Exact eigenstates for trapped weakly interacting bosons in two dimensions

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A system of N two-dimensional weakly interacting bosons in a harmonic trap is considered. When the two-particle potential is a δ function, Smith and Wilkin have proved analytically that the elementary symmetric polynomials of particle coordinates measured from the center of mass are exact eigenstates. In this study, we point out that their proof works equally well for an arbitrary two-particle potential that possesses translational and rotational symmetries. We find that the interaction energy associated with the eigenstate with angular momentum L is equal to $aN(N-1)/2+(b-a)NL/2$, where a and b are the interaction energies of two bosons in the lowest-energy one-particle state with zero and one unit of angular momentum, respectively. Additionally, we study briefly the case of attractive quartic interactions. We prove rigorously that the lowest-energy state is the one in which all angular momentum is carried by the center of mass motion.

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The study of Bose-Einstein condensation in trapped atomic gases has attracted a great deal of attention in the past few years [1–4]. One of the central issues has been the possibility of creating quantized vortices in these dilute atomic gases. Recently, vortex states in various systems have been experimentally observed [5,6]. Some theoretical investigations on quantized vortices and on rotating Bose condensates have also been carried out both in the Thomas-Fermi limit of strong interactions [7–10] and in the limit of weak interactions [11–18] between the atoms. For the weakly interacting bosons one is naturally led to study the model of N two-dimensional bosons in a harmonic trap with weak repulsive δ-function interactions [11]. One important theoretical problem here is to understand the properties of the system with a given total angular momentum L. Numerical studies have shown that when $L>N$ the lowest-energy state is the one where an array of singly quantized vortices is formed [15,8]. However, due to the complexity of these states, very few analytic results are known. In the range $0\leqslant L\leqslant N$ the structure of lowest-energy states turns out to be simpler. Numerical computations by Bertsch and Papenbrock [13] showed that the interaction energy of the lowest-energy states has a very simple form and decreases linearly with L (the only exception being that for $L=1$). They also noted that the wave functions for the lowest-energy states are simply the elementary symmetric polynomials of complex coordinates relative to the center of mass. Very recently, this remarkable formula for interaction energy was derived analytically by Jackson and Kavoulakis [17]. Moreover, Smith and Wilkin [18] proved that these symmetric polynomials are indeed exact eigenstates (see Ref. [19] for an alternative proof). In this note, we point out that the analytic proof by Smith and Wilkin works not only for a δ-function potential but also for an arbitrary potential that possesses translational and rotational symmetries. In this more general case, we find that the formula for the interaction energy still has a similar form to that for a δ-function potential. In addition, we briefly study the case of attractive quartic potential. We prove rigorously that, as in the cases of attractive δ-function and harmonic potentials, the lowest-energy state is the one where all angular momentum is carried by the center of mass motion.

Our model Hamiltonian is $\hat{H}=\hat{H}_0+\hat{V}$, where

$$\hat{H}_0=\sum_{i=1}^{N}\left[-\frac{1}{2}\nabla_i^2+\frac{1}{2}\mathbf{r}_i^2\right] \tag{1}$$

is the one-particle part and

$$\hat{V}=\sum_{i<j} v(|\mathbf{r}_i-\mathbf{r}_j|) \tag{2}$$

is the two-particle interaction. Note that the two-particle potential has been taken to possess translational and rotational symmetries. We also assume that the interaction term is weak and does not make the system unstable. It is known that in the absence of interaction the one-particle spectrum is given by

$$E_{n_r,l}=n_r+|l|+1, \tag{3}$$

where n_r is the radial quantum number and l is the angular momentum. For a system of N noninteracting bosons, the lowest-energy states with a given total angular momentum L can be obtained by putting all bosons in the states with $n_r=0$, and with l of the same sign as L. Obviously, there is a huge degeneracy, which equals the number of ways to distribute L units of angular momentum among N bosons.

