Phase measurements at the theoretical limit

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It is well known that the result of any phase measurement on an optical mode made using linear optics has an introduced uncertainty in addition to the intrinsic quantum phase uncertainty of the state of the mode. The best previously published technique [H. M. Wiseman and R. B. Killip, Phys. Rev. A **57**, 2169 (1998)] is an adaptive technique that introduces a phase variance that scales as $\bar{n}^{-1.5}$, where $\bar{n}$ is the mean photon number of the state. This is far above the minimum intrinsic quantum phase variance of the state, which scales as $\bar{n}^{-2}$. It has been shown that a lower limit to the phase variance that is introduced scales as $\ln(\bar{n})/\bar{n}^2$. Here we introduce an adaptive technique that attains this theoretical lower limit.

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I. INTRODUCTION

The phase of an electromagnetic field cannot be measured directly using linear optics and photodetectors. Rather than measuring phase directly, phase measurement schemes rely on measuring quadratures of the field and inferring the phase from these measurements. In a typical experimental implementation, the mode to be measured is passed through a 50/50 beam splitter, in order to combine it with a much stronger local oscillator field. The difference photocurrent from the two output ports of the beam splitter yields a measurement of a particular quadrature.

The standard technique for measuring a completely unknown phase is heterodyne detection, where all quadratures are sampled with equal probability. This is achieved by using a local oscillator field with a frequency slightly different from the signal's frequency, so its phase changes linearly with respect to the phase of the signal. More accurate phase measurements can be made using the homodyne technique, where the local oscillator phase is $\Phi=\varphi+\pi/2$, with φ the phase of the signal. The problem with this is that it requires initial knowledge of the phase of the signal, and so is not a phase measurement in the strict sense.

To maintain the unbiased nature of heterodyne phase measurements but obtain the increased sensitivity of homodyne measurements, an adaptive dyne technique can be used [1–4]. Here ''dyne'' detection is used to mean photodetection using a strong local oscillator at a beam splitter. The idea behind adaptive phase measurement schemes is to use the information gained so far during the measurement to estimate the system phase φ. This is then used to adjust the local oscillator phase Φ to approximate a homodyne measurement as above.

The apparatus for performing these measurements is shown schematically in Fig. 1. The signal and a local oscillator with amplitude β are combined at the beam splitter and the outputs are measured with photodetectors. The outputs from the photodetectors, δN_+ and δN_-, are subtracted and then fed into a digital signal processor that uses these measurements to estimate the phase of the system, and adjusts the phase of the local oscillator via an electro-optic phase modulator. The signal is shown here as from a cavity with a half-silvered mirror, as this is what is considered in the theory in Sec. IV.

The signal of interest is the difference between the photocurrents at the two ports. We therefore define the signal as

$$I(t)\,\delta t=\frac{\delta N_+-\delta N_-}{\beta e^{t/2}}. \tag{1.1}$$

Here we have used units of time such that the decay constant of the cavity is unity. We multiply by a factor of $e^{-t/2}$ because this is the mode function of the signal. We can take account of signals with more general mode functions $u(t)$ in a similar way [2].

When we take the limit of very large local oscillator amplitude and small time intervals δt, we find that

$$I(v)dv=2\mathrm{Re}(\alpha_v^S e^{-i\Phi(v)})dv+dW(v), \tag{1.2}$$

where v is time scaled to the unit interval and α_v^S is the scaled mean amplitude of the *system* (for which the S super-

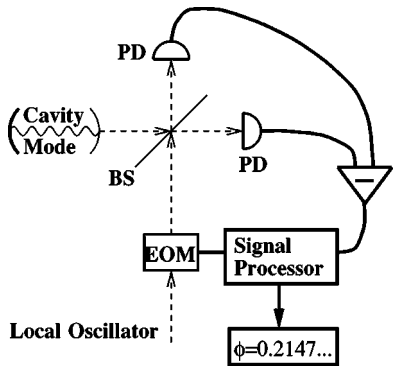

FIG. 1. Diagram of the apparatus for making an adaptive phase measurement. The signal from the cavity is combined with the local oscillator field at a 50/50 beam splitter (BS) and the outputs are detected by photodetectors (PD). The signals from these photodetectors are subtracted, and the difference signal is processed by the digital signal processor, which determines a phase estimate and adjusts the electro-optic phase modulator (EOM) accordingly.

script stands). The systematic variation in the coherent amplitude with time due to the mode shape is scaled out. This scaling is explained in more detail in Sec. IV. The final term $dW(v)$ is an infinitesimal Wiener increment such that $\langle dW(v)^2\rangle = dv$ [5].

