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Multipartite entanglement that is robust against disposal of particles

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We investigate the entanglement properties of multiparticle systems, concentrating on the case where the entanglement is robust against the disposal of particles. Two qubits—belonging to a multipartite system—are entangled in this sense if their reduced density matrix is entangled. We introduce a family of multiqubit states, for which one can choose for any pair of qubits independently whether they should be entangled or not as well as the relative strength of the entanglement, thus providing the possibility to construct all kinds of ''entanglement molecules.'' For some particular configurations, we also give the maximal amount of entanglement achievable.

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Entanglement is at the heart of quantum information theory. In recent years, there has been an ongoing effort to characterize quantitatively and qualitatively entanglement. While for bipartite systems this problem is essentially solved, it remains still open for multipartite systems. In this case, there exist several possible approaches to identify different kinds of multipartite entanglement, and many interesting phenomena related to multipartite entanglement have been discovered [1].

Here we concentrate on bipartite aspects of multipartite entanglement; in particular, on bipartite entanglement that is robust against the disposal of particles. We consider N spatially separated parties $A_1, \ldots, A_N$, each possessing a qubit. We first investigate the N-party Greenberger-Horne-Zeilinger (GHZ) state [2],

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + |1^{\otimes N}\rangle), \qquad (1)$$

which is considered to be a maximally entangled state (MES) of N particles in several senses. For example, one can create a MES shared by any two of the parties with the help of a (local) measurement performed by the remaining ones. Thus any two particles are potentially entangled, i.e., when allowing for the assistance of the other parties, bipartite entanglement can be obtained from state $|\mathrm{GHZ}\rangle$. However, it is essential that the remaining $(N-2)$ parties perform measurements to assist the other two parties to share entanglement. If, however—for some reason—the information about only one of the particles, say A_N, is lost (or party A_N decides not to cooperate with the remaining ones), the state of the remaining parties is only classically correlated and thus not entangled. In particular, the reduced density operator of any two parties is separable.[1] When considering the reduced density operator of two parties, we deal with the situation where the information about all remaining particles is not accessible (or the remaining parties are not willing to cooperate).

We thus say that two particles are entangled if their reduced density operator is nonseparable, i.e., the two particles share entanglement, independent what happens to the remaining particles. Such a definition is very suitable from a practical point of view, as there are certain multipartite scenarios where one is interested in entanglement properties of pairs of parties, which are independent of other parties. Note that in this sense, the state $|\mathrm{GHZ}\rangle$ contains no (bipartite) entanglement at all.

But are there N-particle states that are still entangled when tracing out *any* $(N-2)$ particles; i.e., are there states where all particles are entangled with all other particles? And if this is possible, what is the maximal amount of entanglement the remaining two parties can share? In this paper, we will answer these questions and we will consider the more general setup where some parties are entangled, while some others are not. For example, for $N=3$ one may have that the reduced density operators ρ_{12} and ρ_{13} remain entangled, but ρ_{23} is separable. We will show that one can have all possible configurations of this kind; i.e., there exist states where one can choose for each of the reduced density operators $\rho_{kl}, k < l \in \{1, \ldots, N\}$ independently of whether they should be entangled or not. This allows one to build general structures of N particle states, which we call ''entanglement molecules'' in the spirit of the generalization of Wootters' idea of an ''entangled chain'' [3], where one has a string of qubits, each qubit being entangled only with its nearest neighbors. We are considering more general setups; e.g., closed rings of particles where one only has nearest-neighbor entanglement, (finite) strings with distance-dependent entanglement (also second, third, and so on neighborhood entanglement), entanglement fullerenes (like the C_{60} molecule), or more generally all possible setups of this kind one may imagine.

There are $N(N-1)/2$ different bipartite reduced density operators ρ_{kl} that may be either separable or not. If the reduced density operator ρ_{kl} is nonseparable, this automatically implies that a MES shared between parties A_k and A_l can be distilled (when allowing for several copies of the state), even without the help of the remaining parties. This is due to the fact that for two-qubit systems, inseparability is

[1]Given an N-partite state ρ, the reduced density operator ρ_{12} of party A_1 and A_2 is defined as $\rho_{12} \equiv \mathrm{tr}_{3,\ldots,N}(\rho)$. The operator ρ_{12} is separable if it can be written as a convex combination of product states.

