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Mixing group for relativistic two-particle quantum states

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Within the framework of relativistic Schrödinger theory (RST), the problem of bound two-particle states is studied and compared to the analogous results of conventional quantum theory. The standard dichotomy of symmetric and antisymmetric quantum states finds its RST analog in the form of the positive and negative mixtures. Similarly the conventional exchange degeneracy has its RST counterpart in the form of a certain mixture degeneracy which, however, is not broken by the interparticle interactions as in the standard quantum theory. The corresponding group of quasilinear mixing transformations turns out to be continuous [in contrast to the discrete operations of (anti)symmetrization in the conventional theory] and is closely related to the Lorentz group SO(1,1) in two dimensions. The group properties can be exploited to generate exact solutions for the mixture dynamics from the pure-state configurations where, however, certain physical quantities remain invariant: scalar densities, currents, energy eigenvalues.

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I. INTRODUCTION AND SURVEY OF RESULTS

Though nearly a century has passed since the invention of quantum theory, there still are arising certain controversies about the right interpretation of this theory. However, despite the persistence of some controversial viewpoints referring mainly to the right understanding of the decoherence phenomenon [1,2], there is general agreement about certain features of the quantum world which are believed to be correctly described by the conventional quantum theory. One of these points of general agreement refers to the well-known spin-statistics theorem [3]. This theorem says (in popular language) that particles with half-integer spin are to be described by antisymmetric wave functions and therefore obey the Fermi-Dirac statistics (fermions) whereas particles with integer spin occur in form of symmetric wave functions and obey the Bose-Einstein statistics (bosons). Obviously the crucial point with the spin-statistics theorem lies in that specific way in which the particle permutation symmetry of the wave functions (or the commutation relations of the field operators) is combined with the particle spin. This logic link is nontrivial only if the notions of symmetry of the wave functions and of particle spin are first introduced independently. However, this independent occurrence of both concepts actually did occur during the historical course of the development of quantum theory: namely, Pauli first had to introduce the concept of spin in order to be able to explain the fine structure of the atomic spectral lines [4] and then Heisenberg [5] and Dirac [6] independently established the doubling of the atomic energy level schemes (*ortho*- and *para*-helium) as a consequence of the nonobservability of a particle permutation operation upon the wave functions which immediately led to their symmetry and antisymmetry property. The subsequent unification of both aspects into the spin-statistics theorem has been the concern of many brilliant theoreticians [3] but even today the status of this theorem seems not completely clear; the general belief seems to be that it is not the true origin of the Pauli exclusion principle but rather some kind of consistency requirement. (For a survey of the research for violations of the Pauli principle see Ref. [7].)

In any case, the two prerequisites of the spin-statistics theorem [namely, (i) the occurrence of many-particle systems in either symmetric or antisymmetric states and (ii) integrity or half-integrity of the particle spin] stand on a safe ground. Thus, any conceivable form of quantum theory surely must take account of these two facts which are experimentally well established. The present paper also deals with one of these two fundamental prerequisites, namely, the matter dichotomy in form of the symmetric and antisymmetric states, but not within the framework of the conventional quantum theory but rather within the relativistic Schrödinger theory (RST) recently established [8–10] (for a RST treatment of Dirac particles see, e.g., Refs. [11,12]). Especially we shall be concerned here with the phenomenon of mixture degeneracy as the RST counterpart of the well-known exchange degeneracy in the conventional theory, see, e.g., Ref. [13].

The essential difference between the conventional theory and RST consists in the specific kinematical setting for the treatment of many-particle systems (in the present paper we restrict ourselves to two-particle systems [14,15] but the generalization to many particles is self-evident). Whereas in the conventional theory (as a probabilistic theory) the many-particle Hilbert space is taken as the tensor product of the single-particle spaces, in RST (as a fluid-dynamic theory) the many-particle fibre bundle of possible wave functions is the Whitney sum of the single-particle bundles over pseudo-Riemannian space-time. For instance, the RST counterpart of a simple product state $\Psi(1,2)$ for independent particles with individual wave functions $\psi_{\mathrm{I}}, \psi_{\mathrm{II}}$ in the conventional theory

$$\Psi(1,2)=\psi_{\mathrm{I}}(1)\,\psi_{\mathrm{II}}(2) \tag{1.1}$$

is now a $\mathbb{C}^2$-valued bundle section Ψ

$$\Psi(x)=\begin{pmatrix}\psi_{\mathrm{I}}(x)\\ \psi_{\mathrm{II}}(x)\end{pmatrix}. \tag{1.2}$$

Furthermore, the RST counterpart of the conventional Schrödinger equation

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$$i\hbar\dot{\Psi}(1,2)=\hat{H}(1,2)\Psi(1,2) \tag{1.3}$$

is now the relativistic Schrödinger equation (RSE)

$$i\hbar c\mathcal{D}_\mu\Psi=\mathcal{H}_\mu\Psi \tag{1.4}$$

where the relativistic Hamiltonian $\mathcal{H}_\mu$ is a gl(2,$\mathbb{C}$)-valued one-form to be determined by its field equations [see Eqs. (2.2) and (2.3) below].

A. Exchange degeneracy

The solutions $\Psi(1,2)$ of the conventional Schrödinger equation (1.3) will in general not be of that simple product form (1.1), not even when the two-particle Hamiltonian $\hat{H}(1,2)$ is the sum of the individual one-particle Hamiltonians, i.e.,

$$\hat{H}(1,2)\Rightarrow\hat{H}(1)+\hat{H}(2). \tag{1.5}$$

It is true, for this situation the simple product state $\Psi(1,2)$ (1.1) could represent a solution of the conventional energy eigenvalue problem

$$\hat{H}(1,2)\Psi(1,2)=E_{1,2}\Psi(1,2) \tag{1.6}$$

in the form

$$\hat{H}(1)\psi_{\mathrm{I}}(1)=E_1\psi_{\mathrm{I}}(1), \tag{1.7a}$$

$$\hat{H}(2)\psi_{\mathrm{II}}(2)=E_2\psi_{\mathrm{II}}(2) \tag{1.7b}$$

with the total energy $E_{1,2}(=E_1+E_2)$ being the sum of both individual eigenvalues E_1 and E_2; but the entangled states $\Psi_\pm$, being defined through

$$\Psi_\pm \doteqdot \frac{1}{\sqrt{2}}[\psi_{\mathrm{I}}(1)\psi_{\mathrm{II}}(2)\pm\psi_{\mathrm{II}}(1)\psi_{\mathrm{I}}(2)], \tag{1.8}$$

would equally well solve the eigenvalue problem (1.6),(1.7) with the same energy eigenvalue $E_{1,2}$ (exchange degeneracy). And additionally the latter states (1.8) would obey the requirement of permutation symmetry $[\Psi(1,2)=\pm\Psi(2,1)]$ for indistinguishable particles which is mandatory for physical reasons.

The present exchange degeneracy will be broken by some interparticle interaction $\hat{V}(1,2)[=\hat{V}(2,1)]$ which is not of the additive form (1.5) e.g., the Coulomb interaction

$$\hat{V}(1,2)\Rightarrow\frac{e^2}{|\vec{r}_1-\vec{r}_2|}. \tag{1.9}$$

For such a situation, the solutions $\Psi(1,2)$ of the conventional eigenvalue problem (1.6) will still obey the symmetry requirement with respect to particle permutation albeit the eigenfunctions will not be found to be of that simple entanglement form (1.8). Nevertheless the latter form (1.8) may be used for obtaining lowest-order perturbative results for the splitting of the degenerate levels due to the interparticle interactions $\hat{V}(1,2)$

$$\begin{aligned}\Delta E_\pm&=\langle\Psi_\pm|\hat{V}|\Psi_\pm\rangle\\&=\langle\psi_{\mathrm{I}}(1)\psi_{\mathrm{II}}(2)|\hat{V}(1,2)|\psi_{\mathrm{I}}(1)\psi_{\mathrm{II}}(2)\rangle\\&\quad\pm\langle\psi_{\mathrm{I}}(1)\psi_{\mathrm{II}}(2)|\hat{V}(1,2)|\Psi_{\mathrm{II}}(1)\psi_{\mathrm{I}}(2)\rangle\\&\doteqdot E_C\pm E_G\end{aligned} \tag{1.10}$$

Here, the first part E_C is an energy shift common for both kind of states, e.g., for the electrostatic Coulomb interactions (1.9)

$$E_C=e^2\int\int d^3\vec{r}_1d^3\vec{r}_2\frac{|\psi_{\mathrm{I}}(\vec{r}_1)|^2|\psi_{\mathrm{II}}(\vec{r}_2)|^2}{|\vec{r}_1-\vec{r}_2|}, \tag{1.11}$$

and thus does not yet break the degeneracy. This term could have been obtained also by use of the simple product form (1.1). The breaking of degeneracy occurs via the second term E_G in Eq. (1.10), e.g., for the Coulomb potential (1.9)

$$E_G=e^2\int\int d^3\vec{r}_1d^3\vec{r}_2\frac{\psi^*_{\mathrm{I}}(\vec{r}_1)\psi_{\mathrm{II}}(\vec{r}_1)\psi^*_{\mathrm{II}}(\vec{r}_2)\psi_{\mathrm{I}}(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|}, \tag{1.12}$$

and this results in the well-known energy shift of the *ortho*-states relative to the *para*-states (e.g., in the helium atom [13]).

B. Relativistic Schrödinger theory

How does this conventional picture of exchange degeneracy reappear within the quite different framework of RST? Remember here that the conventional quantum theory of *relativistic* two-particle systems is plagued with many serious difficulties (see, e.g., Ref. [16–18]) which spoil the immediate transcription of those nonrelativistic results to the relativistic domain. Therefore it becomes highly instructive to see how the alternative approach of RST deals with these notorious two-body problems.

First, one has to find the RST counterpart of the conventional (anti)symmetrization process (1.8). This is done by taking the step from the pure RST states $\Psi(x)$ (1.2) to the mixtures which are to be described by the intensity matrix $\mathcal{I}$ (a Hermitian 2×2 matrix, $\mathcal{I}=\overline{\mathcal{I}}$). A pure RST state Ψ (1.2) may then be understood as a degenerate form of $\mathcal{I}$, namely, the tensor product of Ψ (i.e., $\mathcal{I}\Rightarrow\Psi\otimes\overline{\Psi}$). Correspondingly, the RSE (1.4) for the pure two-particle state $\Psi(x)$ (1.2) has to be generalized to the relativistic von Neumann equation (RNE) for the intensity matrix $\mathcal{I}$, see Eq. (2.4) below. The latter equation is considered as the RST counterpart of the conventional 2-particle Schrödinger equation (1.3). The entangling effect of the conventional Hamiltonian $\hat{H}(1,2)$, forbidding association of an individual one-particle wave function to anyone of the two particles, is transcribed to RST via the action of the nondiagonal elements (exchange fields) of the relativistic Hamiltonian $\mathcal{H}_\mu$, see the Hamiltonian decomposition (2.25),(2.26) below. Thus the use of the simply entangled states (1.8) of the conventional theory [in place of the product state (1.1)] would correspond to using an RST mixture $\mathcal{I}$ in place of a pure RST state Ψ (1.2), together with the assumption of vanishing exchange fields. The general form of an RST mixture is specified by Eq. (2.21) below.

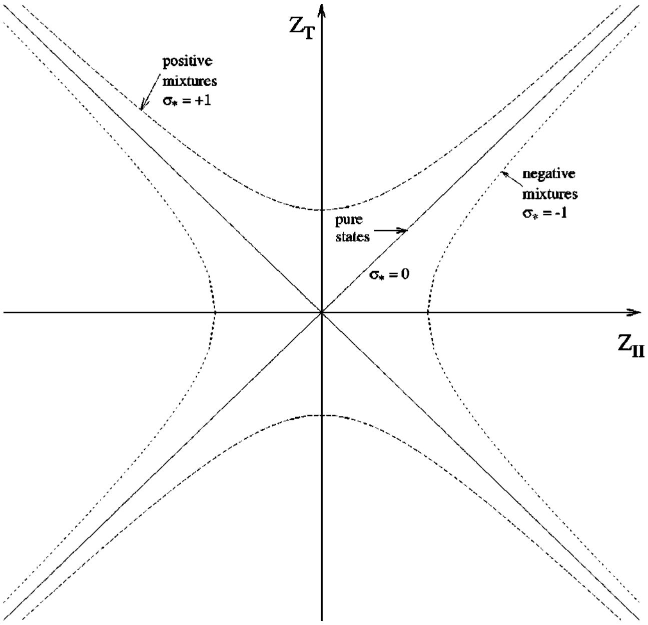

FIG. 1. Mixtures and pure states. The relativistic von Neumann equation (2.4) subdivides the density configuration space into three subsets: the pure states occupy the Fierz cone $(\sigma_* = 0)$, positive mixtures $(\sigma_* = 1)$ are geometrically represented by the two-parted hyperboloid and the negative mixtures $(\sigma_* = -1)$ by the one-parted hyperboloid. The mixtures approach the pure states for $\zeta \to \infty$. The general RST dynamics forbids a change of the mixture type, see Eq. (2.33). The positive (negative) mixtures may be considered as the RST counterparts of the symmetric and antisymmetric states of the conventional quantum theory.