It can be shown [6,3] that just two complex numbers are necessary to encapsulate all of the relevant information in the photocurrent record up to a given time. These are

$$A_v = \int_0^v I(u) e^{i\Phi(u)} du, \tag{1.3}$$

$$B_v = -\int_0^v e^{2i\Phi(u)} du. \tag{1.4}$$

For convenience, we often replace B_v by a third complex number defined in terms of A_v and B_v,

$$C_v = A_v v + B_v A_v^*. \tag{1.5}$$

Generally the best estimate of the phase at time v is $\arg(C_v)$ [2]. The subscripts are omitted for the final values ($v=1$).

In adaptive measurement schemes the phase of the local oscillator is generally taken to be

$$\Phi(v) = \hat{\varphi}(v) + \frac{\pi}{2}, \tag{1.6}$$

where $\hat{\varphi}(v)$ is the estimated phase of the system at time v using the measurement results A_v and C_v. There are a number of possible phase estimates, giving different adaptive schemes. For the mark I scheme [2,3], both the running phase estimate $\hat{\varphi}(v)$ and the final phase estimate are taken to be $\arg(A_v)$. This is better than heterodyne measurements only if the field is very weak [1,3]. For the mark II adaptive phase measurements [2,3] the best phase estimate $\arg(C)$ is used at the end of the measurement, but for the intermediate phase estimate $\arg(A_v)$ is used. This is better than heterodyne measurements for all field strengths. If $\arg(C_v)$ is generally the best phase estimate, it is apparent from Eq. (1.5) that $\arg(A_v)$ will only be the best phase estimate if B_v is negligible (as it is in the case of heterodyne measurements). For adaptive phase measurements B_v does not vanish and $\arg(A_v)$ is generally a much worse phase estimate than $\arg(C_v)$.

This raises the question of why this relatively poor intermediate phase estimate is used. There are two main reasons for this: (i) it is possible to obtain direct analytic results for this case, whereas using a better intermediate phase estimate requires numerical evaluation; (ii) the apparatus required to implement this method is much simpler than that required for a better intermediate phase estimate.

Even with the relatively poor intermediate phase estimate, the mark II adaptive scheme introduces a phase variance of just $\frac{1}{8}(\overline{n^S})^{-1.5}$, a good improvement over the heterodyne result of $\frac{1}{4}(\overline{n^S})^{-1}$. Here $\overline{n^S}$ is the mean photon number of the field being measured, and the actual measured phase variance is the introduced phase variance plus the intrinsic phase variance. The intrinsic phase variance for a state of mean photon number $\overline{n^S}$ can be as small as of order $(\overline{n^S})^{-2}$ [7,4]. This is far smaller than the introduced phase variance, so the latter is what limits the accuracy of phase measurements. Although the mark II results are far superior to the standard result of heterodyne detection, it is still possible to improve on the mark II result, and it is shown in Ref. [3] that a theoretical lower limit to the phase variance that is introduced by an arbitrary phase measurement scheme (based on linear optics and photodetection) is $\frac{1}{4}\ln(\overline{n^S})\times(\overline{n^S})^{-2}$.

In improving on the mark II result, the obvious thing to do is to use a better intermediate phase estimate. It turns out that using the best phase estimate $\arg(C_v)$ actually gives a worse result than the mark II case, for reasons that we will explain later. The phase estimates that we consider in this paper are therefore intermediate between $\arg(A_v)$ and the best phase estimate:

$$\hat{\varphi}(v) = \arg(C_v^{1-\epsilon(v)} A_v^{\epsilon(v)}). \tag{1.7}$$

It is possible to obtain a marked improvement over the mark II case by using constant values of ϵ. We show in Sec. V that a scaling of roughly $(\overline{n^S})^{-1.68}$ is possible. One drawback is that the value of ϵ required depends on the photon number.

We can obtain an even better result if we allow ϵ to have a variation in time, and we show in Sec. V that we can obtain phase estimates very close to the theoretical limit if we use

$$\epsilon(v) = \frac{v^2 - |B_v|^2}{C_v} \sqrt{\frac{v}{1-v}}. \tag{1.8}$$

This expression does not explicitly depend on the photon number. This method works best if the phase estimates are updated in discrete time steps, and the magnitude of the steps depends weakly on the photon number. A more serious problem with this method is that it tends to produce values of $|B|$ that are too close to 1. This means that final phase estimates with an error close to π occur sufficiently frequently to make a significant contribution to the phase uncertainty. We will show how this problem can be corrected.