When the two-particle interaction is added the degeneracy will, in general, be lifted. If the interaction is weak enough, it suffices to analyze this problem by using first-order degenerate perturbation theory. We therefore need to diagonalize $\hat{V}$ in the restricted Hilbert space of lowest-energy states with angular momentum L. We shall do it in the second quantized form. For a positive L (as we always assume), the relevant normalized one-particle wave functions for the states with $n_r=0$ are given by

$$\psi_l(r,\theta)=\frac{1}{\sqrt{l!\,\pi}}r^l e^{il\theta}e^{-r^2/2}=\frac{1}{\sqrt{l!\,\pi}}z^l e^{-|z|^2/2}, \tag{4}$$

where $l \geqslant 0$ and $z=x+iy$ is the complex coordinate. We denote by a_l and $a_l^\dagger$ the operators that annihilate and create one boson in the one-particle state ψ_l. Our Fock space is spanned by basis states of the form

$$|n_0,n_1,n_2,\ldots\rangle \equiv \Pi_k \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle, \tag{5}$$

where $|0\rangle$ is the Fock vacuum and the occupation numbers n_i satisfy

$$\sum_k n_k = N, \quad \sum_k k n_k = L. \tag{6}$$

The second quantized form of $\hat{V}$ is

$$\hat{V} = \frac{1}{2}\sum_{i,j,k,l} V_{ijkl} a_i^\dagger a_j^\dagger a_l a_k, \tag{7}$$

where the matrix elements are given by

$$V_{ijkl} = \int\int d^2z d^2z' \psi_i^*(z)\psi_j^*(z') v(|z-z'|)\psi_k(z)\psi_l(z'). \tag{8}$$

Most of these matrix elements actually vanish. Indeed, the rotational symmetry of $\hat{V}$ implies that $V_{ijkl}=0$ unless $i+j=k+l$. This, of course, corresponds to the conservation of total angular momentum in two-particle collisions, an important feature of a rotationally symmetric potential. The translational symmetry of $\hat{V}$ also leads to a set of constraints on V_{ijkl}. Since the potential is real, this symmetry can be simply expressed by

$$\sum_{i=1}^N \frac{\partial}{\partial z_i}\hat{V} = 0, \tag{9}$$

where $\partial/\partial z_i = \frac{1}{2}(\partial/\partial x_i - i\partial/\partial y_i)$. It is easy to check that the second quantized form of this condition reads

$$[\hat{L}_-, \hat{V}] = 0, \tag{10}$$

where $\hat{L}_- = \Sigma_k \sqrt{k+1}\, a_k^\dagger a_{k+1}$. To work out the constraints imposed by Eq. (10), we write

$$\hat{V} = v_0 a_0^\dagger a_0^\dagger a_0 a_0 + v_1 a_1^\dagger a_1^\dagger a_1 a_1 + v_{10} a_1^\dagger a_0^\dagger a_1 a_0 + v_{11} a_2^\dagger a_0^\dagger a_1 a_1 + v_{11}^* a_1^\dagger a_1^\dagger a_2 a_0 + \cdots. \tag{11}$$

Here we have intentionally displayed only five terms since the other terms turn out to be irrelevant for our analysis (see also Ref. [17]). Because $\hat{V}$ is Hermitian, v_0, v_1, and v_{10} must all be real. For later use, we also note that

$$v_0 = \frac{1}{2}V_{0000}, \quad v_1 = \frac{1}{2}V_{1111}. \tag{12}$$

Substituting Eq. (11) into Eq. (10) yields

$$0 = [\hat{L}_-, \hat{V}] = (v_{10}-2v_0) a_0^\dagger a_0^\dagger a_1 a_0 + (2v_1 - v_{10} + \sqrt{2}v_{11}) a_1^\dagger a_0^\dagger a_1 a_1 + \cdots. \tag{13}$$

Thus, the constraints, which are completely determined by the displayed terms in Eq. (11), are

$$v_{10} = 2v_0, \quad v_{11} = \sqrt{2}(v_0 - v_1). \tag{14}$$

These two constraints are direct consequences of the translational symmetry and will play a crucial role in later analysis.

