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Playing a quantum game with a corrupted source

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The quantum advantage arising in a simplified multiplayer quantum game is found to be a *disadvantage* when the game's qubit source is corrupted by a noisy ''demon.'' Above a critical value of the corruption rate, or noise level, the coherent quantum effects impede the players to such an extent that the ''optimal'' choice of game changes from quantum to classical.

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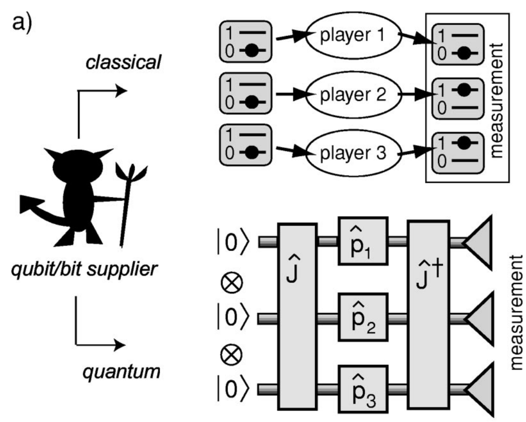
Information plays a fundamental role in both quantum mechanics [1] and games [2]. Recently, some pioneering advances have been made in the field of quantum games [3,4]. Eisert *et al.* [3] considered a quantum version of the famous $N=2$ player prisoner's dilemma [2]. The game showed a fascinating ''quantum advantage'' as a result of a novel payoff equilibrium. Benjamin and Hayden subsequently argued that this equilibrium [3] results from an asymmetric restriction in the strategy set; with unrestricted strategies, it is impossible for such special ''coherent quantum equilibria'' (CQE's) to arise in the maximally entangled $N=2$ player game [5]. Following our conjecture [6] that CQE's arise for $N \geqslant 3$ players, Benjamin and Hayden [7] created a prisoner's dilemma–like game for $N=3$ with a high payoff CQE [7]. This effect of ''two's company, three's a crowd'' is quite familiar in physical systems (both classical and quantum) where complex behaviors tend to emerge only for $N \geqslant 3$ interacting particles.

In this paper the quantum advantage arising in a simplified multiplayer quantum game, is found to be a *disadvantage* when the game's qubit source is corrupted by a noisy ''demon'' whose activity is unknown to the players. Above a critical value of the corruption rate, or noise level, the coherent quantum effects impede the players to such an extent that the classical game outperforms the quantum game; given the choice, the multiplayer system does better if it adopts classical rather than quantum behavior.

Following Ref. [7], $N=3$ players (or ''agents'') each receive a qubit in state $|0\rangle$ (or 0). The quantum-game qubits pass through an entangling $\hat{J}$ gate [3,7] [see Fig. 1(a)]. Without loss of generality [7], we take $\hat{J}=(1/\sqrt{2})(\hat{I}^{\otimes 3}+i\hat{F}^{\otimes 3})$ where $\hat{F}=\hat{\sigma}_x$. Hence, the input state $|0\rangle \otimes |0\rangle \otimes |0\rangle \equiv |000\rangle$ becomes $(1/\sqrt{2})(|000\rangle + i|111\rangle)$. The ith player's strategy s_i is her procedure for deciding which action to play. The strategy profile $s=(s_1,s_2,s_3)$ assigns one strategy to each player, and an equilibrium is a strategy profile with a degree of stability [7]; e.g., in a Nash equilibrium no player can improve her expected payoff by unilaterally changing her strategy. The payoff table [see Fig. 1(b)] bears some resemblance to the ''el farol'' bar problem [8]—the analogy is not strictly correct; however, it aids in understanding the payoff table. A (small) bar has seating capacity for two people, yet three people want to go. Action 0 (1) means do not go (go). State $|000\rangle$ means everyone stayed away. No one gains, but no one is annoyed that others gained while they lost: the net payoff is zero per player. State $|100\rangle$ means one person attended, had plenty of seats (i.e., two) but no company; her payoff is 1. The other two are annoyed that they didn't attend and gain from the available seat, hence each gets -9. State $|110\rangle$ means two attend; they each have a seat and have company, so they get 9. The third person, while not getting maximum enjoyment, is at least relieved that she did not make the effort to attend (making the bar overfull); she gets 1. State $|111\rangle$ means they all attend. They benefit from lots of company, but not enough seating; they all get 2. Input qubits must be supplied for each turn of the game. The players assume that the input qubits are always $|0\rangle$ (or 0), hence yielding the payoffs in Fig. 1(b). The classical game involves

a)



b)