The next step must now refer to the emergence of a certain matter dichotomy which arises in the conventional theory as a consequence of the Dirac-Heisenberg (anti) symmetrization postulate [5,6]. Such a dichotomy exists in RST in a very natural way (and thus is of far less postulative character as in the conventional theory), namely, the RNE (2.4) as the basic matter equation divides the density configuration space *automatically* into two subsets: positive and negative mixtures (Fig. 1), with the pure states as the separating limit for both mixture configurations. Just as the symmetric and antisymmetric states $\Psi_\pm$ (1.8) of the conventional theory induce a certain tendency of the particles to come close together $[\Psi_+(x,x) \neq 0]$ or to recede from one another $[\Psi_-(x,x) = 0]$, the RST amplitude fields of positive and negative mixtures exhibit the effects of fusion and separation (Fig. 3).

C. Mixture degeneracy

However, a difference of both theories exists with respect to the kinematical generation of the matter dichotomy: whereas in the conventional theory the (anti)symmetrization procedure [leading from the simple product $\Psi(1,2)$ (1.1) to $\Psi_\pm(1,2)$ (1.8)] is a discrete operation, the corresponding transition from the RST pure states (1.2) to the RST mixtures is a continuous set of operations which additionally owns the properties of constituting a group (mixing group). More concretely, for a general two-particle mixture there exist two amplitude fields L_a $(a=1,2)$; but it is *only* for the degenerate case $(\det \mathcal{I} = 0)$ that they can be used to construct a two-particle RST wave function Ψ (1.2) as the unification of two one-particle wave functions ψ_a $(a=1,2)$, see Eq. (2.53) below. However, for a true RST mixture $(\det \mathcal{I} \neq 0)$ the two amplitude fields L_a do not integrate to two one-particle wave functions ψ_a; this is the RST analog of the fact that a conventional two-particle state $\Psi_{1,2}$ (1.3) cannot in general be specified in terms of two one-particle wave functions ψ_a. Nevertheless the two amplitude fields L_a provide us with a common parametrization of both mixtures and pure states which thus can be continuously connected by a mixing transformation of the amplitude pair (L_1, L_2), see Eq. (3.23) below.

Clearly, the properties of the mixing group are crucial for both the effects of fusion and separation and also for the existence of the mixture degeneracy. In contrast to the conventional theory, where the tendency of fermions to recede from one another and the tendency of bosons to conglomerate (Bose-Einstein condensation) cannot be explained in terms of real, existing forces, the separating and fusing RST forces can be clearly identified in form of the mixture potentials (Figs. 4, 5). In this context, the interesting point with the mixing group is its close relationship with the group of Lorentz transformations SO(1,1) of a $(1+1)$-dimensional space-time, see Eq. (3.26) below. Indeed, both groups consist of quasilinear transformations, share the same orbits, and leave the light-cones invariant (Fig. 2). However, the decisive difference lies in the specific way in which the points within the orbits are transformed: whereas the Lorentz transformations are of translational character, with no nontrivial fix points, the mixing transformations are of either expansive type [Fig. 2(a)] or of contractive type [Fig. 2(b)] which explains the fusion and separation effects (Fig. 3) brought about in the conventional theory by the symmetrization and antisymmetrization processes (1.8).

Perhaps the most interesting property of the mixing transformations refers to the fact that they do preserve the physical quantities carried by the field configurations such as the scalar densities ρ_a, currents $j_{a\mu}$, and mass eigenvalues M_a. Thus the positive and negative mixtures have the same energy eigenvalues as the pure RST states though the corresponding amplitude fields look very different in shape (Fig. 3) (mixture degeneracy). This result is the RST analog of the conventional exchange degeneracy. The existence of such an exact degeneracy in RST could be guessed on the basis of some earlier results:

(i) The first-order perturbation results for the energy eigenvalues suggested the existence of a mixture degeneracy in the vicinity of the pure states (Fig. 3) of Ref. [14]. (ii) A rigorous perturbative computation (based on the Lewis-Dalgarno method) actually yielded an asymptotic mixture degeneracy in the vicinity of the pure states [15]; the present mixing transformations convert now these asymptotic results into exact ones.

The final point to be left for future clarification refers to the breaking of the mixture degeneracy. Contrary to the conventional exchange degeneracy, which is broken by the in-

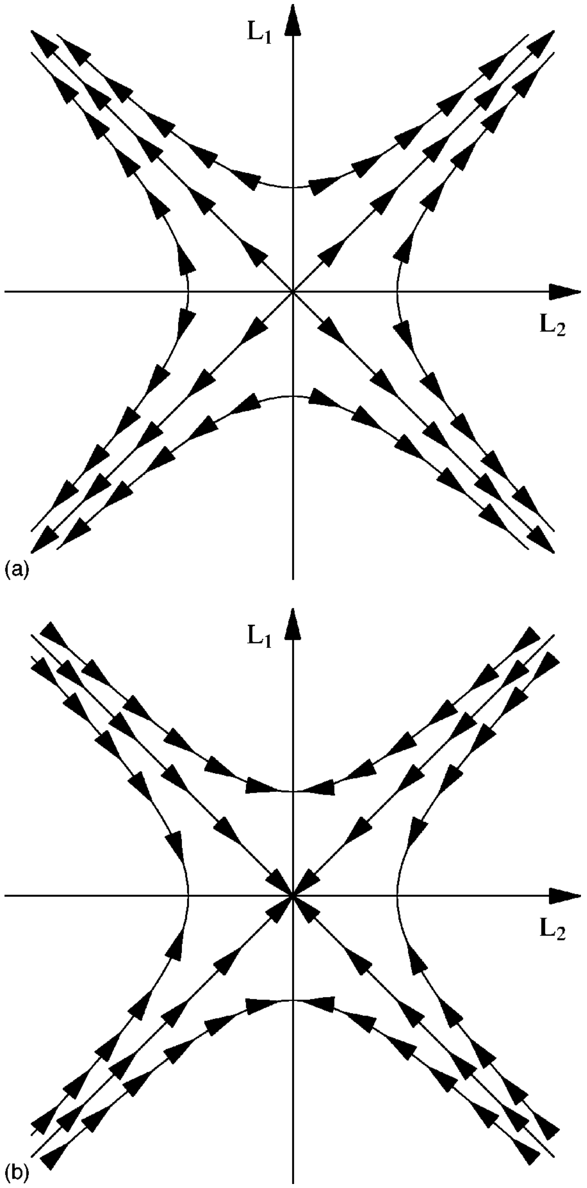

FIG. 2. Expansive and contractive mixing transformation. Fusion and separation of amplitudes. The group of mixing transformations $[\Lambda]$ (3.22) divides into the expansive $[\lambda_* > 1$, (a)] and contractive $[\lambda_* < 1$, (b)] semigroups. The invariant set of points is constituted by the light cones $(L_1^2 = L_2^2)$ and by the axes $(L_1 \equiv 0$ or $L_2 \equiv 0)$. Both invariant sets are approached by the transformed points: the light cones for an expansive transformation [leading to amplitude fusion, see Fig. 3(b) below] and the axes for a contractive transformation [leading to amplitude separation, see Fig. 3(c) below]. But the orbits as a whole are the same for $[\Lambda]$ (3.23) as for the Lorentz group SO(1,1) (3.26).

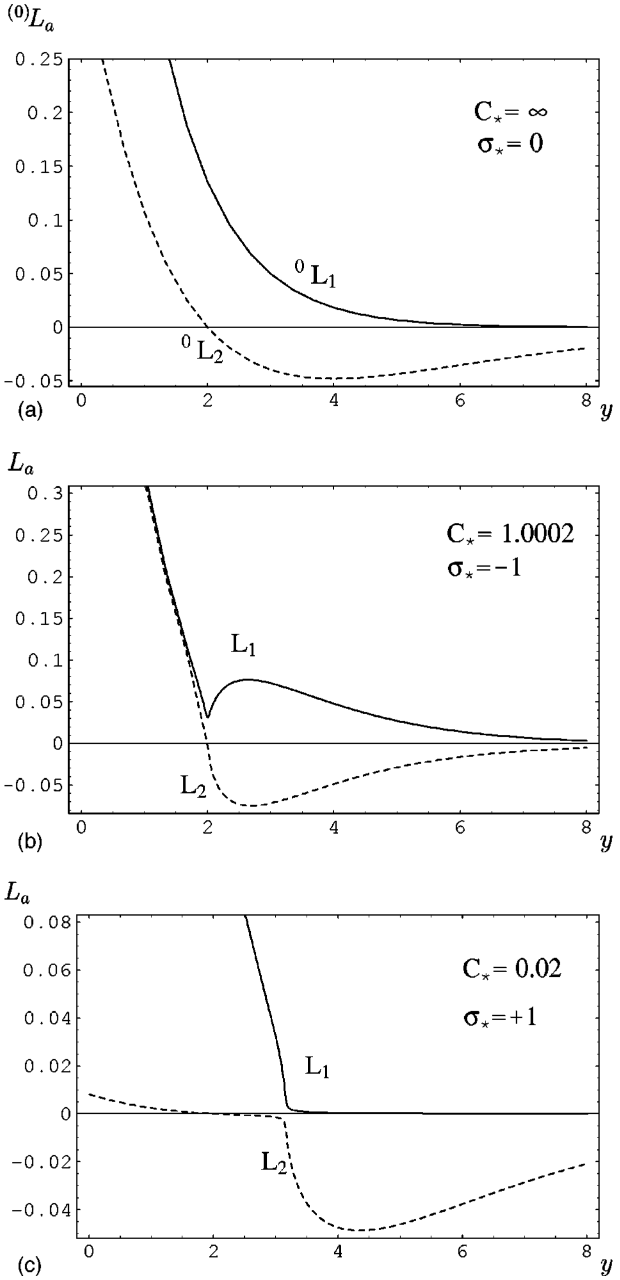

FIG. 3. Expansive and contractive mixing transformation. Fusion and separation of amplitudes. The mixing transformations $[\Lambda]$ (3.22) deform the original (dimensionless) amplitudes $^{(0)}L_a$ (3.33) (a), in such a way that for the expansive case $(\lambda_* > 1)$ the amplitudes are adopting the same shape (up to sign): fusion $(L_1 = L_2)$ or antifusion $(L_1 = -L_2)$, (b); whereas the contractive case $(\lambda_* < 1)$ lets the amplitudes recede from one another: separation (c).

terparticle interactions [see Eq. (1.10)], the present mixture degeneracy is *not* broken by this type of interactions. However, such a breaking is expected when the exchange fields (neglected in the present paper) are fully taken into account.

II. GENERAL TWO-PARTICLE THEORY

In order to make the subsequent elaborations sufficiently self-contained, it seems adequate to first display the most important features of RST in a concise form. For the sake of generality, we want to demonstrate the theory's capability of including also the gravitational interactions. Thus all the interaction forces are incorporated into the theory via the generalized equivalence principle (i.e., minimal coupling). This means that we work over a pseudo-Riemannian space-time with the coordinate-covariant derivative ∇ referring to the Levi-Civita connection Γ of the Riemannian metric g in the tangent bundle; the gauge-plus coordinate-covariant derivative is denoted by $\mathcal{D}$, and ∂ is as usual the ordinary derivative to be applied to gauge-plus coordinate-invariant objects.