To proceed further, it is necessary to introduce the elementary symmetric polynomials

$$e_L = \sum_{i_1<i_2<\cdots<i_L} z_{i_1} z_{i_2} \cdots z_{i_L}, \tag{15}$$

where $L \leqslant N$. The polynomial e_L is related to the state $|N-L,L\rangle$ by

$$\langle z_1, z_2, \ldots, z_N | N-L, L\rangle = \left[\frac{(N-L)!L!}{\pi^N N!}\right]^{1/2} e_L \times \exp\left(-\frac{1}{2}\sum_{i=1}^N |z_i|^2\right). \tag{16}$$

Therefore, we shall use e_L to represent the state $|N-L,L\rangle$. In fact, it is not hard to see that there is a one-to-one correspondence between the *monomial* symmetric polynomials of degree L and the basis states in our Fock space [18]. Consequently, each symmetric polynomial uniquely specifies a quantum state. We would also like to remark that the coordinate representation of the operator $\hat{L}_+ \equiv \hat{L}_-^\dagger$ is simply NR, where $R = \Sigma_i z_i/N$ is the center of mass coordinate. Thus, the symmetric polynomial $R^k e_L$ $(k \geqslant 0)$ represents the state $\hat{L}_+^k |N-L,L\rangle$. Furthermore, since $\hat{L}_+$ commutes with $\hat{V}$, we have $\hat{V}\hat{L}_+^k |N-L,L\rangle = \hat{L}_+^k \hat{V}|N-L,L\rangle$. Equivalently, we may write it as

$$\hat{V} R^k e_L = R^k \hat{V} e_L. \tag{17}$$

The elementary symmetric polynomials measured from the center of mass are defined by

$$\tilde{e}_M = \sum_{i_1<i_2<\cdots<i_M} (z_{i_1}-R)(z_{i_2}-R)\cdots(z_{i_M}-R). \tag{18}$$

Note that $\tilde{e}_1$ is the trivial zero function. It has been proved by Smith and Wilkin [18] that the state represented by Eq. (18) is an exact eigenstate when $v(|\mathbf{r}|)$ is a δ function. Here we follow the steps in their proof. First, we write [18]

$$\tilde{e}_M = \sum_{L=2}^M (-1)^{M-L} \frac{(N-L)!}{(N-M)!(M-L)!} R^{M-L} e_L + (-1)^{M-1} \frac{N!(M-1)}{(N-M)!M!} R^M. \tag{19}$$

Next, by using Eq. (11) we obtain

$$\hat{V}|N-L,L\rangle=[v_0(N-L)(N-L-1)+v_1L(L-1)+v_{10}(N-L)L]|N-L,L\rangle+v_{11}\sqrt{L(L-1)(N-L+1)}|N-L+1,L-2,1\rangle. \tag{20}$$

The monomial symmetric polynomial representing the state $|N-L+1,L-2,1\rangle$ is [18]

$$\sum_{i_1<i_2<\cdots<i_{L-2}}^{j\neq i_1,i_2,\ldots,i_{L-2}} z_{i_1}z_{i_2}\cdots z_{i_{L-2}}z_j^2=NRe_{L-1}-Le_L. \tag{21}$$

In terms of symmetric polynomials, Eq. (20) becomes

$$\hat{V}e_L=\left[v_0(N-L)(N-L-1)+v_1L(L-1)+v_{10}(N-L)L-\frac{1}{\sqrt{2}}v_{11}L(N-L+1)\right]e_L+\frac{1}{\sqrt{2}}v_{11}N(N-L+1)Re_{L-1}. \tag{22}$$

It should be noted that even though Eq. (21) is valid only for $L\geqslant 2$ Eq. (22) holds in the whole range $0\leqslant L\leqslant N$ if the convention $e_{-1}=0$ is taken. Up to this point, we have not made any use of the conditions given by Eqs. (14). To uncover the role played by these conditions, we point out that $\tilde{e}_M$ is a genuine eigenstate if Eq. (22) has the form

$$\hat{V}e_L=[f(N)+Lg(N)]e_L-(N-L+1)g(N)Re_{L-1}, \tag{23}$$

where f and g are two arbitrary functions of N. This statement can easily be proved by operating $\hat{V}$ on the right hand side of Eq. (19) and using Eqs. (17) and (23). We also find that when Eq. (23) holds the eigenvalue associated with $\tilde{e}_M$ is given by

$$\varepsilon_{M,N}=f(N)+g(N)M. \tag{24}$$

It is remarkable that Eqs. (14) are precisely the required conditions for making Eq. (22) of the form given by Eq. (23). Indeed, by substituting Eqs. (14) into Eq. (22) we get

$$\hat{V}e_L=[v_0N(N-1)+(v_1-v_0)NL]e_L-(v_1-v_0)\times N(N-L+1)Re_{L-1}. \tag{25}$$