The paper is structured as follows. In Sec. II we rederive the ultimate theoretical limit to phase measurements of Ref. [3]. This is necessary to understand how the improved feedback algorithm of Eq. (1.7) can approach the theoretical limit, as explained in Sec. III. In Sec. IV we derive the results necessary for a numerical simulation of this algorithm, and in Sec. V present the results of those simulations. The problem of infrequent results with large errors is identified in Sec. VI and a solution proposed and evaluated in Sec. VII. We conclude with a summary and discussion in Sec. VIII.

II. THE THEORETICAL LIMIT

In order to understand how to attain the theoretical limit, we must first understand the reason for the theoretical limit. It can be shown [6] that the probability of obtaining the results A, B from an arbitrary (adaptive or nonadaptive) measurement is

$$P(A,B)d^2A\,d^2B = \mathrm{Tr}[\rho G(A,B)]d^2A\,d^2B, \tag{2.1}$$

where ρ is the state of the mode being measured. Here $G(A,B)$ is the POM (probability operator measure) for the measurement, and is given by

$$G(A,B) = Q(A,B)|\tilde{\psi}(A,B)\rangle\langle\tilde{\psi}(A,B)|, \tag{2.2}$$

where $Q(A,B)$ is what the probability distribution $P(A,B)$ would be if ρ were the vacuum state $|0\rangle\langle 0|$, and $|\tilde{\psi}(A,B)\rangle$ is an unnormalized ket defined by

$$|\tilde{\psi}(A,B)\rangle = \exp\left[\frac{1}{2}B(a^\dagger)^2 - Aa^\dagger\right]|0\rangle. \tag{2.3}$$

This is proportional to a squeezed state [8]:

$$\exp\left[\frac{1}{2}B(a^\dagger)^2 - Aa^\dagger\right]|0\rangle = (1-|B|^2)^{-1/4}\exp(A\alpha^*/2)|\alpha,\xi\rangle, \tag{2.4}$$

where

$$|\alpha,\xi\rangle = \exp(\alpha a^\dagger - \alpha^* a)\exp\left[\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi(a^\dagger)^2\right]|0\rangle, \tag{2.5}$$

and the squeezing parameters are

$$\alpha = \frac{A + BA^*}{1-|B|^2}, \tag{2.6}$$

$$\xi = -\frac{B\,\mathrm{atanh}|B|}{|B|}, \tag{2.7}$$

where atanh is the inverse hyperbolic tan function. In terms of these the POM is given by

$$G(A,B) = Q'(A,B)|\alpha,\xi\rangle\langle\alpha,\xi|, \tag{2.8}$$

where

$$Q'(A,B) = Q(A,B)(1-|B|^2)^{-1/2}\exp[\mathrm{Re}(A\alpha^*)]. \tag{2.9}$$

If the system state is pure, $\rho = |\psi\rangle\langle\psi|$ and the probability distribution is given by

$$P(A,B) = Q'(A,B)|\langle\psi|\alpha,\xi\rangle|^2. \tag{2.10}$$

For an unbiased measurement scheme the probability distribution for the phase resulting from this equation depends entirely on the inner product between the two states, and not on $Q'(A,B)$. To see this, note first that if the measurement is unbiased the vacuum probability distribution $Q(A,B)$ will be independent of the phase. Second, for the squeezed state $|\alpha,\xi\rangle$, $\xi\alpha^*/\alpha$ is independent of the phase $\arg(\alpha)$. This in turn means that BA^*/A is independent of the phase. Since

$$A\alpha^* = (1+BA^*/A)^*\frac{|A|^2}{1-|B|^2}, \tag{2.11}$$

$A\alpha^*$ and therefore $Q'(A,B)$ are independent of the phase.

Since the probability distribution for the phase depends on the inner product between the two states, the variance in the measured phase will approximately be the sum of the intrinsic phase variance and the phase variance of the squeezed state $|\alpha,\xi\rangle$. The maximum overlap between the states will be when the squeezed state has about the same photon number as the input state. This means that the theoretical limit to the phase variance that is introduced by the measurement is the phase variance of the squeezed state that has the same photon number as the input state and has been optimized for minimum intrinsic phase variance. Since the phase variance of a squeezed state optimized for minimum intrinsic phase variance is $\ln\bar{n}/(4\bar{n}^2)$ in the limit of large $\bar{n}$ [9], this is also the limit to the introduced phase variance.