A. Matter dynamics

The basic RST objects are the Hamiltonian $\mathcal{H}_\mu$, the intensity matrix $\mathcal{I}$ which is taken to be Hermitian ($\mathcal{I}=\bar{\mathcal{I}}$), furthermore the gauge potential $\mathcal{A}_\mu(=-\bar{\mathcal{A}}_\mu)$ and its field strength $\mathcal{F}_{\mu\nu}(=-\bar{\mathcal{F}}_{\mu\nu})$:

$$\mathcal{F}_{\mu\nu}=\nabla_\mu\mathcal{A}_\nu-\nabla_\nu\mathcal{A}_\mu+[\mathcal{A}_\mu,\mathcal{A}_\nu]. \tag{2.1}$$

These objects are bound together by the RST dynamics, namely, the integrability condition for the Hamiltonian $\mathcal{H}_\mu$

$$\begin{gathered}\mathcal{D}_\mu\mathcal{H}_\nu-\mathcal{D}_\nu\mathcal{H}_\mu+\frac{i}{\hbar c}[\mathcal{H}_\mu,\mathcal{H}_\nu]=i\hbar c\mathcal{F}_{\mu\nu},\\(\mathcal{D}_\mu\mathcal{H}_\nu\doteqdot\nabla_\mu\mathcal{H}_\nu+[\mathcal{A}_\mu,\mathcal{H}_\nu]),\end{gathered} \tag{2.2}$$

the conservation equation

$$\mathcal{D}^\mu\mathcal{H}_\mu-\frac{i}{\hbar c}\mathcal{H}^\mu\mathcal{H}_\mu=-i\hbar c\left(\frac{\mathcal{M}c}{\hbar}\right)^2, \tag{2.3}$$

and finally the relativistic von Neumann equation (RNE) for the intensity matrix $\mathcal{I}$

$$\mathcal{D}_\mu\mathcal{I}=\frac{i}{\hbar c}[\mathcal{I}\bar{\mathcal{H}}_\mu-\mathcal{H}_\mu\mathcal{I}]. \tag{2.4}$$

Here the mass operator $\mathcal{M}$ is taken to be covariantly constant

$$\mathcal{D}_\mu\mathcal{M}\equiv 0, \tag{2.5}$$

especially for the present situation of identical particles we simply take $\mathcal{M}$ to be of the self-suggesting form

$$\mathcal{M}=M\cdot\mathbf{1}, \tag{2.6}$$

where M is the invariant rest mass of the particles. For special matter distributions (the pure states), the intensity matrix $\mathcal{I}$ obeys the Fierz identity [8]

$$\mathcal{I}^2-\mathcal{I}\,\mathrm{tr}\,\mathcal{I}=0 \tag{2.7}$$

and thus can be written as the tensor product of some wave function Ψ

$$\mathcal{I}\Rightarrow\Psi\otimes\bar{\Psi}. \tag{2.8}$$

The wave function Ψ then obeys the relativistic Schrödinger equation (RSE)

$$\begin{gathered}i\hbar c\mathcal{D}_\mu\Psi=\mathcal{H}_\mu\Psi,\\(\mathcal{D}_\mu\Psi\doteqdot\partial_\mu\Psi+\mathcal{A}_\mu\Psi)\end{gathered} \tag{2.9}$$

and also the Klein-Gordon equation (KGE)

$$\mathcal{D}^\mu\mathcal{D}_\mu\Psi+\left(\frac{Mc}{\hbar}\right)^2\Psi=0 \tag{2.10}$$

which can easily be deduced from the RSE (2.9) by means of the conservation equation (2.3). Thus the Hamiltonian $\mathcal{H}_\mu$ acquires the status of a hidden variable from the point of view of the Klein-Gordon theory. However if the intensity matrix $\mathcal{I}$ does not obey the Fierz condition (2.7), one has to deal with a (RST) mixture in place of a pure state.

B. Gauge forces

The field strength $\mathcal{F}_{\mu\nu}$ is generated by the current $\mathcal{J}_\mu$ according to the (generalized) Maxwell equations

$$\mathcal{D}^\mu\mathcal{F}_{\mu\nu}=4\pi\alpha_*\mathcal{J}_\nu \tag{2.11}$$

where α_* is the electromagnetic coupling constant ($\alpha_*=e^2/\hbar c$). This arrangement implies that the current $\mathcal{J}_\mu$ is sourceless

$$\mathcal{D}^\mu\mathcal{J}_\mu\equiv 0. \tag{2.12}$$

Thus the final problem for the construction of a matter theory is to find the right form of the current $\mathcal{J}_\mu$. In RST, this problem is solved by decomposing first the gauge objects with respect to the generators $\tau_a(=-\bar{\tau}_a)$ of the gauge group as usual

$$\mathcal{A}_\mu=A_{a\mu}\tau^a, \tag{2.13a}$$

$$\mathcal{F}_{\mu\nu}=F_{a\mu\nu}\tau^a, \tag{2.13b}$$

$$\mathcal{J}_\mu=j_{a\mu}\tau^a, \tag{2.13c}$$

and then constructing the currents $j_{a\mu}$ by means of certain velocity operators $v_{a\mu}(=\bar{v}_{a\mu})$ in the following way:

$$j_{a\mu}=\mathrm{tr}(\mathcal{I}v_{a\mu}). \tag{2.14}$$

The source equation (2.12) reads now in component form

$$\nabla^\mu j_{a\mu}\equiv 0 \tag{2.15}$$

and the right choice of the velocity operators is given in terms of the Hamiltonian $\mathcal{H}_\mu$ as

$$v_{a\mu}=\frac{i}{2Mc^2}(\bar{\mathcal{H}}_\mu\tau_a+\tau_a\mathcal{H}_\mu). \tag{2.16}$$

One can show [9] that this choice closes the dynamical system and thus provides us with the possibility of normalizing the currents $j_{a\mu}$ to unity ($z_1=z_2=1$):

$$z_a=\int_{(S)}j_{a\mu}dS^\mu, \tag{2.17}$$

where the integral runs over some three-dimensional hypersurface (S).

C. Reference frames

Both for practical computations and a deeper understanding of RST, it is convenient to select an adequate operator basis for the decomposition of the operator-valued objects. Here the first possibility refers to the single-particle basis

(SPB) which for the case of two particles $(a,b|=1,2)$ consists of two projectors $\mathcal{P}_a(=\overline{\mathcal{P}_a})$

$$\mathcal{P}_a\mathcal{P}_b=\delta_{ab}\mathcal{P}_a, \tag{2.18a}$$

$$\mathcal{P}_1+\mathcal{P}_2=\mathbf{1} \tag{2.18b}$$

and of two permutators $\Pi_a(=\overline{\Pi_a})$ such that

$$\{\Pi_a,\mathcal{P}_b\}=\Pi_a, \tag{2.19a}$$

$$[\mathcal{P}_1,\Pi^a]=-[\mathcal{P}_2,\Pi^a]=i\epsilon^a{}_b\Pi^b, \tag{2.19b}$$

$$\{\Pi_a,\Pi_b\}=2\delta_{ab}\cdot\mathbf{1}, \tag{2.19c}$$

$$[\Pi_a,\Pi_b]=2i\epsilon_{ab}\mathcal{Q}, \tag{2.19d}$$

$$(\mathcal{Q}\doteqdot\mathcal{P}_1-\mathcal{P}_2). \tag{2.19e}$$

Evidently, the generators τ_a of the two-particle gauge group U(1)×U(1) are then given by

$$\tau_a=-i\mathcal{P}_a. \tag{2.20}$$

An alternative choice of frame refers to the extended-particle basis (EPB) which is still based upon the two permutators Π_a (2.19) but relies upon the operators **1** (2.18b) and $\mathcal{Q}$ (2.19e) in place of the projectors $\mathcal{P}_a$ (2.18). Thus, any operator-valued object can be decomposed either in the SPB or in the EPB formalism, e.g., for the intensity matrix $\mathcal{I}$

$$\mathcal{I}=\begin{cases}\rho_a\mathcal{P}^a+\frac{1}{2}s_a\Pi^a & \text{(SPB)},\\ \frac{1}{2}(\rho\cdot\mathbf{1}+q\mathcal{Q}+s_a\Pi^a) & \text{(EPB)}.\end{cases} \tag{2.21}$$

Obviously the single-particle densities ρ_a $(a=1,2)$ are related to their EPB counterparts ρ (total density) and q (internal density) via

$$\rho=\rho_1+\rho_2, \tag{2.22a}$$

$$q=\rho_1-\rho_2. \tag{2.22b}$$

Two further possibilities for the choice of a reference system arise with the emergence of the overlap densities s_a $(a=1,2)$ in connection with the two former reference systems (2.21). More precisely, one may choose also the rotating permutators $\hat{\Pi},\tilde{\Pi}$:

$$\hat{\Pi}\doteqdot\hat{s}_a\Pi^a, \tag{2.23a}$$

$$\tilde{\Pi}\doteqdot\epsilon^{ab}\hat{s}_a\Pi_b, \tag{2.23b}$$

$$\left(\hat{s}_a\doteqdot\frac{s_a}{s};\quad s\doteqdot\sqrt{s^as_a}\right) \tag{2.23c}$$

in place of the original permutators Π_a (2.19). Clearly the new permutators $\hat{\Pi},\tilde{\Pi}$ may now be combined into a complete basis [rotating basis (RTB)] either together with the projectors $\mathcal{P}_a$ (2.18) or with their combinations **1**, $\mathcal{Q}$. Thus the RTB form of the intensity matrix reads

$$\mathcal{I}=\tfrac{1}{2}(\rho\cdot\mathbf{1}+q\mathcal{Q}+s\hat{\Pi}). \tag{2.24}$$

Actually the RTB formalism turns out to be the most convenient one for our subsequent discussions. To give an example, consider the Hamiltonian $\mathcal{H}_\mu$ which we first split up into its Hermitian part $\mathcal{K}_\mu(=\overline{\mathcal{K}_\mu})$ (kinetic field) and anti-Hermitian part $i\mathcal{L}_\mu$ (localization field $\mathcal{L}_\mu=\overline{\mathcal{L}_\mu}$) as follows:

$$\mathcal{H}_\mu=\hbar c(\mathcal{K}_\mu+i\mathcal{L}_\mu). \tag{2.25}$$

The RTB decomposition of these objects then reads

$$\mathcal{K}_\mu=K_{a\mu}\mathcal{P}^a+{}^{(\|)}Q_\mu\hat{\Pi}+{}^{(\perp)}Q_\mu\tilde{\Pi}, \tag{2.26a}$$

$$\mathcal{L}_\mu=L_{a\mu}\mathcal{P}^a+{}^{(\|)}N_\mu\hat{\Pi}+{}^{(\perp)}N_\mu\tilde{\Pi}. \tag{2.26b}$$

Similarly, the RTB form of the currents $j_{a\mu}$ (2.14) is found as

$$j_{1\mu}=\frac{\hbar}{Mc}\left(\rho_1K_{1\mu}+\tfrac{1}{2}s^{(\|)}Q_\mu+\tfrac{1}{2}s^{(\perp)}N_\mu\right), \tag{2.27a}$$

$$j_{2\mu}=\frac{\hbar}{Mc}\left(\rho_2K_{2\mu}+\tfrac{1}{2}s^{(\|)}Q_\mu-\tfrac{1}{2}s^{(\perp)}N_\mu\right). \tag{2.27b}$$

D. Density dynamics

The general matter dynamics (2.2)–(2.4) must now be specialized to the present two-particle situation. First consider the integrability condition (2.2) which yields for both localization components $L_{a\mu}$ of the Hamiltonian $\mathcal{H}_\mu$ (2.25), (2.26) the following curl relations [10]:

$$\nabla_\mu(L_{1\nu}+L_{2\nu})-\nabla_\nu(L_{1\mu}+L_{2\mu})=0. \tag{2.28}$$

Thus the sum of both localization vectors $L_{a\mu}$ is revealed to be a gradient field

$$L_\mu\doteqdot L_{1\mu}+L_{2\mu}=\frac{\partial_\mu L^2}{L^2}=2\,\frac{\partial_\mu L}{L}. \tag{2.29}$$

This new gradient field $L(x)$ (amplitude field) plays an important part when considering now the density dynamics which is given in abstract form by the RNE (2.4).

Indeed the latter equation is nothing else than a dynamical system for the densities ρ,q,s as the RTB components of the intensity matrix $\mathcal{I}$ (2.24). This density dynamics is most effectively discussed in terms of the renormalization factors Z_T,Z_R,Z_O which are introduced now through the following definitions:

$$\rho=Z_TL^2, \tag{2.30a}$$

$$q=Z_RL^2, \tag{2.30b}$$

$$s=Z_OL^2. \tag{2.30c}$$

The meaning of this construction of renormalization variables is as usual, namely, that the physical densities ρ,q,s as the true observables of the theory remain invariant with respect to a renormalization transformation of the (unobservable) auxiliary variables Z_T, Z_R, Z_O, L which, however, are better suited to specify the dynamical equations [10]. Actually, the RNE (2.4) is transcribed to the renormalization factors as follows:

$$\partial_\mu Z_T = Z_R l_\mu + 2 Z_O {}^{(\parallel)}N_\mu , \tag{2.31a}$$

$$\partial_\mu Z_R = Z_T l_\mu - 2 Z_O {}^{(\perp)}Q_\mu , \tag{2.31b}$$

$$\partial_\mu Z_O = 2(Z_T {}^{(\parallel)}N_\mu + Z_R {}^{(\perp)}Q_\mu). \tag{2.31c}$$

Here we have introduced the EPB variable l_μ in a similar way as was done for L_μ (2.29), i.e.,

$$l_\mu \doteq L_{1\mu} - L_{2\mu} . \tag{2.32}$$

Now the crucial point with the renormalization dynamics is that it admits a first integral in the form [10]

$$Z_T^2 - (Z_R^2 + Z_O^2) = \sigma_* , \tag{2.33}$$

where the integration constant σ_* (mixture index) can be taken (without loss of generality) as $0, \pm 1$. Putting

$$Z_{\text{II}} \doteq \sqrt{Z_R^2 + Z_O^2} \tag{2.34}$$

the constraint (2.33) reads

$$Z_T^2 - Z_{\text{II}}^2 = \sigma_* \tag{2.35}$$

and thus it is suggestive to parametrize the three renormalization factors by only two variables, the mixture variable ζ and the overlap angle ξ_o, in the following way:

$$Z_R = Z_{\text{II}}(\zeta) \cos \xi_o , \tag{2.36a}$$

$$Z_O = Z_{\text{II}}(\zeta) \sin \xi_o . \tag{2.36b}$$

The pure states have mixture index $\sigma_* = 0$ and therefore occupy the Fierz cone in density configuration space (Fig. 1) being specified by

$$Z_T = Z_{\text{II}} = \tfrac{1}{2} e^{\zeta}, \qquad (\sigma_* = 0). \tag{2.37}$$

Similarly the positive mixtures ($\sigma_* = +1$) are geometrically characterized by the two-parted hyperboloid

$$Z_T = \pm \cosh \zeta, \tag{2.38a}$$

$$Z_{\text{II}} = \sinh \zeta, \tag{2.38b}$$

$$(\sigma_* = +1),$$

and the negative mixtures by the one-parted hyperboloid

$$Z_T = \sinh \zeta, \tag{2.39a}$$

$$Z_{\text{II}} = \cosh \zeta, \tag{2.39b}$$

$$(\sigma_* = -1).$$

Clearly, for $\zeta \to \infty$ both mixtures (2.38),(2.39) approach the pure states (2.37), or in geometric terms, the hyperboloids come close to the Fierz cone (Fig. 1).

