Preparation of arbitrary pure states of two-dimensional motion of a trapped ion

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A scheme is proposed to generate an arbitrary entangled state of the two-dimensional vibrational motion of a trapped ion. In the scheme the ion is excited by a sequence of laser pulses tuned to the appropriate vibrational sidebands. In order to generate a pure state with upper phonon numbers M and N in the X and Y directions, respectively, we require no more than $(M+2)(N+1)$ operations.

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The generation of an arbitrary state of a quantum system occupies a central position in quantum optics. In the context of cavity QED, a number of schemes have been proposed for generating various nonclassical states of a single-mode electromagnetic field [1–5]. Recently, the experimental realization of superpositions of two coherent states separated by a small distance in a cavity field has been reported [6].

On the other hand, recent advances in laser cooling and ion trapping have opened prospects in quantum state engineering. The external and internal degrees of freedom of a trapped ion can be coupled via the momentum exchange between the ion and a driving laser field, which make it possible to manipulate the motional state of the ion. The extremely weak coupling between the vibrational motion of the trapped ion and the external environment provides the possibility of generating various nonclassical states of the ion motion with high efficiency. Recently, proposals have been made for generating some nonclassical vibrational states of a trapped ion, such as Fock [7], squeezed [8], Schrödinger cat [9–12], pair coherent [13], and SU(1,1) intelligent [14] states. To date, motional Schrödinger cat [15], Fock, squeezed, and coherent [16] states have been observed.

Recently, Gardiner, Cirac, and Zoller [17] have presented a scheme for the construction of arbitrary one-dimensional (1D) motional states of a trapped ion based on alternating excitations of the ion by two laser pulses. In order to generate a superposition of the first $N+1$ Fock states, $2N+1$ ion-laser interactions are required. We [18] proposed an alternative method to control the vibrational states of a trapped ion by driving the ion with a traveling-wave field sequentially tuned to the appropriate vibrational sidebands. In our scheme we require only $N+1$ ion-laser interaction times to generate a superposition of the first $N+1$ Fock states.

Gardiner *et al.* [17] have also generalized their method to the generation of an arbitrary pure state for the two-dimensional (2D) motion of a trapped ion,

$$|\Psi_d\rangle=\sum_{m=0}^{M}\sum_{n=0}^{N} d_{mn}|m\rangle_x|n\rangle_y, \tag{1}$$

where $|m\rangle_x$ and $|n\rangle_y$ denote Fock states of the motions along the X and Y axes, respectively. However, the number of required laser operations depends exponentially on the upper phonon numbers M and N. Recently, Kneer and Law [19] presented a scheme for synthesizing any entangled state of the 2D motion of a trapped ion. In the scheme the number of operations is reduced to $(2M+1)(N+1)+2N$. Drobny, Hladky, and Buzek [20] have proposed another scheme, which involves $2(M+N)^2$ operations. In this paper, I propose an alternative scheme to generate an arbitrary entangled state for the two-dimensional motion of a trapped ion. In this scheme we require only $(M+2)(N+1)$ operations. The great reduction of the number of operations is important for experiments.

We consider an ion with one excited electronic state $|e\rangle$ and two ground states $|g\rangle$ and $|g'\rangle$ with different magnetic quantum numbers. Suppose the ion is trapped in a two-dimensional harmonic potential and interacting with a traveling-wave laser field, propagating along the X axis and tuned to the nth upper vibrational sideband with respect to the transition $|g\rangle\rightarrow|e\rangle$. Assume the laser is of σ^+ polarization and thus the state $|g'\rangle$ cannot be coupled to $|e\rangle$ [15]. In the rotating-wave approximation, the Hamiltonian for such a system is given by

$$H_n=\nu_x a^\dagger a+\nu_y b^\dagger b+\omega_e|e\rangle\langle e|+\omega_g|g\rangle \times\langle g|+[\lambda E_n^+(x,t)|e\rangle\langle g|+\text{H.c.}], \tag{2}$$

where $a^\dagger$ and $b^\dagger$ are the creation operators for the motions along the X and Y axes, ω_e and ω_g are the energies for the corresponding electronic levels, ν_x and ν_y are the trap frequencies in the corresponding directions, and λ is the dipole matrix element characterizing the transition $|g\rangle\rightarrow|e\rangle$. $E^+(x,t)$ is the positive part of the frequency of the classical field,

$$E_n^+(x,t)=E_n e^{-i[(\omega_e-\omega_g+n\nu)t-k_n x+\phi_n]}, \tag{3}$$

where E_n, ϕ_n, and k_n are the amplitude, phase, and wave vector for the driving light field. The operator of the center-of-mass position x can be reexpressed as

$$x=\frac{1}{\sqrt{2\nu M}}(a+a^\dagger), \tag{4}$$

with M being the mass of the ion.