As a result, $\tilde{e}_M$ is an exact eigenstate and the corresponding eigenvalue is

$$\varepsilon_{M,N}=[v_0N(N-1)+(v_1-v_0)NM]=\frac{1}{2}[V_{0000}N(N-1)+(V_{1111}-V_{0000})NM]. \tag{26}$$

We still have a spectrum that varies linearly with the angular momentum M. It is worth noting that V_{0000} (V_{1111}) represents the interaction energy of two bosons in the lowest-energy state with $l=0$ ($l=1$). Equation (26) represents the main result of the present work. As a verification of this formula, we consider the δ-function potential

$$v(\mathbf{r}-\mathbf{r}')=2\pi\eta\delta(\mathbf{r}-\mathbf{r}'), \tag{27}$$

where η is a small dimensionless parameter. Simple calculations show that

$$V_{0000}=\eta, \quad V_{1111}=\frac{1}{2}\eta. \tag{28}$$

Consequently, Eq. (26) gives

$$\varepsilon_{L,N}^{\delta}=\frac{\eta}{2}\left[N(N-1)-\frac{1}{2}NL\right], \tag{29}$$

in agreement with the results of Refs. [13,17].

Finally, we consider the case of the quartic potential

$$v(\mathbf{r}-\mathbf{r}')=\frac{\eta}{8}|\mathbf{r}-\mathbf{r}'|^4. \tag{30}$$

When $\eta<0$ the repulsive force between two particles that are moved away from the trap center in opposite directions would eventually be stronger than the trapping forces acting on them. We thus expect that in this case the harmonic trap is unable to stably confine the bosons in a finite region of space. To avoid instability we assume that the interaction is attractive; i.e., $\eta>0$. The purpose of studying this case is twofold. First, it serves as an additional example to which our results are applied. Secondly, it provides another case, in addition to the cases of δ-function and harmonic potentials, where the lowest-energy states can be analytically determined. After some straightforward algebra we get the following expression for the two-particle interaction:

$$\hat{V}=\eta\left[\frac{1}{2}\hat{N}(\hat{N}-1)+\frac{7}{8}\hat{N}\hat{L}+\frac{1}{4}\hat{L}(\hat{L}-1)+\frac{1}{8}\hat{N}\hat{J}-\frac{1}{4}(\hat{J}_+\hat{L}_-+\hat{L}_+\hat{J}_-)+\frac{1}{8}\hat{L}_{++}\hat{L}_{--}\right], \tag{31}$$

where $\hat{N}=\Sigma_k a_k^\dagger a_k$, $\hat{L}=\Sigma_k k a_k^\dagger a_k$, and we have defined

$$\hat{J}=\sum_k k^2 a_k^\dagger a_k, \quad \hat{J}_+=\hat{J}_-^\dagger=\sum_k (k+2)\sqrt{k+1}\,a_{k+1}^\dagger a_k,$$

$$\hat{L}_{++}=\hat{L}_{--}^\dagger=\sum_k \sqrt{(k+2)(k+1)}\,a_{k+2}^\dagger a_k. \tag{32}$$

In this case the interaction energy associated with $\tilde{e}_L$ is found to be

$$\varepsilon_{L,N}^{quartic}=\frac{\eta}{2}\left[N(N-1)+\frac{5}{2}NL\right]. \tag{33}$$

The interaction energy increases linearly with L. Now $\tilde{e}_L$ is no longer the lowest-energy state with angular momentum L.

We find that, as in the case of the attractive δ-function potential [11], the lowest-energy state is the one where all angular momentum is absorbed by the center of mass motion. However, the proof used in Ref. [11] is not applicable here since the matrix elements V_{ijkl} are not all nonpositive. We prove it by deriving a lower bound for the interaction energy. To this end, we first note that operating $\hat{L}_+$ on a eigenstate amounts to adding an unit of angular momentum to the center of mass coordinate, and the resulting state is still an eigenstate with the same eigenvalue. Therefore, only the states, called intrinsic states and denoted by $|L,N\rangle_{int}$, which have no center of mass excitation, need to be considered. Such states are characterized by [17]