The field equations for the new variables ζ and ξ_o are easily deduced from the original renormalization dynamics (2.31) as

$$\partial_\mu \zeta = h_\mu , \tag{2.40a}$$

$$\partial_\mu \xi_o = 2\left({}^{(\perp)}Q_\mu + \frac{Z_T}{Z_{\text{II}}} g_\mu \right), \tag{2.40b}$$

where the vector fields h_μ and g_μ stand for certain linear combinations of the localization components l_μ (2.32) and ${}^{(\parallel)}N_\mu$ (2.26b):

$$h_\mu = 2 {}^{(\parallel)}N_\mu \sin \xi_o + l_\mu \cos \xi_o , \tag{2.41a}$$

$$g_\mu = 2 {}^{(\parallel)}N_\mu \cos \xi_o - l_\mu \sin \xi_o . \tag{2.41b}$$

E. Curl relations

The present parametrizations may now be used in order to conveniently transcribe the abstract Hamiltonian dynamics (2.2),(2.3) to the equations of motion for the remaining component fields. First consider the integrability condition (2.2) which yields for the kinetic fields $K_{a\mu}$ (2.26a) [10]

$$\nabla_\mu K_{1\nu} - \nabla_\nu K_{1\mu} = F_{1\mu\nu} + G_{\mu\nu} , \tag{2.42a}$$

$$\nabla_\mu K_{2\nu} - \nabla_\nu K_{2\mu} = F_{2\mu\nu} - G_{\mu\nu} , \tag{2.42b}$$

where the exchange field strength $G_{\mu\nu}$ is composed of the former exchange fields ${}^{(\parallel)}N_\mu , {}^{(\perp)}N_\mu , {}^{(\parallel)}Q_\mu , {}^{(\perp)}Q_\mu$ in the following way:

$$\begin{aligned} G_{\mu\nu} = 2(& {}^{(\parallel)}Q_\mu {}^{(\perp)}Q_\nu - {}^{(\parallel)}Q_\nu {}^{(\perp)}Q_\mu - {}^{(\parallel)}N_\mu {}^{(\perp)}N_\nu \\ & + {}^{(\parallel)}N_\nu {}^{(\perp)}N_\mu). \end{aligned} \tag{2.43}$$

Here it is easy to prove that the two-form $G_{\mu\nu}$ obeys the identity

$$\nabla_\mu G_{\nu\lambda} + \nabla_\nu G_{\lambda\mu} + \nabla_\lambda G_{\mu\nu} \equiv 0 \tag{2.44}$$

and thus can be generated by a (gauge invariant) exchange vector potential G_μ

$$G_{\mu\nu} = \nabla_\mu G_\nu - \nabla_\nu G_\mu . \tag{2.45}$$

Furthermore such an exchange potential can easily be found as

$$G_\mu = \frac{Z_R}{Z_O} {}^{(\parallel)}Q_\mu - \frac{Z_T}{Z_O} {}^{(\perp)}N_\mu . \tag{2.46}$$

In order to verify that the present proposition (2.46) actually is a possible solution of the curl relation (2.45) one must know also the curl relations for the other exchange fields which, however, are easily deduced from the integrability condition (2.2):

$$\nabla_\mu {}^{(\|)}Q_\nu - \nabla_\nu {}^{(\|)}Q_\mu = l_\mu {}^{(\perp)}N_\nu - l_\nu {}^{(\perp)}N_\mu + 2({}^{(\perp)}Q_\mu G_\nu - {}^{(\perp)}Q_\nu G_\mu), \tag{2.47a}$$

$$\nabla_\mu {}^{(\perp)}Q_\nu - \nabla_\nu {}^{(\perp)}Q_\mu = -l_\mu {}^{(\|)}N_\nu + l_\nu {}^{(\|)}N_\mu - 2({}^{(\|)}Q_\mu G_\nu - {}^{(\|)}Q_\nu G_\mu), \tag{2.47b}$$

$$\nabla_\mu {}^{(\|)}N_\nu - \nabla_\nu {}^{(\|)}N_\mu = -l_\mu {}^{(\perp)}Q_\nu + l_\nu {}^{(\perp)}Q_\mu + 2({}^{(\perp)}N_\mu G_\nu - {}^{(\perp)}N_\nu G_\mu), \tag{2.47c}$$

$$\nabla_\mu {}^{(\perp)}N_\nu - \nabla_\nu {}^{(\perp)}N_\mu = l_\mu {}^{(\|)}Q_\nu - l_\nu {}^{(\|)}Q_\mu - 2({}^{(\|)}N_\mu G_\nu - {}^{(\|)}N_\nu G_\mu). \tag{2.47d}$$

Clearly in place of the fields l_μ, ${}^{(\|)}N_\mu$ one can also work with the new vectors g_μ, h_μ (2.41) whose curl relations are

$$\nabla_\mu h_\nu - \nabla_\nu h_\mu = 0 \tag{2.48a}$$

$$\begin{aligned}\nabla_\mu g_\nu - \nabla_\nu g_\mu = {} & \frac{Z_T}{Z_{\mathrm{II}}}(h_\mu g_\nu - h_\nu g_\mu) \\ & + 4\{{}^{(\perp)}N_\mu[(\cos\xi_o)G_\nu + (\sin\xi_o){}^{(\|)}Q_\nu] \\ & - {}^{(\perp)}N_\nu[(\cos\xi_0)G_\mu + (\sin\xi_o){}^{(\|)}Q_\mu]\}.\end{aligned} \tag{2.48b}$$

Here, the first curl relation (2.48a) is trivial because the vector h_μ has already been found to be a gradient field, see Eq. (2.40a).

F. Source equations

Whereas the meaning of the curl relations mainly is of kinematical nature, namely, to guarantee the (local) existence of solutions, the proper field equations are specified by the conservation equation (2.3). Here the most important ones are the wave equations (amplitude equations) for the two single-particle amplitude fields $L_a(x)$ $(a=1,2)$ which are defined as follows:

$$L_1 \doteqdot \left(\cos\frac{\xi_o}{2}\right)\sqrt{Z_{\mathrm{II}}}L \equiv \left(\cos\frac{\xi_o}{2}\right){}'L, \tag{2.49a}$$

$$L_2 \doteqdot \left(\sin\frac{\xi_0}{2}\right)\sqrt{Z_{\mathrm{II}}}L \equiv \left(\sin\frac{\xi_o}{2}\right){}'L, \tag{2.49b}$$

where $L(x)$ is the amplitude field introduced through Eq. (2.29). For the sake of convenience one may refer also to the modified amplitude field ${}'L = \sqrt{Z_{\mathrm{II}}}L$ in place of the original L. Now, when the conservation equation (2.3) is transcribed to the amplitudes L_a (2.49), there arise wave equations of the following form (amplitude equations):

$$\Box L_1 + L_1\left\{\left(\frac{Mc}{\hbar}\right)^2 - K_{1\mu}K_1^\mu + X_o + W_1\right\} = X_1 L_2, \tag{2.50a}$$

$$\Box L_2 + L_2\left\{\left(\frac{Mc}{\hbar}\right)^2 - K_{2\mu}K_2^\mu + X_0 + W_2\right\} = X_2 L_1. \tag{2.50b}$$

If both the mixture potentials W_a and the (scalar) exchange potentials X_0 and X_a $(a=1,2)$ (to be explained below) would vanish: $W_a = X_a = X_0 \equiv 0$, then we would be left with two uncoupled wave equations $(a=1,2)$

$$\Box L_a + L_a\left\{\left(\frac{Mc}{\hbar}\right)^2 - K_{a\mu}K_a^\mu\right\} = 0. \tag{2.51}$$

These are equivalent to the two conventional Klein-Gordon equations

$$\begin{gathered} D^\mu D_\mu \psi_a + \left(\frac{Mc}{\hbar}\right)^2 \psi_a = 0, \\ (D_\mu \psi_a \doteqdot \partial_\mu \psi_a - i A_{a\mu}\psi_a), \end{gathered} \tag{2.52}$$

where the $\mathbb{C}^1$-valued single-particle wave functions ψ_a are built up by the amplitude fields L_a and phases α_a in the usual way

$$\psi_a = L_a e^{-i\alpha_a} \tag{2.53}$$

with the (gauge dependent) phases α_a being given through

$$\alpha_a(x) = \int_x (K_{a\mu} - A_{a\mu})dx^\mu, \tag{2.54}$$

see Refs. [19,20]. Clearly, the two Klein-Gordon equations (2.52) are nothing else than the component version of the abstract KGE (2.10). Thus, this system describes two individual Klein-Gordon particles which may be thought to be coupled through the ordinary gauge forces inherent in the conventional quantum theory via the electromagnetic vector potentials $A_{a\mu}$. These conventional gauge forces are described in RST by the kinetic fields $K_{a\mu}$ coupling to the electromagnetic field strengths $F_{a\mu\nu}$ in the usual way [apart from the presence of the exchange field strengths $G_{\mu\nu}$ in the Maxwell equations (2.42)].

However, the presence of the mixture potentials W_a and exchange potentials X_a in the amplitude equations (2.50) as well as G_μ in the Maxwell equations (2.42) signals the occurrence of unconventional nongauge interactions among the two particles which we take as the RST counterpart of the entanglement phenomenon in the conventional theory. Clearly, the physical implications of such an RST entanglement of both particles will strongly depend upon the nature of the mixture and exchange potentials W_a and X_a. Concerning the mixture case, one finds [14,15]

$$W_1=\frac{\sigma_*}{(2Z_{\mathrm{II}})^2}\left\{h^\mu h_\mu-g^\mu g_\mu-2g^\mu h_\mu\frac{L_2}{L_1}\right\}$$
$$-g_s(K_{1\mu}K_1^\mu-K_{2\mu}K_2^\mu), \tag{2.55a}$$

$$W_2=\frac{\sigma_*}{(2Z_{\mathrm{II}})^2}\left\{h^\mu h_\mu-g^\mu g_\mu+2g^\mu h_\mu\frac{L_1}{L_2}\right\}$$
$$+g_s(K_{1\mu}K_1^\mu-K_{2\mu}K_2^\mu), \tag{2.55b}$$

and similarly for the (scalar) exchange potentials

$$X_0={}^{(\perp)}N_\mu{}^{(\perp)}N^\mu-{}^{(\parallel)}Q_\mu{}^{(\parallel)}Q^\mu \tag{2.55c}$$

$$X_1=\frac{Z_T}{Z_{\mathrm{II}}}({}^{(\parallel)}Q_\mu K^\mu+{}^{(\perp)}N_\mu\ {}'k^\mu)+({}^{(\parallel)}Q_\mu\ {}'k^\mu+{}^{(\perp)}N_\mu K^\mu) \tag{2.55d}$$

$$X_2=\frac{Z_T}{Z_{\mathrm{II}}}({}^{(\parallel)}Q_\mu K^\mu+{}^{(\perp)}N_\mu\ {}'k^\mu)-({}^{(\parallel)}Q_\mu\ {}'k^\mu+{}^{(\perp)}N_\mu K^\mu). \tag{2.55e}$$

Here, for the representation of the mixture potentials W_a (2.55) we have made use of the structure function g_s being defined in terms of the renormalization factors through [14]

$$g_s\doteqdot\frac{1}{2}\left(\frac{Z_T}{Z_{\mathrm{II}}}-1\right). \tag{2.56}$$

For later purposes it is convenient to complement this by a further structure function g_w to be defined through

$$g_w\doteqdot\frac{1}{2}\left(\frac{Z_T}{Z_{\mathrm{II}}}+1\right)=g_s+1 \tag{2.57}$$

such that both structure functions have their characteristic pure-state limits

$$\lim_{\zeta\to\infty}g_s=0, \tag{2.58a}$$

$$\lim_{\zeta\to\infty}g_w=1. \tag{2.58b}$$

Furthermore it is convenient also to use the EPB forms K_μ, k_μ of the SPB kinetic fields $K_{a\mu}$, i.e.,

$$K_\mu=K_{1\mu}+K_{2\mu}, \tag{2.59a}$$

$$k_\mu=K_{1\mu}-K_{2\mu}, \tag{2.59b}$$

$${}'k_\mu=k_\mu-2G_\mu. \tag{2.59c}$$

It is important to remark that those mixture potentials W_a (2.55a),(2.55b) vanish ($W_a\Rightarrow 0$) for the pure states, namely, either by directly putting the mixture index to zero [$\sigma_*=0$, see Eq. (2.37)] or by letting tend the mixture variable ζ to infinity [$\zeta\to\infty\Leftrightarrow Z_{\mathrm{II}}\to\infty$, see Eqs. (2.38),(2.39)]. Thus the pure states may also be reached dynamically as the limit configuration of the mixtures, namely, when the dynamics guides the mixture variable ζ asymptotically to its limit value $\zeta=\infty$.