Measured State	Payoff to Players
$\|000\rangle$	(0, 0, 0)
$\|100\rangle$	(1,-9,-9)
$\|010\rangle$	(-9, 1,-9)
$\|001\rangle$	(-9,-9, 1)
$\|011\rangle$	(1, 9, 9)
$\|101\rangle$	(9, 1, 9)
$\|110\rangle$	(9, 9, 1)
$\|111\rangle$	(2, 2, 2)

FIG. 1. Three-player game: (a) classical (top) and quantum (bottom) with input qubits/bits supplied by a (demonic) external source. (b) Payoff table.

TABLE I. Average payoffs for the classical game (players or ''agents'' denoted by ''a''). p is the probability of *not* flipping the input qubit. Average payoffs for input qubit 0 are shown as (. . .), those for input qubit 1 are shown as [. . .], while those for 50:50 mixture (i.e., $x=0.5$) of input qubits are shown without parentheses. \$ is the payoff averaged over the three players, for a given input qubit (0 or 1). $\overline{\$}$ is \$ averaged over the input qubit.

Class	$p=0$	$p=1/2$	$p=1$	C	\$	$\overline{\$}$
(i)		aaa(1/2)[1/2]1/2		1	(1/2)[1/2]	1/2
(ii)	a(21/4)[−17/4]1/2	aa(3/4)[1/4]1/2		3	(9/4)[−5/4]	1/2
(iii)	aa(11/2)[−9/2]1/2	a(3/2)[1/2]1		3	(25/6)[−17/6]	2/3
(iv)	aaa(2)[0]1			1	(2)[0]	1
(v)			aaa(0)[2]1	1	(0)[2]	1
(vi)	a(1)[1]1		aa(−9)[9]0	3	(−17/3)[19/3]	1/3
(vii)	aa(9)[−9]0		a(1)[1]1	3	(19/3)[−17/3]	1/3
(viii)	a(5)[−4]1/2	a(0)[0]0	a(−4)[5]1/2	6	(1/3)[1/3]	1/3
(ix)		aa(1/4)[3/4]1/2	a(−17/4)[21/4]1/2	3	(−5/4)[9/4]	1/2
(x)		a(1/2)[3/2]1	aa(−9/2)[11/2]1/2	3	(−17/6)[25/6]	2/3

only one of two possible states for each player's qubit at each stage (0 or 1) and hence one particular outcome in the payoff table of Fig. 1(b). In the quantum game a superposition of qubit states is possible and hence a superposition of outcome states will generally arise—the classical game is therefore embedded in the quantum game. As in conventional game theory [2], average payoffs are given by an expectation value over the possible measurement results.