The photon number of the squeezed state at maximum overlap will be mainly determined by the photon number of the input, but the degree and direction of squeezing (parametrized by ξ) will be determined by the multiplying factor $Q'(A,B)$. The multiplying factor can be expressed as a function of $\bar{n}$ and ζ, for which we will use the same symbol Q', even though it is a new function $Q'(\bar{n},\zeta)$. Here $\bar{n}$ is the mean photon number for the state $|\alpha,\xi\rangle$ (and will be close to the photon number $\bar{n}^S$ of the input state), and $\zeta = \xi\alpha^*/\alpha$ is ξ with the phase of α scaled out. The multiplying factor will tend to be concentrated along a particular line, effectively giving ζ as a function of $\bar{n}$. In order to obtain the theoretical limit, the measurement scheme must give a multiplying factor $Q'(\bar{n},\zeta)$ that tends to give values of ζ for each $\bar{n}$ that are the same as for optimized squeezed states.

We can determine the approximate variation of ζ with $\bar{n}$ in the multiplying factor if we can estimate how it varies for measurements on a coherent state. If we consider measurements on a coherent state with real amplitude α^S, then the maximum overlap with the state $|\alpha,\xi\rangle$ will be for $\alpha^S \approx \alpha$. We use α^S without a subscript to indicate the initial coherent amplitude before the measurement.

If we are using an adaptive scheme with intermediate phase estimates that are unbiased, it is easy to see that the maximum probability will be for B real and therefore also A real. These results imply that

$$\alpha \approx \frac{A(1+B)}{1-B^2} = \frac{A}{1-B}. \tag{2.12}$$

In turn this gives ζ as

$$\zeta \approx -\,\mathrm{atanh}(1-A/\alpha) \tag{2.13}$$

$$\approx \frac{1}{2}\ln\frac{A}{2\alpha} \approx \frac{1}{2}\ln\frac{A}{2\sqrt{\bar{n}}}. \tag{2.14}$$

Since the value of ζ is governed by the multiplying factor $Q'(\bar{n},\zeta)$, this result for ζ should hold for more general input states.

From Ref. [9] the phase variance of a squeezed state is

$$\langle \Delta\phi^2 \rangle \approx \frac{n_0+1}{4\bar{n}^2} + 2\,\mathrm{erfc}(\sqrt{2n_0}), \tag{2.15}$$

where $n_0 = \bar{n}e^{2\zeta}$ for real ζ. This is minimized asymptotically as

$$\frac{\ln \bar{n} + \Delta}{4\bar{n}^2}, \tag{2.16}$$

where $\Delta \approx 2.43$, for

$$n_0 \approx \ln(4\bar{n}) - \frac{1}{4}\ln(2\pi). \tag{2.17}$$

If we use the result obtained for ζ in Eq. (2.14) we find that

$$n_0 \approx \frac{1}{2}|A|\sqrt{\bar{n}}. \tag{2.18}$$

This result means that in order for the measurement to be optimal, $|A|$ should scale with $\bar{n}$ as

$$|A| \propto \frac{\ln \bar{n}}{\sqrt{\bar{n}}}. \tag{2.19}$$

For the case of mark II measurements we have the result that $|A|=1$ [2], which is why these measurements are not optimal. Note that if we substitute $|A|=1$ into the expression (2.18) to find n_0, and substitute that into Eq. (2.15), we obtain the correct result for the mark II introduced phase variance,

$$\langle \Delta\phi^2 \rangle \approx \frac{1}{8}\bar{n}^{-1.5}. \tag{2.20}$$

III. IMPROVED FEEDBACK

Now we have the result that for optimal feedback $|A|$ should decrease with photon number. Therefore in order to improve the phase measurement scheme we want one that gives $|A|<1$. To see in general how this can be achieved, consider a coherent state with amplitude α^S and determine the Ito SDE (stochastic differential equation) for $|A|^2$:

$$d|A_v|^2 = A_v^*(dA_v) + (dA_v^*)A_v + (dA_v^*)(dA_v) \tag{3.1}$$

$$= A_v^* e^{i\Phi(v)} I(v)\,dv + e^{-i\Phi(v)} I(v)\,dv A_v + dv \tag{3.2}$$

$$= [|A_v| I(v) 2\,\mathrm{Re}(e^{i\Phi(v)} e^{-i\varphi_v^A}) + 1]\,dv, \tag{3.3}$$

where $\varphi_v^A = \arg A_v$. In terms of the phase estimate $\hat{\varphi}_v = \Phi(v) - \pi/2$ this becomes

$$d|A_v|^2 = [1 + 2|A_v| I(v) \sin(\varphi_v^A - \hat{\varphi}_v)]\,dv. \tag{3.4}$$