In order to close the dynamical system, one must specify the dynamical equations of all the other fields emerging in the amplitude equations (2.50). First consider the case of the mixture variable ζ whose field equation is obtained by combining the source equations of the exchange fields l_μ and ${}^{(\parallel)}N_\mu$

$$\nabla^\mu l_\mu+L^\mu l_\mu-K^\mu k_\mu=0, \tag{2.60a}$$

$$\nabla^{\mu(\parallel)}N_\mu-{}^{(\perp)}N_\mu\ {}'k^\mu+{}^{(\parallel)}N_\mu L^\mu-{}^{(\parallel)}Q_\mu K^\mu=0, \tag{2.60b}$$

into the source equation ($\nabla^\mu h_\mu\equiv\Box\zeta$) for the gradient field h_μ (2.40a):

$$\Box\zeta+{}'L^\mu\partial_\mu\zeta-\frac{Z_T}{Z_{\mathrm{II}}}\{\partial^\mu\zeta\partial_\mu\zeta+g^\mu g_\mu\}=\frac{L_1^2-L_2^2}{L_1^2+L_2^2}(K^\mu k_\mu)$$
$$+2{}^{(\perp)}Q_\mu g^\mu+4\frac{L_1L_2}{L_1^2+L_2^2}({}^{(\parallel)}Q_\mu K^\mu+{}^{(\perp)}N_\mu\ {}'k^\mu). \tag{2.61}$$

Here the modified localization vector ${}'L_\mu$ is related to the modified amplitude field ${}'L(x)$ (2.49) just in the same way as for the corresponding unmodified objects (2.29), i.e.,

$${}'L_\mu=2\frac{\partial'_\mu L}{{}'L}. \tag{2.62}$$

In a similar way, one finds the source equation for the vector field g_μ as

$$\nabla^\mu g_\mu+{}'L^\mu g_\mu+2\frac{L_1L_2}{L_1^2+L_2^2}(K^\mu k_\mu)=-2{}^{(\perp)}Q_\mu h^\mu$$
$$+2\frac{L_1^2-L_2^2}{L_1^2+L_2^2}({}^{(\perp)}N_\mu{}'k^\mu k^\mu+{}^{(\parallel)}Q_\mu K^\mu). \tag{2.63}$$

Next, one computes the source equations for the remaining exchange fields ${}^{(\perp)}N_\mu$, ${}^{(\perp)}Q_\mu$, and ${}^{(\parallel)}Q_\mu$ as [10]

$$\nabla^{\mu(\perp)}Q_\mu+{}^{(\parallel)}Q_\mu\ {}'k^\mu+{}^{(\perp)}Q_\mu L^\mu+{}^{(\perp)}N_\mu K^\mu=0 \tag{2.64a}$$

$$\nabla^{\mu(\parallel)}Q_\mu-{}^{(\perp)}Q_\mu\ {}'k^\mu+{}^{(\parallel)}Q_\mu L^\mu+{}^{(\parallel)}N_\mu K^\mu=0 \tag{2.64b}$$

$$\nabla^{\mu(\perp)}N_\mu+{}^{(\parallel)}N_\mu\ {}'k^\mu+{}^{(\perp)}N_\mu L^\mu-{}^{(\perp)}Q_\mu K^\mu=0. \tag{2.64c}$$

G. Conservation laws

Finally, the source equations for the kinetic fields must be specified. The significance of these equations becomes immediately obvious from the conservation laws for the currents $j_{a\mu}$ (2.15) which are essentially built up by the kinetic fields $K_{a\mu}$, see Eq. (2.27); especially this applies to the situation of vanishing exchange fields (${}^{(\parallel)}Q_\mu={}^{(\perp)}Q_\mu={}^{(\parallel)}N_\mu={}^{(\perp)}N_\mu\equiv 0$), a situation to be considered subsequently in great detail. The desired source equations are deduced again from the conservation equation (2.3) and look as follows ($a=1,2$):

$$\nabla^{\mu}K_{a\mu}+2L^{\mu}_{a}K_{a\mu}=-2({}^{(\parallel)}Q_{\mu}{}^{(\parallel)}N^{\mu}+{}^{(\perp)}Q_{\mu}{}^{(\perp)}N^{\mu}). \qquad (2.65)$$

Here it is now an instructive exercise to explicitly verify the conservation laws (2.15) by computing the sources of both currents $j_{a\mu}$ with the help of all the preceding source equations.

Thus our dynamical two-particle system is now closed and evidently comprises the following dynamical variables.

(i) Two amplitude fields L_a obeying the amplitude equations (2.50).

(ii) Two kinetic fields $K_{a\mu}$ obeying the Maxwell equations (2.42) and the source relations (2.65).

(iii) The mixture variable ζ to be determined from its wave equation (2.61).

(iv) The vector field g_{μ} whose source is determined by Eq. (2.63) and its curl by Eq. (2.48b).

(v) The exchange fields ${}^{(\parallel)}Q_{\mu}$, ${}^{(\perp)}Q_{\nu}$, ${}^{(\parallel)}N_{\mu}$, and ${}^{(\perp)}N_{\mu}$ obeying the curl relations (2.47) and the source equations (2.64) together with Eq. (2.60).

III. MIXTURE DEGENERACY

The preceding list of dynamical variables demonstrates that the most general situation comprehends two distinct phenomena: (i) the mixture effect and (ii) the action of the exchange forces lying beyond the gauge interactions. Therefore it seems advisable to consider both effects separately, i.e., we study now the mixture effect alone by putting all the exchange fields to zero: ${}^{(\parallel)}Q_{\mu}={}^{(\perp)}Q_{\mu}={}^{(\parallel)}N_{\mu}={}^{(\perp)}N_{\mu}\equiv 0$. Furthermore we first neglect also the gauge interactions among the two particles and retain exclusively some external force ${}^{(\mathrm{ex})}F_{\mu\nu}$ in order to either hold both particles together (bound solutions, e.g., electrons around a nucleus) or to let them freely move over some background force field (unbound solutions). Thus we will be able to study the mixture effect in its pure form and we shall find here the phenomenon of (continuous) mixture degeneracy as the RST counterpart of the (discrete) exchange degeneracy in conventional quantum theory.

A. Mixture dynamics

For cutting down the general two-particle dynamics to the exclusive mixture case (i.e., for vanishing exchange fields and interparticle interactions) one first observes that the exchange field strength $G_{\mu\nu}$ (2.43) vanishes and therefore the modified Maxwell equations (2.42) adopt their ordinary form ($a=1,2$):

$$\nabla_{\mu}K_{a\nu}-\nabla_{\nu}K_{a\mu}={}^{(\mathrm{ex})}F_{\mu\nu}. \qquad (3.1)$$

Here the formal solution in terms of (gauge dependent) scalar fields $\alpha_a(x)$ and external gauge potentials ${}^{(\mathrm{ex})}A_{\mu}$ reads

$$K_{a\mu}=\partial_{\mu}\alpha_a+{}^{(\mathrm{ex})}A_{\mu}. \qquad (3.2)$$

Observe that a gauge transformation

$${}^{(\mathrm{ex})}A'_{\mu}={}^{(\mathrm{ex})}A_{\mu}-\partial_{\mu}a(x), \qquad (3.3a)$$

$$\alpha'_a(x)=\alpha_a(x)+a(x). \qquad (3.3b)$$

leaves the kinetic fields $K_{a\mu}$ (as some of the Hamiltonian component fields) invariant. The reason is that the external gauge group (3.3) is abelian and the Hamiltonian $\mathcal{H}_{\mu}$ as well as its Hermitian part $\mathcal{K}_{\mu}$ (2.26a) transform homogeneously under a change of gauge [10].

Next, observe that our assumption of vanishing exchange fields gives rise to introduce a new scalar field (χ, say) which generates the vector field g_{μ} (2.41b) in a way similar to the ray its companion h_{μ} (2.41a) is generated by the mixture variable ζ, see Eq. (2.40a). This claim is immediately verified by reconsidering the general curl relation for g_{μ} (2.48b) which now simplifies to

$$\nabla_{\mu}g_{\nu}-\nabla_{\nu}g_{\mu}=\frac{Z_T}{Z_{\mathrm{II}}}(h_{\mu}g_{\nu}-h_{\nu}g_{\mu}). \qquad (3.4)$$

However, this truncated form admits the formal solution for g_{μ}

$$g_{\mu}=Z_{\mathrm{II}}\partial_{\mu}\chi \qquad (3.5)$$

which proves our claim. Once the scalar χ is at hand, one would like to transcribe the source equation for g_{μ} (2.63) into a wave equation for the new scalar field χ:

$$\Box\chi+{}'L^{\mu}\partial_{\mu}\chi+\frac{Z_T}{Z_{\mathrm{II}}}\partial^{\mu}\zeta\partial_{\mu}\chi+2\,\frac{L_1L_2}{L_1^2+L_2^2}\,\frac{K^{\mu}k_{\mu}}{Z_{\mathrm{II}}}=0. \qquad (3.6)$$

With these presumptions, the wave equation for the mixture variable ζ (2.61) adopts a similar form, namely,

$$\Box\zeta+{}'L^{\mu}\partial_{\mu}\zeta-\frac{Z_T}{Z_{\mathrm{II}}}\times\left[\partial^{\mu}\zeta\partial_{\mu}\zeta+Z_{\mathrm{II}}^2\partial^{\mu}\chi\partial_{\mu}\chi-\frac{L_1^2-L_2^2}{L_1^2+L_2^2}(K^{\mu}k_{\mu})\right]=0. \qquad (3.7)$$

Now a pleasant property of the mixture configurations consists in the fact that both wave equations (3.6),(3.7) are not needed at all! The reason is that both scalars χ and ζ can be expressed by the amplitudes L_1 and L_2 so that the wave equations for χ (3.6) and ζ (3.7) are automatically satisfied whenever the corresponding amplitude equations are valid, see (2.50)

$$\Box L_1+L_1\left\{\left(\frac{Mc}{\hbar}\right)^2-K_{1\mu}K_1^{\mu}+W_1\right\}=0, \qquad (3.8a)$$

$$\Box L_2+L_2\left\{\left(\frac{Mc}{\hbar}\right)^2-K_{2\mu}K_2^{\mu}+W_2\right\}=0. \qquad (3.8b)$$

The origin of this nice result traces back to the renormalization dynamics (2.31c) which requires the renormalization factor Z_O to be a constant over space-time

$$Z_O = \text{const} \quad (\doteqdot C_*). \tag{3.9}$$

However, when Z_O is constant, the overlap angle ξ_0 is linked rigidly to the mixture variable ζ, see Eq. (2.36b):

$$Z_{\text{II}}(\zeta) = \frac{C_*}{|\sin \xi_o|} = C_* \left| \frac{L_1^2 + L_2^2}{2L_1L_2} \right|. \tag{3.10}$$

Thus the mixture variable ζ is uniquely determined by the amplitudes L_a and the mixture index C_*. On the other hand, the vanishing of both exchange fields ${}^{(\parallel)}N_\mu$ and ${}^{(\perp)}Q_\mu$ yields the following system for the scalars ζ and χ, see Eqs. (2.40b) and (2.41):

$${}^{(\parallel)}N_\mu \equiv \tfrac{1}{2}\left[((\cos \xi_o) Z_{\text{II}} \partial_\mu \chi + (\sin \xi_o) \partial_\mu \zeta) \right] = 0, \tag{3.11a}$$

$$2^{(\perp)}Q_\mu \equiv \partial_\mu \xi_o - Z_T \partial_\mu \chi = 0, \tag{3.11b}$$

the solution of which is easily found as

$$\cos \xi_o = \frac{L_1^2 - L_2^2}{L_1^2 + L_2^2} = -\begin{cases} \sqrt{C_*^2 + 1}\,\sin(\chi - \chi_*), & \sigma_* = +1 \\ \sqrt{C_*^2 - 1}\,\sinh(\chi - \chi_*), & \sigma_* = -1 \end{cases}. \tag{3.12}$$

Thus, also the scalar χ is determined (up to an integration constant χ_*) by the amplitude fields L_a. As a consequence, one can do away completely with the wave equations for χ (3.6) and ζ (3.7) and one can concentrate exclusively upon the amplitude equations (3.8).