In the resolved sideband limit, where the trapping frequency is much larger than other characteristic frequencies,

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the ion-laser interaction can be described by the nonlinear Jaynes-Cummings model [9,21]. Then the Hamiltonian of Eq. (2), in the interaction picture, can be simplified to

$$H_{i,n}=\Omega_n e^{-i\phi_n}e^{-\eta_n^2/2}|e\rangle\langle g| \times\sum_{k=0}^{\infty}\frac{(i\eta_n)^{2k+n}}{k!(k+n)!}a^{\dagger(k+n)}a^k+\text{H.c.}, \tag{5}$$

where $\Omega_n=\lambda E_n$ is the Rabi frequency of the laser field and $\eta_n=k_n/\sqrt{2\nu_x M}$ the Lamb-Dicke parameter. This Hamiltonian connects the transition between $|e\rangle|m+n\rangle_x$ and $|g\rangle|m\rangle_x$ with the effective Rabi frequency

$$g_n=\Omega_n e^{-i\phi_n}e^{-\eta_n^2/2}\sum_{k=0}^{m}\frac{(i\eta_n)^{2k+n}}{k!(k+n)!}\frac{\sqrt{(m+n)!m!}}{(m-k)!}. \tag{6}$$

For the case $m=0$ the effective Rabi frequency reduces to

$$g_n=\Omega_n e^{-i\phi_n}e^{-\eta_n^2/2}\frac{(i\eta_n)^n}{\sqrt{n!}}. \tag{7}$$

We now consider another case, where the ion is driven by a laser beam propagating along the Y direction and tuned to the nth lower vibrational sideband with respect to the transition $|e\rangle\rightarrow|g'\rangle$. Assume the laser is of σ^- polarization and thus the state $|g\rangle$ is not coupled to $|e\rangle$. Then the Hamiltonian in the interaction picture is given by

$$H'_{i,n}=\Omega'_n e^{-i\phi'_n}e^{-\eta_n'^2/2}|e\rangle\langle g'| \times\sum_{k=0}^{\infty}\frac{(i\eta'_n)^{2k+n}}{k!(k+n)!}b^{+k}b^{k+n}+\text{H.c.}, \tag{8}$$

where Ω'_n, ϕ'_n, and η'_n are the Rabi frequency, phase, and Lamb-Dicke parameter for the laser field. The Hamiltonian connects the states $|e\rangle|m\rangle_y$ and $|g'\rangle|m+n\rangle_y$ with the effective Rabi frequency

$$g'_n=\Omega'_n e^{-i\phi'_n}e^{-\eta_n'^2/2}\sum_{k=0}^{m}\frac{(i\eta'_n)^{2k+n}}{k!(k+n)!}\frac{\sqrt{(m+n)!m!}}{(m-k)!}. \tag{9}$$

For the case $m=0$ the effective Rabi frequency reduces to

$$g'_n=\Omega'_n e^{-i\phi'_n}e^{-\eta_n'^2/2}\frac{(i\eta'_n)^n}{\sqrt{n!}}. \tag{10}$$

Suppose we desire to generate a target state of the form of Eq. (1). The system is initially in the ground state $|g\rangle|0\rangle_x|0\rangle_y$. We assume that the spontaneous emission rate from the excited electronic state is much smaller than the parameters $|g_n|$ and $|g'_n|$ and thus the spontaneous emission can be neglected during the ion-laser interactions. First, we drive the ion with a traveling-wave field, propagating along the X direction and tuned to the ion transition $|g\rangle\rightarrow|e\rangle$. After an interaction time τ_0, the system evolves into

$$|\Phi_{1,1}\rangle=[\cos(|g_0|\tau_0)|g\rangle-ie^{-i\phi_0} \times\sin(|g_0|\tau_0)|e\rangle]|0\rangle_x|0\rangle_y. \tag{11}$$