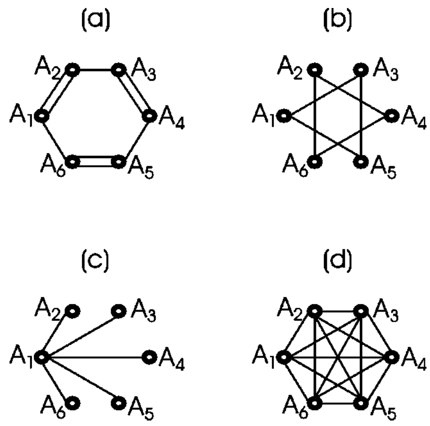

FIG. 1. Several kinds of entanglement molecules for $N=6$. (a) Entanglement ring—double lines indicate stronger entanglement between particles. (b) All even (odd) particles are entangled respectively. (c) One particle A_1 equally entangled with five others. (d) All particles equally entangled.

equivalent to distillability [4]. In fact, the remaining parties can by no means prevent parties A_k and A_l from distilling a MES. In a diagram, this will be visualized by a line between particles A_k and A_l that represents entanglement between the two parties in question (see Fig. 1). Each of the particles may be entangled with one (or more) of the remaining $(N-1)$ particles. In particular, one can have that any particle is entangled with all the remaining ones. Clearly, it is interesting to ask how strong these ''bindings'' (the entanglement between two particles) can be. Therefore, one has to quantify the entanglement of the bipartite reduced density operators ρ_{kl}. In this work, we choose as a measure of entanglement the concurrence $\mathcal{C}$ (for a definition of the concurrence see, e.g., [5]). On the one hand, we follow the lines suggested in [3,5]; on the other, we have that the entanglement of formation[2]—the amount of entanglement required to prepare a state ρ—is monotonically increasing with the concurrence $\mathcal{C}$, and thus the concurrence itself may be used to measure the strength of the bindings. For two special cases of particular interest, we will give the states with the maximal achieveable strength of the bindings.

Let us start by introducing a family of N qubit states, which includes all possible configurations of ''entanglement molecules.'' First we specify for each of the reduced density operators ρ_{kl} whether it should be distillable or not, i.e., whether entanglement between the parties A_k and A_l can be distilled—without the help of the remaining parties—or not. Let $I=\{k_1 l_1, \ldots, k_M l_M\}$ be the set of all those pairs where distillation should be possible; i.e., for $kl \in I$, we have that ρ_{kl} is distillable. We define the state

$$|\Psi_{ij}\rangle \equiv |\Psi^+\rangle_{ij} \otimes |0\ldots 0\rangle_{\text{rest}}, \tag{2}$$

that is, the particles A_i and A_j are in a MES, namely $|\Psi^+\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$, and the remaining particles are disentangled from each other and from $A_i A_j$. We now introduce a family of states that has the desired properties:

$$\rho_I = \frac{1}{M} \sum_{kl \in I} x_{kl} |\Psi_{kl}\rangle\langle\Psi_{kl}|, \tag{3}$$

where $M \equiv \Sigma_{kl \in I} x_{kl}$ is a normalization factor. It is straightforward to calculate the reduced density operators ρ_{kl}. In the standard basis, one can check for $kl \in I$ that ρ_{kl} is of the form

$$\rho_{kl} = \frac{1}{M} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & x_{kl}/2 & 0 \\ 0 & x_{kl}/2 & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

while for $mn \notin I$ one obtains

$$\rho_{mn} = \frac{1}{M} \begin{pmatrix} \tilde{a} & 0 & 0 & 0 \\ 0 & \tilde{b} & 0 & 0 \\ 0 & 0 & \tilde{c} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

It is simple to calculate the concurrence in both cases:

$$\mathcal{C}_{kl} = \frac{x_{kl}}{M} \quad \text{iff} \quad kl \in I,$$

$$\mathcal{C}_{mn} = 0 \quad \text{iff} \quad mn \notin I, \tag{6}$$

where we denote $\mathcal{C}_{kl} \equiv \mathcal{C}(\rho_{kl})$. We have that in $\mathbb{C}^2 \otimes \mathbb{C}^2$ systems, nonzero concurrence automatically implies distillability of the state, while zero concurrence implies separability. Thus it is already clear that ρ_I has the desired properties, i.e., entanglement between A_k and A_l can be distilled if $kl \in I$. We see that we can arbitrarily choose the relative strength of the bindings (measured by the concurrence) via the positive coefficients x_{kl}. It is now straightforward to explicitly construct the examples illustrated in Fig. 1: In (a) we have that $x_{12}=x_{34}=x_{56}=2/9$, $x_{23}=x_{45}=x_{16}=1/9$ (entanglement ring); in (b) $x_{kl}=1/6$ if both k and l are even (odd), respectively and zero otherwise [all even (odd) particles are equally entangled]; in (c) we have that $x_{1l}=1/5$, while all other $x_{kl}=0$ (one particle equally entangled with five other ones); finally, in (d) $x_{kl}=1/15$ (all particles equally entangled). In a similar way, one can construct all examples mentioned in the introduction, such as strings with distance dependent entanglement or entanglement fullerenes.

Note that in this construction, we have that

$$\sum_{i<k} \mathcal{C}_{ik} = 1, \tag{7}$$

which shows that in a situation where all bindings have the same strength, the concurrence of the corresponding reduced density operators is determined by the total number of bind-

[2]The entanglement of formation is given by $E_f(\rho)=h(\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathcal{C}^2})$, where $\mathcal{C}$ is the concurrence and h is the binary entropy function $h(x)=-x\log_2 x-(1-x)\log_2(1-x)$.

ings, i.e., the number of elements in I. Hence, a state constructed in this way, where each particle is entangled with all others (i.e., all possible reduced density operators are distillable) and the strength of all bindings is equal has $\mathcal{C}_{kl} = 2/[N(N-1)]$. As we shall see next, this is, however, not the maximum value one can achieve for this particular configuration. In the following, we will analyze the two situations illustrated in Figs. 1(c) and 1(d), namely, (i) all particles are equally entangled and (ii) one particle that is equally entangled with $(N-1)$ others, and determine the maximum strength of the bindings for $N=3$.

(i) All particles pairwise entangled. We start with the case where all particles are equally entangled with each other [see also Fig. 1(d)]. As shown in [6], the state

$$|W\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle) \tag{8}$$

is the state of three qubits whose entanglement has the highest degree of endurance against the loss of one of the three qubits. In particular, $|W\rangle$ maximizes the function

$$\mathcal{E}_{\min}(\psi) \equiv \min(\mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}), \tag{9}$$

and has $\mathcal{C}_W \equiv \mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 2/3$. It follows that $|W\rangle$ is the three-qubit state where all particles are equally entangled and the entanglement—measured by the concurrence $\mathcal{C}$—is maximal. Comparing $|W\rangle$ with ρ_I of the form (3) with $I = \{12,13,23\}$, one immediately observes that $\mathcal{C}_{\rho_I} = 1/3$, while $\mathcal{C}_W = 2/3$.

More generally, let us consider the N-party form $|W_N\rangle$ of the state $|W\rangle$, defined as

$$|W_N\rangle \equiv 1/\sqrt{N}|N-1,1\rangle, \tag{10}$$

where $|N-1,1\rangle$ denotes the (unnormalized) totally symmetric state including $N-1$ zeros and one 1, e.g., $|2,1\rangle = |001\rangle + |010\rangle + |100\rangle$. As shown in [6], state $|W_N\rangle$ is an N qubit state with all reduced density operators equal and the concurrences given by $\mathcal{C}_{kl} = 2/N$, which has to be compared to $\mathcal{C}_{\rho_I} = 2/[N(N-1)]$, ρ_I being a state of the family (3) where all particles are equally entangled. Recently it was shown that $\mathcal{C}_{kl} = 2/N$, $\forall k \neq l$ is indeed the maximal value achievable [7].

(ii) One particle entangled with $N-1$ others. We consider now the case where one particle is equally entangled with $(N-1)$ others and determine the maximal possible strength $\mathcal{C}$ of the bindings [see Fig. 1(c)]. A related problem of this kind, namely, the question of optimal entanglement splitting, i.e., the optimal way for a party B to equally distribute its initial entanglement (shared with a party A) among several partners, was recently analyzed by Bruß in [8].