$$\hat{L}_-|L,N\rangle_{int}=0. \tag{34}$$

By using Eq. (34), ${}_{int}\langle L,N|\hat{L}_{++}\hat{L}_{--}|L,N\rangle_{int}\geqslant 0$, and

$${}_{int}\langle L,N|\hat{J}|L,N\rangle_{int}\geqslant L\ {}_{int}\langle L,N|L,N\rangle_{int}, \tag{35}$$

we obtain

$$\frac{{}_{int}\langle L,N|\hat{V}|L,N\rangle_{int}}{{}_{int}\langle L,N|L,N\rangle_{int}}\geqslant \eta\left[\frac{1}{2}N(N-1)+NL+\frac{1}{4}L(L-1)\right]. \tag{36}$$

This inequality holds for all $L\geqslant 0$. Since $|N,0\rangle$ is the only eigenstate corresponding to the eigenvalue $\eta N(N-1)/2$, the lowest-energy state with a given angular momentum L must be $\hat{L}_+^L|N,0\rangle$, the state in which all angular momentum is in the center of mass motion. It has been argued [11] (see Ref. [20], however) that this state is uncondensed and is an example of the fragmented condensate of Nozières and Saint James [21]. Moreover, the presence of such an uncondensed lowest-energy state is believed to be a general feature of attractive interactions [11] . Our study supplies one more example supporting this belief.

In conclusion, we have extended the work by Smith and Wilkin [18] to the model of N bosons in a two-dimensional harmonic trap interacting via arbitrary rotationally and translationally symmetric potentials. We have shown that $\tilde{e}_L$ remains an exact eigenstate and the associated interaction energy varies linearly with the angular momentum. Our analysis reveals the importance of rotational and translational symmetries in proving this result and makes clear the physical meaning of the coefficients appearing in the formula for the interaction energy. We also briefly discuss the problem of attractive quartic interactions. A lower bound for the interaction energy of intrinsic states is derived. Based on this, we find that the lowest-energy state for a given angular momentum L is the one in which all angular momentum is carried by the center of mass motion.

Note added. A large class of interacting boson systems is considered in a recent report by Papenbrock and Bertsch [22]. Our main results are in agreement with their findings.

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[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).
[2] C. C. Bradley, C. A. Sackett, J. J. Tollet, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).
[3] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).
[4] F. Dalfovo, S. Giogini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[5] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **83**, 2498 (1999).
[6] K. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).
[7] D. S. Rokhsar, Phys. Rev. Lett. **79**, 2164 (1997).
[8] D. A. Butts and D. S. Rokhsar, Nature (London) **397**, 327 (1999).
[9] D. L. Feder, C. W. Clark, and B. I. Schneider, Phys. Rev. Lett. **82**, 4956 (1999).
[10] A. A. Svidzinsky and A. L. Fetter, Phys. Rev. Lett. **84**, 5919 (2000).
[11] N. K. Wilkin, J. M. F. Gunn, and R. A. Smith, Phys. Rev. Lett. **80**, 2265 (1998).
[12] B. Mottelson, Phys. Rev. Lett. **83**, 2695 (1999).
[13] G. F. Bertsch and T. Papenbrock, Phys. Rev. Lett. **83**, 5412 (1999).
[14] N. K. Wilkin and J. M. F. Gunn, Phys. Rev. Lett. **84**, 6 (2000).
[15] G. M. Kavoulakis, B. Mottelson, and C. J. Pethick, e-print cond-mat/0004307.
[16] A. D. Jackson, G. M. Kavoulakis, B. Mottelson, and S. M. Reimann, e-print cond-mat/0004309.
[17] A. D. Jackson and G. M. Kavoulakis, Phys. Rev. Lett. **85**, 2854 (2000).
[18] R. A. Smith and N. K. Wilkin, e-print cond-mat/00005230.
[19] T. Papenbrock and G. F. Bertsch, e-print cond-mat/0005480.
[20] C. J. Pethick and L. P. Pitaevskii, Phys. Rev. A **62**, 033609 (2000).
[21] P. Noziéres and D. Saint James, J. Phys. (Paris) **43**, 1133 (1982).
[22] T. Papenbrock and G. F. Bertsch, e-print cond-mat/0008286.