Classical game players either leave the input qubit 0 unchanged, or flip it to 1. Allowing full knowledge of the payoff table, classical game players will search for the dominant strategy payoff (2,2,2) and hence choose action 1 [7]. Following the approach to $N=3$ player classical games of Ref. [9], each player is assigned a p value, where p is the probability of leaving the input qubit unflipped, i.e., *not* flipping the input qubit. For simplicity, suppose $p=0$, 1/2, or 1 instead of being continuous [9]. There are $3^3=27$ possible profiles or ''configurations'' (p_1,p_2,p_3). These yield ten ''classes'' each containing $C\geqslant 1$ configurations that are equivalent under the interchange of player label [9]. Table I shows the average payoffs for each configuration class. Given that the input is 0, the dominant strategy equilibrium corresponds to all players choosing $p=0$, i.e., class (iv) in Table I. Hence, although the continuous-parameter p space has been discretized to only three values, this description includes the desired dominant strategy equilibrium. The quantum game players, having followed the analysis of Ref. [7] in which the special (5,9,5) ''quantum'' payoff is presented, independently decide to play for the CQE given there. In particular, Ref. [7] shows that the strategies $\hat{I}$, $\hat{\sigma}_x$, and $(1/\sqrt{2})(\hat{\sigma}_x+\hat{\sigma}_z)$ yield a novel, high payoff CQE [7] given input qubit $|0\rangle$. We will assume that the set of $3^3=27$ strategy profiles formed from these three simple strategies contain the *only* strategy profiles subsequently chosen by the quantum game players. Again this choice is restricted—in particular the quantum game should include *all* SU(2) operations [5]. However, it allows for a straightforward comparison between quantum and classical games without the complication of continuous-parameter sets. The resulting Table II provides a simple quantum analog of Table I. $\hat{\sigma}_x$ corresponds to not qubit flipping with probability $p=0$, hence we denote it as ''$\hat{p}\equiv 0$.'' $(1/\sqrt{2})(\hat{\sigma}_x+\hat{\sigma}_z)$ corresponds to not qubit flipping with probability $p=1/2$, hence we denote it as ''$\hat{p}\equiv 1/2$.'' $\hat{I}$ corresponds to not qubit flipping with probability $p=1$, hence we denote it as ''$\hat{p}\equiv 1$.'' (This correspondence can be established by imagining switching off the J gates.) In both quantum and classical games, players are unable to communicate between themselves, hence they cannot coordinate which player picks which strategy. In the quantum game, this is more critical since the CQE [i.e., the Nash equilibrium given by class (viii) in Table II] involves players using different $\hat{p}$'s. [Although class (vii) has the same average payoff $\langle\hat{\$}\rangle=19/3$, it is not ''fair'' to all players and is not a Nash equilibrium.] The payoffs in Tables I and II [indicated by (. . .)] are in general quite different, i.e., the quantum and classical systems behave differently.

Now consider the effect of a noisy source created by an external ''demon'' [Fig. 1(a)]. The demon controls the input qubit corruption level; however, the players are unaware of his presence. This is reminiscent of a ''crooked house'' in gambling—players assume the source (e.g., deck of cards) is clean even though it may have been corrupted by the supplier (e.g., dealer). In Table I (II), the average payoffs with input qubits always 1 ($|1\rangle$) are shown as [. . .]. Again, the quantum and classical payoffs are generally quite different. Comparing the (. . .) and [. . .] entries in column $\langle\hat{\$}\rangle$ of Table II, and repeating this for column \$ of Table I, we see that the quantum game exhibits a *lower* symmetry than the classical game under interchange of the input qubit; e.g., there are two entries $\langle\hat{\$}\rangle=(19/3)$ in Table II, but only one entry [19/3]. A remarkable result is obtained if we now assume that the source contains equal numbers of $|0\rangle$ and $|1\rangle$ (or 0 and 1) qubits on average: the quantum and classical games now produce identical payoffs for a given class (i.e., $\overline{\langle\hat{\$}\rangle}=\overline{\$}$). Also, the resulting payoff entries for *each* p value

TABLE II. Average payoffs for the quantum game (players or ''agents'' denoted by ''a''). $\hat{p}\equiv 0$ corresponds to $\hat{\sigma}_x$; $\hat{p}\equiv 1/2$ corresponds to $1/\sqrt{2}(\hat{\sigma}_x+\hat{\sigma}_z)$; $\hat{p}\equiv 1$ corresponds to $\hat{I}$ (see text). Average payoffs for the input qubit $|0\rangle$ are shown as (. . .), those for the input qubit $|1\rangle$ are shown as [. . .], while those for 50:50 mixture (i.e., $x=0.5$) of input qubits are shown without parentheses. $\langle\hat{\$}\rangle$ is the payoff averaged over the three players for a given input qubit ($|0\rangle$ or $|1\rangle$). $\overline{\langle\hat{\$}\rangle}$ is $\langle\hat{\$}\rangle$ averaged over the input qubit.

