes\cdots\otimes\mathbf{I}_n\otimes\mathbf{E}_2^j\right]\exp(-it\mathbf{H}_0)$$
$$+\cdots+\exp(it\mathbf{H}_0)\left[\sum_j \mathbf{I}_1\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{Q}_n^j\otimes\mathbf{E}_n^j\right]$$
$$\times\exp(-it\mathbf{H}_0)$$
$$\equiv V_1(t)+V_2(t)+\cdots+V_n(t). \tag{9}$$

We consider the relation

$$U_I(t)=U_I^1(t)U_I^2(t)\ldots U_I^n(t)+O(t^2), \tag{10}$$

where

$$U_I^{\alpha}(t)=1+\Sigma_{m=1}^{\infty}(-i)^m\int_0^t dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m \times V_\alpha(t_1)V_\alpha(t_2)\ldots V_\alpha(t_m)$$

and $O[f(x)]$ means asymptotically less than a constant operator times $f(x)$. However, since $|\psi\rangle_S=\exp(-it\mathbf{H}_0)|\psi\rangle_I$ and the operator $\exp(-it\mathbf{H}_0)$ do not entangle qubits with environments, it is sufficient for us to consider only $U_I(t)$. We can see that each $U_I^{\alpha}(t)$ makes the αth qubit to entangle with environment. For example,

$$U_I^1(t)|\psi\rangle_I|e\rangle_I=\sum_{k=0}^{3}(\sigma^k\otimes\mathbf{I}_2\otimes\cdots\otimes\mathbf{I}_n)|\psi\rangle_I|e_k\rangle_I \equiv\sum_{k=0}^{3}\sigma_1^k|\psi\rangle_I|e_k\rangle_I. \quad (11)$$

Here, $|\psi\rangle_I$ and $|e\rangle_I$ denotes qubits and the environment state in the interaction picture, respectively, and $|e_k\rangle_I$ are not normalized and not necessarily orthogonal. By operating all factors in $U_I(t)$ sequentially, we obtain

$$U_I(t)|\psi\rangle_I|e\rangle_I=\sum_{\{k\}}\sigma_1^{k_1}\sigma_2^{k_2}\cdots\sigma_n^{k_n}|\psi\rangle_I|e_{\{k\}}\rangle_I +O(t^2)|\psi\rangle_I|e\rangle_I, \quad (12)$$

where $\{k\}$ is an abbreviation for $k_1,k_2,\ldots,k_n$, and $k_\alpha=0,1,2,3$. Let us consider Eq. (11). As above, the norm of the terms with $\sigma_1^1,\sigma_1^2,\sigma_1^3$ of Eq. (11) are of the first order of time t while the norm of the term with σ_1^0 is of the zeroth order of time t. Therefore,

$$\sum_{k=0}^{3}\sigma_1^k|\psi\rangle_I|e\rangle_I=c_1^0t^0\sigma_1^0|\psi\rangle_I|\overline{e_0}\rangle_I+c_1^kt\sum_{k=1}^{3}\sigma_1^k|\psi\rangle_I|\overline{e_k}\rangle_I +O(t^2)|\psi\rangle_I|e\rangle_I, \quad (13)$$

where $|\overline{e_k}\rangle_I$ is the normalized state of $|e_k\rangle_I$. The same relation is satisfied for other α's. Then,

$$\sum_{\{k\}}\sigma_1^{k_1}\sigma_2^{k_2}\ldots\sigma_n^{k_n}|\psi\rangle_I|e_{\{k\}}\rangle_I+O(t^2)|\psi\rangle_I|e\rangle_I =\sum_{\{k\}}c_{\{k\}}t^{N(\{k\})}\sigma_1^{k_1}\sigma_2^{k_2}\ldots\sigma_n^{k_n}|\psi\rangle_I|\overline{e_{\{k\}}}\rangle_I +O(t^2)|\psi\rangle_I|e\rangle_I, \quad (14)$$

where $N(\{k\})$ is the number of instances when $k_\alpha\neq 0$. Now, we can see that *all terms with more than 1 error [or* $N(\{k\})\geqslant 2$] *are of order* t^2. Thus the independence condition is satisfied to the second order (we can obtain the full independence condition in the case where the $O(t^2)|\psi\rangle_I|e\rangle_I$ term is negligible). So, we can say that any qubit-environment system that satisfies the no-qubits-interaction condition [Eq. (4)] obeys the independence condition to the second order so that the QECC's correcting single-qubit-errors works successfully.

To summarize, we have shown that errors are not generated correlatedly, provided that quantum bits do not directly interact with each other, or that in each term of the qubit-environment interaction Hamiltonian $\mathbf{H}_I$ only one qubit operator is a nonidentity operator [Eq. (4)]. Generally, this no-qubits-interaction condition is assumed except for the case where two-qubit gate operation is being performed. In particular, the no-qubits-interaction condition is satisfied in the collective decoherence models [15–17]. So, current QECC's [6–9], which correct single-qubit-errors, work in most cases including the collective decoherence.

This work was supported by the Korean Ministry of Science and Technology through the Creative Research Initiatives Program under Contact No. 98-CR-01-01-A-20.

[1] R.P. Feynman, Int. J. Theor. Phys. **21**, 467 (1982).
[2] P. Benioff, Phys. Rev. Lett. **48**, 1581 (1982).
[3] D. Deutsch, Proc. R. Soc. London, Ser. A **400**, 97 (1985).
[4] S. Wiesner, SIGACT News **15**, 78 (1983).
[5] C.H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore* (IEEE, New York, 1984), p. 175.
[6] P. Shor, Phys. Rev. A **52**, R2493 (1995).
[7] A.R. Calderbank and P.W. Shor, Phys. Rev. A **54**, 1098 (1996).
[8] A.M. Steane, Phys. Rev. Lett. **77**, 793 (1996).
[9] R. Laflamme, C. Miquel, J.P. Paz, and W.H. Zurek, Phys. Rev. Lett. **77**, 198 (1996).
[10] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
[11] E. Knill and R. Laflamme, e-print quant-ph/9608012 (unpublished) (available at http://xxx.lanl.gov).
[12] E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. **84**, 2525 (2000).
[13] C.H. Bennett, D.P. Divincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[14] J. Preskill, e-print quant-ph/9705031 (unpublished) (available at http://xxx.lanl.gov).
[15] L.M. Duan and G.C. Guo, Phys. Rev. Lett. **79**, 1953 (1997).
[16] P. Zanardi and M. Rassetti, Phys. Rev. Lett. **79**, 3306 (1997).
[17] D.A. Lidar, D. Bacon, and K.B. Whaley, Phys. Rev. Lett. **82**, 4556 (1999).
[18] B. Misra and E.C.G. Sudersan, J. Math. Phys. **18**, 756 (1977).
[19] L. Vaidman, L. Goldenberg, and S. Wiesner, Phys. Rev. A **54**, R1745 (1996).
[20] L.M. Duan and G.C. Guo, Phys. Rev. A **57**, 2399 (1998).
[21] J.J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, MA, 1985).