As shown in [5], we have for $N=3$ that $\mathcal{C}_{12}^2 + \mathcal{C}_{13}^2 \leq 1$; i.e., for $\mathcal{C}_{12} = \mathcal{C}_{13} \equiv \mathcal{C}$ we have that $\mathcal{C} \leq 1/\sqrt{2}$. This value is achieved by the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{2}(|001\rangle + |010\rangle). \tag{11}$$

More generally, it was conjectured in [5] that the inequality $\Sigma_{k=2}^N \mathcal{C}_{1k}^2 \leq 1$ also holds. If we demand again that $\mathcal{C}_{1k} = \mathcal{C}_{1l} \equiv \mathcal{C} \forall\ k,l > 1$, we find that $\mathcal{C} \leq 1/\sqrt{N-1}$. This value is obtained by the state

$$|\psi\rangle = a|1\rangle|00\ldots 0\rangle + b|0\rangle|N-2,1\rangle, \tag{12}$$

with $a = 1/\sqrt{2}$ and $b = 1/\sqrt{2(N-1)}$, where $|N-2,1\rangle$ is again an (unnormalized) totally symmetric state including $(N-2)$ zeros and one 1. For a state of the form (3), where one particle is equally entangled with $(N-1)$ others, one obtains $\mathcal{C}_{1k} = 1/(N-1)$.

For practical purposes, it may sometimes be useful to consider the fidelity of the reduced density operator (i.e., the maximal overlap of the reduced density operator with any MES)—which is generally not a proper measure of entanglement—instead of the concurrence. The fidelity of the reduced density operator, however, indicates the achievable quality to perform certain quantum information tasks, e.g., teleportation [9]. The fidelity of a density operator ρ is defined as

$$F_\rho = \max\langle\Phi|\rho|\Phi\rangle, \tag{13}$$

where the maximum is taken over all maximally entangled states $|\Phi\rangle$. In the situation we consider, where one particle is equally entangled (i.e., F_ρ is equal for all reduced density operators ρ_{1k}) with $(N-1)$ others, one can derive a bound for the maximum fidelity F_ρ of the reduced density operators ρ_{1k} using results from optimal cloning [10]. Given a density operator ρ with a certain fidelity F_ρ, one can use ρ to teleport [9] the (unknown) state of a particle. As shown by the Horodecki [11], one can find a teleportation protocol that works equally well for all input states and has the maximum teleport fidelity $F_t = (2F_\rho + 1)/3$. In our situation, we have that particle A_1 is entangled with $(N-1)$ other particles. Thus, when performing a certain teleportation protocol [12,13], one obtains $(N-1)$ (imperfect) clones of a state at the locations $A_2, \ldots, A_N$, where the quality of the clones is determined by the teleport fidelity F_t of the corresponding reduced density operators. As shown by Werner [10], the maximum cloning fidelity of a $1 \rightarrow (N-1)$ cloner is given by $F_c = [2(N-1)+1]/[3(N-1)]$, from which it follows that the teleport fidelity F_t of the reduced density operators must fulfill $F_t \leq F_c$; otherwise one could construct in this way a cloning machine that works better than the optimal one, which is clearly impossible. We thus have that

$$F_\rho \leq \frac{1}{2} + \frac{1}{2(N-1)}. \tag{14}$$

For the state $|\psi\rangle$ that maximizes the concurrence, we find that it does not obtain the maximal possible fidelity F_ρ. However, there exist states for which the maximal possible value of the fidelity is obtained. For example, a state of the form (12) with $a = 1/\sqrt{N(N-1)}$ and $b = \sqrt{(N-1)/N}$ has the desired properties, as well as the ‘‘telecloning state’’ introduced in [12]. On the other hand, a state of the form (3), where one particle is equally entangled with $(N-1)$ others has $F_\rho=[2+N]/[4(N-1)]$, much smaller than the optimal value.

Note that the maximum value of the entanglement (measured by the concurrence) of a given qubit with its neighbors is not simply determined by the number of entangled neighbors, but also by the properties of the neighbors (i.e., the number of particles to which the neighboring particles are entangled). This can be seen by noting that (i) in the case where one particle is equally entangled with two others (which are disentangled among themselves), the maximal value for the concurrence is given by $\mathcal{C}=1/\sqrt{2}$; and (ii) in the case where three qubits are equally entangled, the maximum value is given by $\mathcal{C}=2/3$, which shows that the entanglement between systems A_2 and A_3 influences the maximum value of the entanglement between systems A_1A_2 and A_1A_3.

In summary, we have provided a family of states that allows us to construct all possible kinds of ''entanglement molecules,'' i.e., one can choose for any pair of qubits independently of whether a MES can be distilled without the help of the remaining parties or not. In addition, the relative strength of the bindings (measured by the concurrence) can be adjusted arbitrarily. We investigated more closely two particular configurations and provided states achieving the maximum value for the strength of the bindings.

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