Class	$\hat{p}\equiv 0$	$\hat{p}\equiv 1/2$	$\hat{p}\equiv 1$	C	$\langle\hat{\$}\rangle$	$\overline{\langle\hat{\$}\rangle}$
(i)		aaa(−15/4)[19/4]1/2		1	(−15/4)[19/4]	1/2
(ii)	a(−15/4)[19/4]1/2	aa(−15/4)[19/4]1/2		3	(−15/4)[19/4]	1/2
(iii)	aa(−7/2)[9/2]1/2	a(3/2)[1/2]1		3	(−11/6)[19/6]	2/3
(iv)	aaa(2)[0]1			1	(2)[0]	1
(v)			aaa(0)[2]1	1	(0)[2]	1
(vi)	a(1)[1]1		aa(−9)[9]0	3	(−17/3)[19/3]	1/3
(vii)	aa(9)[−9]0		a(1)[1]1	3	(19/3)[−17/3]	1/3
(viii)	a(5)[−4]1/2	a(9)[−9]0	a(5)[−4]1/2	6	(19/3)[−17/3]	1/3
(ix)		aa(19/4)[−15/4]1/2	a(19/4)[−15/4]1/2	3	(19/4)[−15/4]	1/2
(x)		a(3/2)[1/2]1	aa(−7/2)[9/2]1/2	3	(−11/6)[19/6]	2/3

within a given class become identical. In short, the quantum and classical games converge to produce *identical* payoffs for a given strategy class.

Since the players are unaware of the demon's presence, they will still try to achieve the dominant strategy equilibrium payoff (2,2,2) for the classical game, i.e., class (iv) in Table I, and the superior CQE payoff (5,9,5) for the quantum game, i.e., class (viii) in Table II. We now examine the average payoff from these two classes to see which game is ''optimal'' from the players' collective perspective. Let x be the input qubit noise level provided by the demon's supply, representing the fraction of $|1\rangle$ (or 1) qubits received by each agent over many turns of the game. For simplicity we assume that the demon supplies identical qubits at each turn, i.e., $|0\rangle\otimes|0\rangle\otimes|0\rangle$ with probability $(1-x)$ and $|1\rangle\otimes|1\rangle\otimes|1\rangle$ with probability x. There is no notion of ''memory'' so far in the system, hence a periodic qubit sequence $\ldots|0\rangle|1\rangle|0\rangle|1\rangle|0\rangle|1\rangle$ supplied to each agent has the same ''noise-level'' ($x=0.5$) as a random sequence produced by a memoryless coin toss. Class (iv) in Table I yields the average payoff per player $\overline{\$}_x=0.x+2.(1-x)=2-2x$, while class (viii) in Table II yields $\overline{\langle\hat{\$}\rangle}_x=(-17/3).x+(19/3).(1-x)=19/3-12x$. Figure 2 shows these average payoffs as a function of x. There is a crossover at $x_{\mathrm{cr}}=13/30=0.433$; the quantum game does better than the classical game for $0\leqslant x<x_{\mathrm{cr}}$, while the classical game does better than the quantum game for $x_{\mathrm{cr}}<x\leqslant 1$. If $\overline{\langle\hat{\$}\rangle}_x>2$, and hence $0\leqslant x<x_-$ where $x_-=13/36=0.361$, then the quantum game does better than the classical game even if the demon reduces the classical game noise level to $x=0$. If $\overline{\langle\hat{\$}\rangle}_x<0$, and hence $x_+<x\leqslant 1$ where $x_+=19/36=0.528$, then the classical game will do better than the quantum game even if the demon increases the classical game noise level to $x=1$. Suppose the demon is replaced by a heat bath at temperature T; using the Boltzmann weighting for a two-level system (energy separation ΔE) yields $k_BT=\Delta E(\ln[(1-x)x^{-1}])^{-1}$. Hence, $k_BT_{\mathrm{cr}}=3.7\Delta E$, $k_BT_-=1.75\Delta E$, while k_BT_+ is unobtainable (i.e., negative). Given the choice, the ''optimal'' game for the players to play therefore changes from being quantum to classical as T (i.e., x) increases. For $T>3.7\Delta E$, the classical game ''takes over,'' which is consistent with a simple-minded notion of a crossover from quantum → classical behavior. From the viewpoint of risk, the class (viii) quantum-game players have high potential gains, but large potential losses—this can lead to large fluctuations in their momentary wealth depending on the demon's actions, and hence large risk. By contrast, the class (iv) classical-game players have a smaller risk because of the potentially smaller wealth fluc-