When we choose the amplitude (or interaction time τ_0) and phase of the classical field in such a way that the conditions

$$-ie^{-i\phi_0}\sin(|g_0|\tau_0)=d_{0,0} \tag{12}$$

are fulfilled, we obtain

$$|\Phi_{1,1}\rangle=(\sqrt{1-|d_{0,0}|^2}|g\rangle+d_{0,0}|e\rangle)|0\rangle_x|0\rangle_y. \tag{13}$$

We then tune the traveling-wave field propagating along the X axis and tuned to the first upper vibrational sideband with respect to the electronic transition $|g\rangle\rightarrow|e\rangle$. After an interaction time τ_1, the system evolves into

$$|\Phi_{2,1}\rangle=d_{0,0}|e\rangle|0\rangle_x|0\rangle_y+\sqrt{1-|d_{0,0}|^2}[\cos(|g_1|\tau_1)|g\rangle|0\rangle_x +e^{-i\phi_1}\sin(|g_1|\tau_1)|e\rangle|1\rangle_x]|0\rangle_y. \tag{14}$$

We adjust the amplitude (or interaction time τ_1) and phase of the driving field to satisfy

$$\sqrt{1-|d_{0,0}|^2}e^{-i\phi_1}\sin(|g_1|\tau_1)=d_{1,0}. \tag{15}$$

Then the state of the whole system evolves into

$$|\Phi_{2,1}\rangle=\{[d_{0,0}|0\rangle_x+d_{1,0}|1\rangle_x]|e\rangle +\sqrt{1-|d_{0,0}|^2-|d_{1,0}|^2}|g\rangle|0\rangle_x\}|0\rangle_y. \tag{16}$$

Repeat the operation $M+1$ times. During the $(k+1)$th time the driving field is tuned to the kth upper vibrational sideband. We choose the amplitude (or interaction time τ_k) and phase of the driving field to satisfy

$$-i^{k+1}\left(1-\sum_{m=0}^{k}|d_{m,0}|^2\right)^{1/2}e^{-i\phi_{k+1}}\sin(|g_k|\tau_{k+1})=d_{k,0}. \tag{17}$$

Thus we obtain the state for the system after $M+1$ interactions:

$$|\Phi_{M+1,1}\rangle=\left\{\sum_{m=0}^{M}d_{m,0}|e\rangle|m\rangle_x +\left(1-\sum_{m=0}^{M}|d_{m,0}|^2\right)^{1/2}|g\rangle|0\rangle_x\right\}|0\rangle_y. \tag{18}$$

We now drive the ion with a laser propagating along the Y axis and tuned to the electronic transition $|e\rangle\rightarrow|g'\rangle$. After an interaction time τ'_0 the system evolves to

$$|\Psi_{M+1,1}\rangle=\sum_{m=0}^{M} d_{m,0}\{\cos(|g_0'|\tau_0')|e\rangle-ie^{i\phi_0'}\times\sin(|g_0'|\tau_0')|g'\rangle\}|m\rangle_x|0\rangle_y+\left(1-\sum_{m=0}^{M}|d_{m,0}|^2\right)^{1/2}|g\rangle|0\rangle_x|0\rangle_y. \quad (19)$$

We adjust the amplitude (or interaction time τ_0') and phase of the driving field to satisfy

$$-ie^{i\phi_0'}\sin(|g_0'|\tau_0')=1. \quad (20)$$

This leads to

$$|\Psi_{M+1,1}\rangle=\sum_{m=0}^{M} d_{m,0}|g'\rangle|m\rangle_x|0\rangle_y+\left(1-\sum_{m=0}^{M}|d_{m,0}|^2\right)^{1/2}|g\rangle|0\rangle_x|0\rangle_y. \quad (21)$$

We repeat the above-mentioned procedure for $N+1$ cycles. During the kth cycle the ion is first driven by $M+1$ laser pulses propagating along the X axis and tuned to the appropriate upper vibrational sidebands with respect to the electronic transition $|g\rangle\to|e\rangle$, then excited by a laser beam propagating along the Y axis and tuned to the $(k-1)$th lower vibrational sideband with respect to the transition $|e\rangle\to|g'\rangle$. Assume that after the kth cycle the system is in the state

$$|\Psi_{M+1,k}\rangle=\sum_{m=0}^{M}\sum_{n=0}^{k-1} d_{m,n}|g'\rangle|m\rangle_x|n\rangle_y+\left(1-\sum_{m=0}^{M}\sum_{n=0}^{k-1}|d_{m,n}|^2\right)^{1/2}|g\rangle|0\rangle_x|0\rangle_y. \quad (22)$$