In order that these amplitude equations form a manifestly covariant system, it remains to express the mixture potentials W_a (2.55) in terms of just those amplitude fields L_a:

$$W_1 = \sigma_* \frac{L_1^2 - 3L_2^2}{(L_1^2 + L_2^2)} \frac{(L_1 \partial_\mu L_2 - L_2 \partial_\mu L_1)(L_1 \partial^\mu L_2 - L_2 \partial^\mu L_1)}{\sigma_*(2L_1L_2)^2 + C_*^2 (L_1^2 + L_2^2)^2}$$
$$- g_s [K_{1\mu} K_1^\mu - K_{2\mu} K_2^\mu], \tag{3.13a}$$

$$W_2 = \sigma_* \frac{L_2^2 - 3L_1^2}{(L_1^2 + L_2^2)} \frac{(L_1 \partial_\mu L_2 - L_2 \partial_\mu L_1)(L_1 \partial^\mu L_2 \text{-} L_2 \partial^\mu L_1)}{\sigma_*(2L_1L_2)^2 + C_*^2 (L_1^2 + L_2^2)}$$
$$+ g_s [K_{1\mu} K_1^\mu - K_{2\mu} K_2^\mu]. \tag{3.13b}$$

Finally, the structure function g_s (2.56) must also be written exclusively in terms of the amplitude fields

$$g_s = \frac{1}{2}\left(\sqrt{1 + \frac{4\sigma_*}{C_*^2} \frac{L_1^2 L_2^2}{(L_1^2 + L_2^2)^2}} - 1 \right). \tag{3.14}$$

In this way the mixture dynamics is actually a closed system for the amplitude fields, since the kinetic fields contain only the external potential when the electromagnetic interactions between both particles are neglected, see Eq. (3.2). Clearly the mixture potentials W_a (3.13) are highly nonlinear functions of the amplitudes L_a and their derivatives and therefore it may seem absolutely hopeless to find exact solutions to the coupled amplitude system (3.8) for a given mixture parameter C_*. However, as we shall readily show, the exact solutions are obtainable from the solutions for the pure-state limit $C_* \to \infty$ (2.51) by a transformation which we are now going to work out in detail.

B. Mixing group

So far we were mainly concerned with the amplitude fields L_a but the key for finding their exact form leads us back to the SPB densities ρ_a (2.21). Observe here that, for our present assumption of vanishing exchange fields, these densities essentially build up the currents $j_{a\mu}$ (2.27) alone, i.e.,

$$j_{1\mu} \Rightarrow \frac{\hbar}{Mc} \rho_1 K_{1\mu}, \tag{3.15a}$$

$$j_{2\mu} \Rightarrow \frac{\hbar}{Mc} \rho_2 K_{2\mu}. \tag{3.15b}$$

However, the crucial point with the densities is now their relationship with the amplitude fields L_a [14] which looks as follows:

$$\rho_1 = \text{tr}(\mathcal{I}\mathcal{P}_1) = g_w L_1^2 + g_s L_2^2, \tag{3.16a}$$

$$\rho_2 = \text{tr}(\mathcal{I}\mathcal{P}_2) = g_w L_2^2 + g_s L_1^2, \tag{3.16b}$$

From the pure-state limit (2.58) of the structure functions g_s and g_w it is concluded that both objects coincide in that limit, i.e., $\rho_a \to {}^{(0)}L_a^2$ $(a = 1,2)$ for $C_* \to \infty$. Here the pure-state amplitudes as solutions of the ordinary decoupled Klein-Gordon system (2.51),(2.52) have been denoted by ${}^{(0)}L_a$. Since for the present case of vanishing exchange fields, the structure functions g_s and g_w become functions of the amplitude fields alone, see Eq. (3.14), the relationship (3.16) acquires the status of a (highly nonlinear) transformation $[R^{(2)}]$ of the amplitudes

$$[R^{(2)}]: (L_1, L_2) \to (\rho_1, \rho_2), \tag{3.17}$$

i.e., in detail

$$\rho_1 = {}^{(2)}R_1(L_1, L_2)$$
$$\doteqdot g_w(L_1, L_2) L_1^2 + g_s(L_1, L_2) L_2^2, \tag{3.18a}$$

$$\rho_2 = {}^{(2)}R_2(L_1, L_2)$$
$$\doteqdot g_w(L_1, L_2) L_2^2 + g_s(L_1, L_2) L_1^2. \tag{3.18b}$$

Defining here two new amplitude fields (up to sign) as

$$(L_1')^2 \doteqdot \rho_1, \tag{3.19a}$$

$$(L_2')^2 \doteqdot \rho_2, \tag{3.19b}$$

one can rewrite the density transformation $[R^{(2)}]$ (3.17) as an amplitude transformation $[R]$

$$[R]: (L_1, L_2) \to (L_1', L_2'), \tag{3.20}$$

i.e.,

$$L_1' = R_1(L_1, L_2) \doteqdot \sqrt{g_w L_1^2 + g_s L_2^2}, \tag{3.21a}$$

$$L_2' = R_2(L_1, L_2) = \sqrt{g_w L_2^2 + g_s L_1^2}. \tag{3.21b}$$

[By taking the roots, the signs of the new amplitudes L_a' have to be chosen in such a way that L_a' remain continuously differentiable over space-time if the original amplitudes $L_a(x)$ are of the corresponding class; observe that the solutions $L_a(x)$ of the quasilinear Klein-Gordon systems (3.8) are determined only up to sign.]

The transformations $[R]$ (3.20) have an interesting property, namely, the effects of fusion $[\rho_1(x) \approx \rho_2(x)]$ for positive mixtures $(\sigma_* = +1)$ and separation $(\rho_1(x)\rho_2(x) \approx 0)$ for negative mixtures $(\sigma_* = -1)$. This phenomenon was studied extensively in a preceding paper [14] but will be considered here for the inverse transformations $[\Lambda] = [R^{-1}]$, i.e., we put

$$[\Lambda]: (L_1, L_2) \rightarrow ({}'L_1, {}'L_2), \tag{3.22}$$

or in detail

$$\begin{aligned} {}'L_1 &= \Lambda_1(L_1, L_2) \\ &\doteqdot \sqrt{\tfrac{1}{2}\,(L_1^2 - L_2^2) + \tfrac{1}{2}\,\sqrt{(L_1^2 - L_2^2)^2 + \lambda_*(2L_1L_2)^2}}, \end{aligned} \tag{3.23a}$$

$$\begin{aligned} {}'L_2 &= \Lambda_2(L_1, L_2) \\ &\doteqdot \sqrt{-\tfrac{1}{2}\,(L_1^2 - L_2^2) + \tfrac{1}{2}\,\sqrt{(L_1^2 - L_2^2)^2 + \lambda_*(2L_1L_2)^2}}. \end{aligned} \tag{3.23b}$$

The most striking feature of the set of transformations (3.22) is here that they form a group with the group parameter λ_*

$$\lambda_* \doteqdot \frac{1}{1 + (\sigma_*/C_*^2)} \tag{3.24}$$

obeying the simple composition rule

$${}^{(3)}\lambda_* = {}^{(1)}\lambda_*\,{}^{(2)}\lambda_* \tag{3.25}$$

for two successive transformations parametrized by ${}^{(1)}\lambda_*$ and ${}^{(2)}\lambda_*$. Thus the group element Λ^{-1} due to λ_*^{-1} is the inverse of that element Λ which is parametrized by λ_*; and the identity is given by $\lambda_* = 1 (\Leftrightarrow C_* = \infty)$.

The mixing group $[\Lambda]$ (3.22) has some striking similarities with the Lorentz group SO(1,1) in $1+1$ dimensions

$${}'L_1 = \cosh \beta_* L_1 + \sinh \beta_* L_2 \tag{3.26a}$$

$${}'L_2 = \sinh \beta_* L_1 + \cosh \beta_* L_2\,. \tag{3.26b}$$

Indeed, in both cases the light cones $(L_1 = \pm L_2)$ remain invariant as a whole and some point $L_1 = L_2 (\doteqdot L_c)$ on a cone is shifted within the cone according to

$${}'L_c = \begin{cases} (4\lambda_*)^{1/4} L_c & \text{for } [\Lambda] \;\; (3.23) \\ e^{\beta_*} L_c\,, & \text{for SO(1,1)} \;\; (3.26). \end{cases} \tag{3.27}$$

Furthermore, the orbit of some point (L_1, L_2) is the same for both group actions and is given by the hyperbola

$${}'L_1^2 - {}'L_2^2 = \text{const}(= L_1^2 - L_2^2), \tag{3.28}$$

see Fig. 2. But clearly, points are shifted within the orbits in a different way for both transformation groups. For instance the ‘‘space’’ and ‘‘time’’ axis $(L_1 \equiv 0$ or $L_2 \equiv 0)$ are transformed to a new axis system for the Lorentz group SO(1,1) in such a way that the well-known effects of Lorentz contraction and time dilatation will arise. In contrast to this, the points on the axes are kept fixed by a transformation of the mixing group [Λ], i.e.,

$$\Lambda_1(L_1, 0) = L_1, \quad \Lambda_2(L_1, 0) = 0, \tag{3.29a}$$

$$\Lambda_1(0, L_2) = 0, \quad \Lambda_2(0, L_2) = L_2\,. \tag{3.29b}$$

Let us remark also that the product of the amplitudes transforms under the mixing group in the following way:

$$({}'L_1\ {}'L_2)^2 = \lambda_*(L_1 L_2)^2 \tag{3.30}$$

which has no counterpart for the Lorentz group but preserves here the orthogonality of the amplitude fields [see Eq. (3.35) below]. Thus the important similarities of both groups mainly refer to the vicinity of the light cones. This becomes especially clear when considering the pullback metric γ_{ab} of the Lorentz metric $\eta_{ab} = \text{diag}[1, -1]$

$$\gamma_{ab} = \frac{\partial\, {}'L_c}{\partial L_a}\,\frac{\partial\, {}'L_d}{\partial L_b}\,\eta_{cd}\,. \tag{3.31}$$

Indeed one finds here for the mixing group by direct computation

$$\gamma_{11} = 1 - (\lambda_* - 1)\,\frac{L_2^2(L_1^2 - L_2^2)}{({}'L_1^2 + {}'L_2^2)^2}, \tag{3.32a}$$

$$\gamma_{22} = -1 - (\lambda_* - 1)\,\frac{L_1^2(L_1^2 - L_2^2)}{({}'L_1^2 + {}'L_2^2)^2}, \tag{3.32b}$$

$$\gamma_{21} = \gamma_{12} = (\lambda_* - 1)\,\frac{L_1 L_2(L_1^2 - L_2^2)}{({}'L_1^2 + {}'L_2^2)^2}. \tag{3.32c}$$

Thus in the vicinity of the light cones $(L_1^2 = L_2^2)$ the pullback metric γ_{ab} coincides with the Lorentz metric η_{ab} for any value of the group parameter λ_*.

It is just this behavior of the mixing group around the light cones and the axes which is responsible for the effects of fusion and separation. Indeed following the orbit of some point (L_1,L_2) during a group transformation with increasing group parameter [expansive transformation: $\lambda_* > 1$, Fig. 2(a)], the light cone $(L_1^2 = L_2^2)$ is approached by the orbit more and more and this means nothing else than that the values of both amplitude fields $L_a(x)$ become identical (up to sign): this is the effect of amplitude fusion $[L_1(x) \approx L_2(x)]$, Fig. 3(b). Since these expansive transformations have $\lambda_* > 1$ which means $\sigma_* = -1$, see Eq. (3.24), the fusion effect for the amplitudes does occur for the negative mixtures only. The opposite effect of amplitude separation [Fig. 3(c)] occurs for the positive mixtures $(\sigma_* = +1)$ because here the mixing transformations act in a contractive manner [Fig. 2(b)] due to $\lambda_* < 1$. This contractive group action lets the smaller one of both amplitudes L_a decay to zero but leaves the larger amplitude with a finite value. Thus if both amplitudes L_a are considered as fields over some region of space-time they must become separated in the sense that they are forming space-time cells which are carrying only one of the two amplitudes with the other one having decayed to zero. The geometric shape of these cells is determined by the surfaces carrying zero values of the amplitudes, e.g., by the two-spheres in three-dimensional space when the field configuration is spherically symmetric.