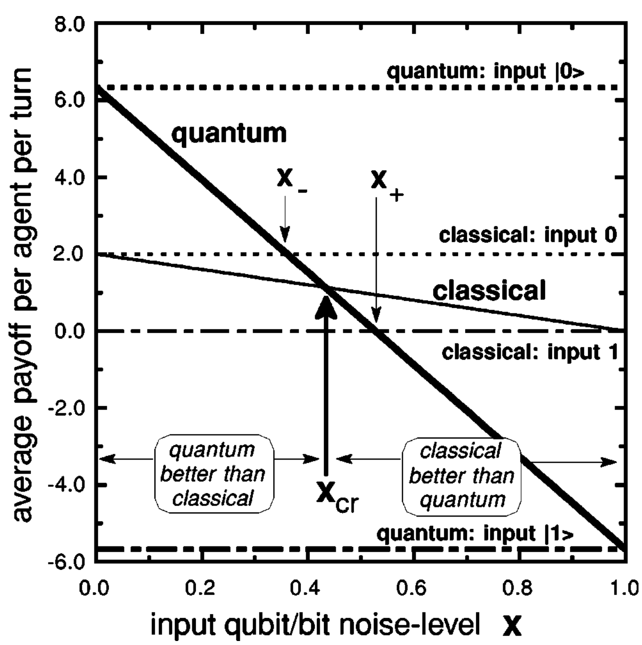


FIG. 2. Average payoff per player (''agent'') per turn for quantum game (thick solid line) and classical game (thin solid line) as a function of input qubit/bit noise level x (i.e., demon's corruption rate). Dotted lines correspond to payoff for pure $|0\rangle$ (or 0) input, while dotted-dashed lines are for pure $|1\rangle$ (or 1) input.

tuations. We emphasize that the degradation of the ''quantum advantage'' discussed here arises *without* any decoherence between the J gates, i.e., there is full coherence within the three-player subsystem. Note that the quantum advantage would also disappear (in a different way) if the quantum correlations *between* the $\hat{J}$ and $\hat{J}^{\dagger}$ gates were destroyed, but this is a trivial limit.

An interesting generalization is to consider an *evolutionary* quantum game in which players may modify their strategies based on information from the past, i.e., they ''learn'' from past mistakes [9]. This introduces a ''memory'' into the system and allows transitions between classes in Tables I and II. The memory in the evolutionary version will have a nontrivial effect on whether the quantum game outperforms the classical one, or vice versa [6]; the quantum and/or classical game [10] may even *freeze* into a given configuration. A deeper understanding of the relative ''advantage'' between such classical and quantum many-player dynamical games may eventually shed light on connections between quantum and classical many-particle, dynamical systems: it is possible that payoffs can be used to represent energies, the entangled state of the many-player quantum game can represent some exotic many-particle wave function, and the demon's actions can mimic environmental decoherence. Interestingly, Frieden *et al.* [11] have proposed that physical laws are derived from an extremum principle for the Fisher information of a measurement and the information bound in the physical quantity being measured [11,12]—this extreme physical information principle represents a game played against Nature. Since the observer can never win [11], the phenomenon of interest takes on an all-powerful, but malevolent, force—this is the information ''demon'' who is looking to increase the degree of ''blur'' of information, and against whom the players are forced to play.

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