Then the excitation of the ion by $M+1$ laser pulses propagating along the X axis and tuned to the appropriate upper vibrational sidebands with respect to the transition $|g\rangle\to|e\rangle$ leads to

$$|\Phi_{M+1,k}\rangle=\sum_{m=0}^{M}\sum_{n=0}^{k-1} d_{m,n}|g'\rangle|m\rangle_x|n\rangle_y+\sum_{m=0}^{M} d_{m,k}|e\rangle|m\rangle_x|0\rangle_y+\left(1-\sum_{m=0}^{M}\sum_{n=0}^{k}|d_{m,n}|^2\right)^{1/2}|g\rangle|0\rangle_x|0\rangle_y. \quad (23)$$

After the ion is driven by the laser beam propagating along the Y axis and tuned to the kth lower vibrational sideband with respect to the transition $|e\rangle\to|g'\rangle$ the system evolves to

$$|\Psi_{M+1,k}\rangle=\sum_{m=0}^{M}\sum_{n=0}^{k-1} d_{m,n}|g'\rangle|m\rangle_x|n\rangle_y+\sum_{m=0}^{M} d_{m,k}[\cos(|g_{k+1}'|\tau_{k+1}')|e\rangle|m\rangle_x|0\rangle_y+(-i)^{k+1}e^{i\phi_k'}\sin(|g_{k+1}'|\tau_{k+1}')|g'\rangle|m\rangle_x|k\rangle_y]+\left(1-\sum_{m=0}^{M}\sum_{n=0}^{k}|d_{m,n}|^2\right)^{1/2}|g\rangle|0\rangle_x|0\rangle_y. \quad (24)$$

With the choice

$$(-i)^{k+1}e^{i\phi_k'}\sin(|g_k'|\tau_k')=1, \quad (25)$$

we obtain

$$|\Psi_{M+1,k}\rangle=\sum_{m=0}^{M}\sum_{n=0}^{k} d_{m,n}|g'\rangle|m\rangle_x|n\rangle_y+\left(1-\sum_{m=0}^{M}\sum_{n=0}^{k}|d_{m,n}|^2\right)^{1/2}|g\rangle|0\rangle_x|0\rangle_y. \quad (26)$$

After the $(N+1)$th cycle the system evolves to the state

$$|\Psi_{M+1,N+1}\rangle=\sum_{m=0}^{M}\sum_{n=0}^{N} d_{m,n}|g'\rangle|m\rangle_x|n\rangle_y. \quad (27)$$

Therefore, the 2D vibrational motion is prepared in the desired state of Eq. (1) with the electronic ground state $|g'\rangle$.

It is necessary to give a brief discussion of the experimental feasibility of the proposed scheme. The effective Rabi frequency g_n of Eq. (7) is proportional to $(i\eta_n)^n/\sqrt{n}$. When n is not small we require η_n not to be too small. Otherwise, the time required to complete the procedure might be too long in view of decoherence. Thus, in order to generate a pure state with the upper phonon numbers M and N not being small we require laser cooling of the ion to a motional ground state beyond the Lamb-Dicke regime. Although this has not been experimentally achieved yet, in a recent paper a scheme was proposed to do it [22].

In conclusions, we have proposed a scheme to drive the 2D vibrational motion of a trapped ion to any pure state. The approach can be used to measure general motional observables and to coherently tailor the shape of the wave function of the ion [17]. We now make a comparison of the present scheme with previous ones. In the scheme of Gardiner *et al.* [17], the number of required laser operations depends exponentially on the upper phonon numbers M and N. The scheme introduced by Kneer and Law [19] involves $(2M+1)(N+1)+2N$ operations, while the scheme proposed by Drobny *et al.* [20] requires $2(M+N)^2$ operations. The present scheme involves only $(M+2)(N+1)$ operations. Furthermore, the number of operations can be further reduced if there are some coefficients in the target state equal to zero. For example, when $d_{m,n}=0$, in the $(n+1)$th cycle we do not need to apply the laser pulse propagating along the X axis and tuned to the mth upper sideband with respect to the transition $|g\rangle \rightarrow |e\rangle$.

The state of Eq. (1) involves $(M+1)(N+1)$ desired complex coefficients. The amplitude (or duration) and phase of each pulse can control one coefficient. Thus, there might exist a better approach that requires only $(M+1)(N+1)$ operations. We hope that someone can develop such an approach.

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