As an example for such a static and spherically symmetric two-particle configuration consider the nonrelativistic ground $({}^{(0)}L_1)$ and first excited state $({}^{(0)}L_2)$ of the one-particle Coulomb force problem [14] which together form a pure two-particle state in RST:

$$ {}^{(0)}L_1(y) = e^{-y}, \tag{3.33a}$$

$$ {}^{(0)}L_2(y) = \frac{1}{\sqrt{8}}(1 - y/2)e^{-y/2}. \tag{3.33b}$$

Here the radial coordinate $r(=\sqrt{|x^i x_i|})$ has been rescaled into the dimensionless variable y

$$ y = z_{\text{ex}} \frac{r}{a_B} \tag{3.34}$$

by means of the Bohr radius $a_B = \hbar^2/Me^2$ and the number z_{ex} of nuclear charge units. Both one-particle states are orthonormal in the following sense:

$$ \int_0^\infty dy y^{2\,(0)}L_a(y)^{(0)}L_b(y) = \tfrac{1}{4}\,\delta_{ab} \qquad (a,b|=1,2). \tag{3.35}$$

Now one takes the pure two-particle state (3.33) as the starting point for a mixing transformation (3.23) and looks for the resulting amplitudes $L_a(y)$ $(a=1,2)$

$$ L_a(y) = \Lambda_a({}^{(0)}L_1(y), {}^{(0)}L_2(y)), \tag{3.36}$$

see Fig. 3. Obviously for negative mixtures $(\sigma_* = -1 \Leftrightarrow \lambda_* > 1)$ the amplitudes $L_a(y)$ are (anti) fusing into one another $[L_1^2(x) \approx L_2^2(x)$, Fig. 3(b)] whereas for the positive mixtures $(\sigma_* = +1 \Leftrightarrow \lambda_* < 1)$ the amplitudes are separating [Fig. 3(c)]. Thus the amplitudes L_a do react in the opposite sense when compared with the charge densities ρ_a (3.18) which are fusing for the positive mixtures and separating for the negative mixtures [14]. Clearly the reason for this is that the densities ρ_a are connected with the inverse mixing transformations $[R]$ (3.20) in place of the present $[\Lambda]$ (3.22) for the amplitudes L_a.

C. Exact solutions

With all these preparations, it has now become easy to find exact solutions $L_a(x)$ for the coupled amplitude equations (3.8) despite the fact that the mixture potentials W_a [as given by Eq. (3.13)] are highly nonlinear functions of the desired amplitudes L_a. The reason is that the solutions of the intricate amplitude system (3.8) are connected with the solutions of the simple uncoupled Klein-Gordon system (2.51) just by an appropriate mixing transformation $[\Lambda]$ (3.22). On the other hand, exact solutions $[{}^{(0)}L_a(x)$, say] of the simple one-particle Klein-Gordon equations (2.51) are known for many situations of physical interest (for the exact solutions of the static relativistic Coulomb force problem see, e.g., Ref. [10]). Therefore the corresponding solutions $L_a(x)$ of the full amplitude equations (3.8) are also known exactly. In this sense, the amplitude fields L_a pictured in Fig. 3 represent the exact two-particle solutions of the full amplitude system (3.8) for that time-independent two-particle state in the Coulomb field which refers to the ground and first excited one-particle states; and it was merely for the sake of convenience that we restricted ourselves to their nonrelativistic limit form (3.33) (differring from the exact relativistic form only in order α_*^2 [10]).

In order to verify our claim, we first have to show that the mixing transformations $[\Lambda]$ (3.22) actually leave form-invariant the amplitude equations (3.8) which then defines the transformation law for the mixture potentials W_a. This goal can be achieved by exploiting the group property of the mixing transformations. More precisely, one composes the general transformation $\Lambda: L_a \to {}'L_a$ by first passing over from the original amplitudes L_a to the pure-state solutions ${}^{(0)}L_a$ of the simple Klein-Gordon system (2.51) which has vanishing mixture potentials; and then one transforms these pure-state amplitudes ${}^{(0)}L_a$ into the desired new amplitudes ${}'L_a: L_a \Rightarrow {}^{(0)}L_a \Rightarrow {}'L_a$. Obviously the desired proof is accomplished by this composition if the final mixture potentials ${}'W_a$, expressed in terms of the final amplitudes ${}'L_a$, are just of the form (3.13) being prescribed by RST. Carrying now through this line of arguments, we shall extensively benefit from the *quasilinearity* of the mixing transformations.

First observe that the d'Alembertian of the new amplitudes $\Box\, {}'L_a$ reads in terms of the old variables L_a $(a=1,2)$

$$ \Box\, {}'L_a = \sum_{b,c|=1}^{2} \frac{\partial^2\, {}'L_a}{\partial L_b \partial L_c}(\partial^\mu L_a)(\partial_\mu L_b) + \sum_{b=1}^{2} \frac{\partial\, {}'L_a}{\partial L_b}\Box L_b. \tag{3.37}$$

Thus if the old amplitudes L_a obey the system (3.8), the new amplitudes ${}'L_a$ must obey the following equations $(d\neq a)$:

$$\Box {}'L_a+\left(L_1\frac{\partial {}'L_a}{\partial L_1}+L_2\frac{\partial {}'L_a}{\partial L_2}\right)\left\{\left(\frac{Mc}{\hbar}\right)^2-K_{a\mu}K_a^\mu+{}'W_a\right\}$$

$$=\left(L_1\frac{\partial {}'L_a}{\partial L_1}+L_2\frac{\partial {}'L_a}{\partial L_2}\right)\ {}'W_a-\sum_{b=1}^{2}L_b\frac{\partial {}'L_a}{\partial L_b}W_b$$

$$+L_d\frac{\partial {}'L_a}{\partial L_d}[K_{1\mu}K_1^\mu-K_{2\mu}K_2^\mu]$$

$$+\sum_{b,c|=1}^{2}\frac{\partial^2 {}'L_a}{\partial L_b\partial L_c}(\partial^\mu L_b)(\partial_\mu L_c). \tag{3.38}$$

Now in order that these new amplitudes ${}'L_a$ obey the same equations as the old ones (3.8), i.e.,

$$\Box {}'L_a+{}'L_a\left\{\left(\frac{Mc}{\hbar}\right)^2-K_{a\mu}K_a^\mu+{}'W_a\right\}=0, \tag{3.39}$$

one requires quasilinearity of the mixing transformations $(a=1,2)$:

$$L_1\frac{\partial {}'L_a}{\partial L_1}+L_2\frac{\partial {}'L_a}{\partial L_2}={}'L_a\,, \tag{3.40}$$

and furthermore one lets the mixture potentials W_a transform in the following way $(d\neq a)$:

$${}'L_a\ {}'W_a=\sum_{b=1}^{2}L_b\frac{\partial {}'L_a}{\partial L_b}W_b-L_d\frac{\partial {}'L_a}{\partial L_d}[K_{1\mu}K_1^\mu-K_{2\mu}K_2^\mu]$$

$$-\sum_{b,c|=1}^{2}\frac{\partial^2 {}'L_a}{\partial L_b\partial L_c}(\partial^\mu L_b)(\partial_\mu L_c). \tag{3.41}$$

Here, for our proof we imagine now that the old amplitudes L_a coincide with the pure-state amplitudes ${}^{(0)}L_a$ obeying the ordinary Klein-Gordon system (2.51) with vanishing mixture potentials $(W_a\equiv 0)$ and the new amplitudes ${}'L_a$ are thought to be just the desired solutions of the amplitude system (3.8) $({}'L_a\rightarrow L_a)$. The corresponding mixture potentials are then obtained from Eq. (3.41) as $(b\neq a)$

$${}'W_a=-(-1)^a\frac{L_b}{L_a}\frac{\partial {}'L_a}{\partial L_b}[K_{1\mu}K_1^\mu-K_{2\mu}K_2^\mu]$$

$$-\frac{1}{{}'L_a}\sum_{b,c|=1}^{2}\frac{\partial^2 {}'L_a}{\partial L_b\partial L_c}(\partial^\mu L_b)(\partial_\mu L_c). \tag{3.42}$$

As explained above, our proof requires now to identify the present mixture potentials W_a (3.42) with their RST form (3.13). Thus it remains for us to prove the following two identifications, where the first one refers to the kinetic part of the mixture potentials

$$\frac{L_1}{{}'L_2}\frac{\partial {}'L_2}{\partial L_1}=\frac{L_2}{{}'L_1}\frac{\partial {}'L_1}{\partial L_2}=-g_s({}'L_1,{}'L_2) \tag{3.43a}$$

and the second one to their derivative part

$$-\frac{1}{{}'L_a}\sum_{b,c|=1}^{2}\frac{\partial^2 {}'L_a}{\partial L_b\partial L_c}(\partial^\mu L_b)\Bigg(\partial_\mu L_c=\sigma_*\frac{T_a}{{}'L_1^2+{}'L_2^2}\,\frac{({}'L_1\partial_\mu {}'L_2-{}'L_2\partial_\mu {}'L_1)({}'L_1\partial^\mu {}'L_2-{}'L_2\partial^\mu {}'L_1)}{\sigma_*(2\,{}'L_1\,{}'L_2)^2+C_*^2({}'L_1^2+{}'L_2^2)^2} \tag{3.43b}$$

with the functions $T_a=T_a({}'L_1,{}'L_2)$ being given by

$$T_1={}'L_1^2-3\,{}'L_2^2, \tag{3.44a}$$

$$T_2={}'L_2^2-3\,{}'L_1^2. \tag{3.44b}$$

The desired proof for the equations (3.43), (3.44) strongly relies now upon the property of quasilinearity (3.40) of the mixing transformations. Indeed this property is the definition for the homogeneous functions of the first degree and it is self-evident that our mixing transformations $[\Lambda]$ (3.22) are just of this type [together with the Lorentz transformations (3.26)]. But this observation leads us also immediately to the desired verification of the first requirement (3.43a). The reason is that the homogeneous functions (3.40) could be defined equally well as the solutions of the following system of partial differential equations:

$$\frac{\partial {}'L_1}{\partial L_1}=\frac{{}'L_1}{L_1}g_w\,, \tag{3.45a}$$

$$\frac{\partial {}'L_1}{\partial L_2}=-\frac{{}'L_1}{L_2}g_s\,, \tag{3.45b}$$

$$\frac{\partial {}'L_2}{\partial L_1}=-\frac{{}'L_2}{L_1}g_s\,, \tag{3.45c}$$

$$\frac{\partial {}'L_2}{\partial L_2}=\frac{{}'L_2}{L_2}g_w\,. \tag{3.45d}$$

Here the functions $g_w(L_1,L_2)$ and $g_s(L_1,L_2)$ must be homogeneous of degree zero

$$L_1\frac{\partial g_w}{\partial L_1}+L_2\frac{\partial g_w}{\partial L_2}=0, \tag{3.46a}$$

$$L_1\frac{\partial g_s}{\partial L_1}+L_2\frac{\partial g_s}{\partial L_2}=0, \tag{3.46b}$$

and must obey the constraint

$$g_w - g_s = 1. \tag{3.47}$$

But clearly our structure functions g_w and g_s (2.56),(2.57) are just of the required type (3.45)–(3.47) and thus the first condition (3.43a) is satisfied almost trivially.

On the other hand, the second condition (3.43b) involves the second-order derivatives of the homogeneous functions $'L_a$ which suggests to consider the second-order counterpart of the first-order relation (3.40):

$$L_1^2 \frac{\partial^2 {}'L_a}{(\partial L_1)^2} + 2L_1 L_2 \frac{\partial^2 {}'L_a}{\partial L_1 \partial L_2} + L_2^2 \frac{\partial^2 {}'L_a}{(\partial L_2)^2} = 0$$

$$(a = 1,2) \tag{3.48}$$

which holds quite generally for all quasilinear functions. But this forces the second-order derivatives of the new amplitudes into the following form ($a = 1,2$):

$$\frac{\partial^2 {}'L_a}{(\partial L_1)^2} = S_a L_2^2, \tag{3.49a}$$

$$\frac{\partial^2 {}'L_a}{\partial L_1 \partial L_2} = -S_a L_1 L_2, \tag{3.49b}$$

$$\frac{\partial^2 {}'L_a}{(\partial L_2)^2} = S_a L_1^2. \tag{3.49c}$$

For the special case of our mixing group $[\Lambda]$ (3.22) the functions $S_a({}'L_1, {}'L_2)$ are found by direct computation as

$$S_a = (\lambda_* - 1)\, {}'L_a \frac{T_a}{({}'L_1^2 + {}'L_2^2)^3}, \tag{3.50}$$

with the functions T_a being already specified through Eq. (3.44). However, the special shape of the second-order derivatives (3.49) together with the present result (3.50) recasts now the second requirement (3.43b) into the form

$$\begin{aligned}(L_1 \partial_\mu L_2 - L_2 \partial_\mu L_1)(L_1 \partial^\mu L_2 - L_2 \partial^\mu L_1) = {}& -\frac{\sigma_*}{\lambda_* - 1} \frac{({}'L_1^2 + {}'L_2^2)^2}{\sigma_* (2\,{}'L_1\,{}'L_2)^2 + C_*^2 ({}'L_1^2 + {}'L_2^2) \ldots)^2} \cdot ({}'L_1 \partial_\mu {}'L_2 - {}'L_2 \partial_\mu {}'L_1) \\ & \times ({}'L_1 \partial^\mu {}'L_2 - {}'L_2 \partial^\mu {}'L_1).\end{aligned} \tag{3.51}$$

Fortunately this is rather a triviality since it is again the homogeneity of the mixing transformations which lets the derivative terms transform in the following pleasant way:

$$L_1 \partial_\mu L_2 - L_2 \partial_\mu L_1 = \frac{1}{D_L} ({}'L_1 \partial_\mu {}'L_2 - {}'L_2 \partial_\mu {}'L_1) \tag{3.52}$$

with D_L being the determinant of the Jacobian

$$D_L = \det\left(\frac{\partial\, {}'L_a}{\partial L_b} \right). \tag{3.53}$$

For our special case of mixing group $[\Lambda]$ (3.22) one finds by direct computation

$$D_L = \frac{{}'L_1\,{}'L_2}{L_1 L_2} \frac{L_1^2 + L_2^2}{{}'L_1^2 + {}'L_2^2}, \tag{3.54}$$

which then immediately validates the remaining claim (3.51) [observe here also the relationship between the mixture index σ_* and group parameter λ_* (3.24) and the kinematical invariant (3.30)].

Thus the proof is perfect and the mixing group $[\Lambda]$ has been actually identified as the invariance group of the amplitude system (3.8) with the invariant form of the mixing potentials being given by Eq. (3.13). The presence of the mixing potentials in the amplitude system (3.8) is necessary in order to achieve the effects of (anti)fusion and separation as pictured in Fig. 3. This becomes immediately evident within the framework of the nonrelativistic limit where the relativistic system (3.8) in its time-independent specialization adopts the well-known Schrödinger eigenvalue form [15]

$$-\frac{\hbar^2}{2M} \Delta L_1 + (E_{B,1} + V_C + V_1) L_1 = 0, \tag{3.55a}$$

$$-\frac{\hbar^2}{2M} \Delta L_2 + (E_{B,2} + V_C + V_2) L_2 = 0. \tag{3.55b}$$

Here one imagines the external electromagnetic potential ${}^{(\mathrm{ex})}A_\mu$ (3.2) to be given in form of some binding potential V_C (e.g., Coulomb potential) confining the two particles to the vicinity of an attractive force center (e.g., the nucleus), furthermore the mixing potentials V_a are defined simply by

$$V_a = \frac{\hbar^2}{2M} W_a, \tag{3.56}$$

and $E_{B,a}$ are the binding energies. According to the present results, the exact solutions $L_a(\vec{r})$ of the coupled Schrödinger system (3.55) can be obtained by first solving the ordinary Schrödinger equations with vanishing mixture potentials ($V_a \equiv 0$)

$$-\frac{\hbar^2}{2M} \Delta^{(0)} L_1 + (E_{B,1} + V_c)^{(0)} L_1 = 0, \tag{3.57a}$$

$$-\frac{\hbar^2}{2M} \Delta^{(0)} L_2 + (E_{B,2} + V_c)^{(0)} L_2 = 0, \tag{3.57b}$$

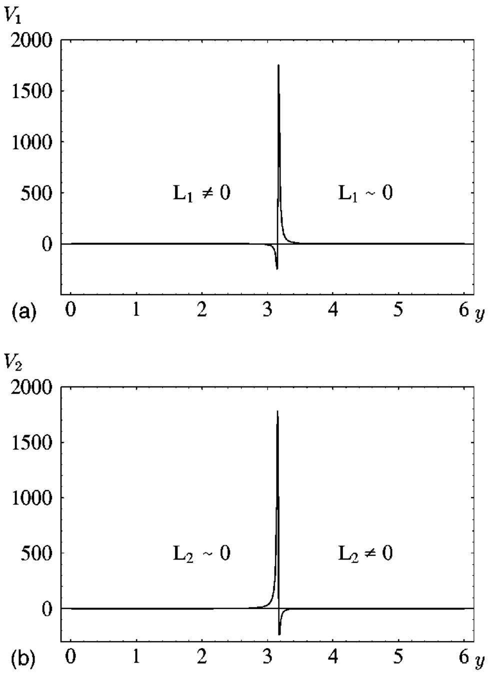

FIG. 4. Separation potentials ($\sigma_* = +1$). For positive mixtures ($\sigma_* = +1$) the amplitude *separation* [see Fig. 3(c)] is achieved by the specific shape of the mixture potentials W_a (3.13) or [V_a (3.56)], which act as repulsive walls confining the first amplitude L_1 to the left of the wall and the second amplitude L_2 to its right. The amplitude field is confined to that side of the repulsive wall where the mixture potential becomes attractive just in front of the wall (choice of mixing parameter: $C_* = 0.02$, potentials V_a measured in atomic energy units of $e^2/2a_B = 13.61\,[\mathrm{eV}]$).

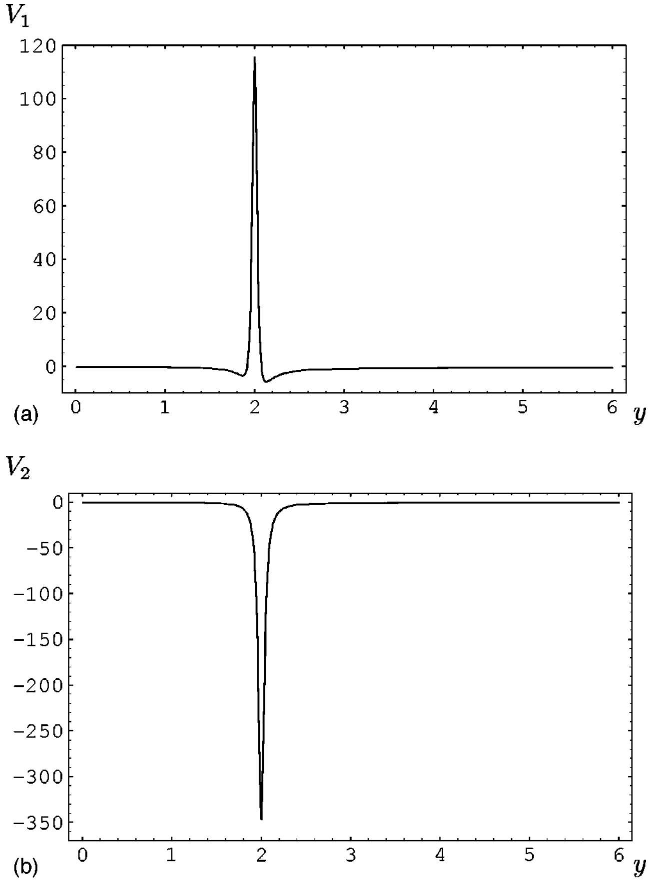

FIG. 5. Fusion potentials ($\sigma_* = -1$). For negative mixtures ($\sigma_* = -1$), the fusion of amplitudes comes about through the attractive and repulsive actions of the mixture potentials W_a, which cause the amplitudes to deform into a common shape (differing only in sign: $L_1^2 \approx L_2^2$). The mixture potential V_1 for the first amplitude L_1 (a) acts similar as a *repulsive* δ function and thus generates the corresponding salient point of L_1 [Fig. 3(b)]. The second mixture potential V_2 (b) acts as an *attractive* δ function but does not generate a salient point for L_2 [Fig. 3(b)] because the second amplitude L_2 is zero at the point where V_2 attains its minimal value (choice of mixture parameter: $C_* = 1.0001$).

which is the nonrelativistic limit of the ordinary Klein-Gordon system (2.51),(2.52), and then applying an appropriate mixing transformation of the kind described above (i.e., ${}^{(0)}L_a \Rightarrow L_a$). The resulting solutions $L_a(\vec{r})$ must then be substituted into the mixture potentials $W_a(\vec{r})$ [or $V_a(\vec{r})$,] and this yields the appropriate mixture forces to be applied for the fusion and separation effects (Figs. 4 and 5). Clearly it is suggestive here to consider these latter effects as the RST counterparts of the analogous effects in conventional quantum theory where they are brought about by (anti)symmetrization of the two-particle wave functions.

D. Degeneracy

It is important to remark that up to now we did neglect the interelectronic interactions (of electromagnetic nature), see Eq. (3.1). This, however, is not mandatory for the present mechanism of generation of exact solutions by means of the mixing group; quite on the contrary: even if we take into account the electromagnetic interparticle forces, the pure-state solutions of the Klein-Gordon system (2.51) are still connected with the corresponding solutions of the coupled amplitude system (3.8) by a mixing transformation. Surprisingly enough even if the interelectronic forces are taken into account, the mass eigenvalues M_a are not changed by the mixing transformations which thus turn out to be of the isospectral type being frequently considered in soliton theory [21].

Thus, e.g., for a static bound two-particle system, the present mixture degeneracy is not broken in the analogous way as the exchange degeneracy of the conventional quantum theory is broken by the interparticle interactions (1.10). This astonishing phenomenon is due to the fact that the kinetic fields $K_{a\mu}$ remain invariant under the mixing transformations, just as is the case for the currents $j_{a\mu}$ and scalar densities ρ_a [otherwise the relationship (3.15) would be inconsistent]. The mixture invariance of the densities ρ_a follows directly from the fact that the mixing group $[\Lambda]$ (3.22) is, by its very definition, the inverse of the group $[R]$ (3.20). Therefore if the pure-state solutions ${}^{(0)}L_a$ of the Klein-Gordon system (2.51) generate the densities ${}^{(0)}\rho_a$ according to the prescription (3.16), then these same densities (and therefore also the currents $j_{a\mu}$) are generated also by the transformed solutions $L_a : \rho_a[L_1, L_2] \equiv \rho_a[{}^{(0)}L_1, {}^{(0)}L_2]$. The only difference to the situation with neglected interparticle forces (3.2) is that the kinetic fields contain now also the potential ${}'A_{a\mu}$ generated by the other particle, i.e., one has now in place of Eq. (3.2)

$$K_{a\mu} = \partial_\mu \alpha_a + {}^{(\mathrm{ex})}A_\mu - {}'A_{b\mu} \quad (a \neq b), \tag{3.58}$$

where the single-particle potentials ${}'A_{a\mu}$ are generated by the single-particle currents $j_{a\mu}$ (3.15) in the usual way:

$$\Box\, {}'A_{a\mu} = 4\pi\alpha_* j_{a\mu}. \tag{3.59}$$

Especially for a static bound system one introduces the mass eigenvalues M_a through specifying the gauge scalars α_a (3.58) through

$$\partial_\mu \alpha_a = \frac{M_a c}{\hbar} \hat{t}_\mu, \tag{3.60}$$

$$(\hat{t}_\mu \doteq \partial_\mu t, \hat{t}^\mu \hat{t}_\mu = +1)$$

and thus the eigenvalues M_a are seen to inherit their mixture invariance directly from the invariance of the kinetic fields under the mixing transformations. This invariance of the relevant physical variables (such as densities ρ_a, currents $j_{a\mu}$, eigenvalues M_a) with respect to the mixing transformations $[\Lambda]$ is what one may understand to be a kind of mixture degeneracy. Thus we arrive at the result that this degeneracy of the mixtures in RST survives the switching on of the electromagnetic interparticle interactions (${}'A_{a\mu}$) which, however, break the exchange degeneracy in conventional quantum theory.

It seems that in RST the mixture degeneracy cannot be broken until the exchange fields $N_{a\mu}, Q_{a\mu}$ are taken into account. The reason is that the latter are able to modify the RST currents [compare the simplified currents (3.15) to their general form (2.27)] and this may be understood as the true RST analog of the redistribution of probability densities in conventional quantum theory (occurring by means of symmetrization and antisymmetrization of the two-particle wave functions). Such a dichotomic redistribution of the densities is not observable (within the framework of the conventional theory) as long as the interparticle interactions remain switched off. However this dichotomy becomes observable when the interparticle interactions are switched on (e.g., via the observation of different frequencies of spectral lines due to the atomic *ortho*- and *para*-states).

Analogous results do hold also in RST: Here the mixture degree of freedom [as being quantified by the value of the mixture parameter C_*, or better λ_* (3.24)] will not be observable as long as the exchange interactions are switched off. In fact, in such a situation all mixtures share the same single-particle densities ρ_a (2.21), current densities $j_{a\mu}$ (3.15), and mass eigenvalues M_a (3.60), which must be conceived as the observable objects also in RST. It is true, the single-particle amplitudes L_a would be different for mixtures with different mixing parameter λ_*, see Fig. 3, but the amplitudes L_a themselves are not counted as observable objects in RST [recall here that the amplitude field L must be multiplied with an appropriate renormalization factor Z in order to get an ‘‘observable’’ density ρ, q, s (2.30)]. However, when the exchange fields are excited nontrivially, there no longer exists such a nice integration constant C_* for parametrizing the continuously degenerate mixtures. But what still remains is the dynamical separation of mixtures of the positive and negative kind which are expected to be equipped now with *different* mass eigenvalues M_a, densities ρ_a, and currents $j_{a\mu}$. This then is of course an observable effect, namely, the RST counterpart of the breaking of the conventional exchange degeneracy by switching on the interparticle interactions.

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