If we take the expectation value of $I(v)$ and simplify we get

$$\langle I(v) \rangle = -2|\alpha^S| \sin(\hat{\varphi}_v - \varphi), \tag{3.5}$$

where $\varphi = \arg \alpha^S$. If we use this result the expectation value for the increment in $|A_v|^2$ is

$$\langle d|A_v|^2 \rangle = [1 - 4|A_v||\alpha^S| \sin(\hat{\varphi}_v - \varphi) \sin(\varphi_v^A - \hat{\varphi}_v)]\,dv. \tag{3.6}$$

The first term on its own will give $|A|=1$, and in order to get $|A|<1$ the two sines must have the same sign. This will be the case if the phase estimate is between the actual phase and the phase of A_v. It is for this reason that we consider phase estimates that are intermediate between the best phase estimate and the phase of A_v, i.e., of the form

$$\hat{\varphi}(v) = \arg[(A_v v + B_v A_v^*)^{1-\epsilon(v)} A_v^{\epsilon(v)}]. \tag{3.7}$$

In general, smaller values of $|A|$ can be obtained by using smaller values of ϵ. This is because φ_v^A tends to be a worse phase estimate, thus making it possible for the sines in Eq. (3.6) to be larger. Note that it is far too simplistic to use the best phase estimate (i.e., with $\epsilon=0$), as we need to adjust ϵ in order to make n_0 closer to optimal.

IV. SIMULATION METHOD

The easiest input states to use for numerical simulations are coherent states, as they remain coherent with a deterministically decaying amplitude. However, in order to estimate the phase variance that is introduced by the measurement this would be very inefficient, as the phase variance would be dominated by the intrinsic phase variance. It is almost as easy (and much more efficient) to perform calculations on squeezed states, as squeezed states remain squeezed states under the stochastic evolution, and only the two squeezing parameters need be kept track of. The best squeezed states to use are those optimized for minimum intrinsic phase variance. For these states the total phase variance will be approximately twice the intrinsic phase variance when the measurements are close to optimal.

To determine the SDE's for the squeezing parameters, we must first consider the SDE for the state. For dyne detection the stochastic evolution of the conditioned state vector is [6]

$$d|\psi(t)\rangle = \left[dt\left(\frac{\langle a^\dagger a \rangle}{2} - \frac{a^\dagger a}{2} + \frac{\langle a^\dagger \gamma + \gamma^* a \rangle}{2} - \gamma^* a \right) + dN(t)\left(\frac{a e^{-i\Phi} + |\gamma|}{\sqrt{\langle (a^\dagger + \gamma^*)(a+\gamma) \rangle}} - 1 \right) \right] |\psi(t)\rangle, \tag{4.1}$$

where a is the annihilation operator for the mode, $|\gamma| \gg 1$ is the amplitude of the local oscillator, and $\Phi = \arg \gamma$ is its phase. Here the mode being measured is assumed to come from a cavity with an intensity decay rate equal to unity. The point process $dN(t)$ has a mean κdt, where

$$\kappa = \langle (a^\dagger + \gamma^*)(a+\gamma) \rangle. \tag{4.2}$$

The equation given in [6] differs from Eq. (4.1) by a trivial phase factor. The form above is given because it is not pos-

sible to directly take the limit of large local oscillator amplitude using the form given in [6]. To take the limit of large local oscillator amplitude we approximate the Poisson process $\delta N(t)$ by a Gaussian process

$$\delta N(t) \approx \kappa \delta t + \sqrt{\kappa}\delta W(t), \tag{4.3}$$

where $\delta W(t)$ is a Gaussian random variable of zero mean and variance δt. Then we find that in the limit of large $|\gamma|$ we have

$$d|\psi(t)\rangle = [(-a^\dagger a/2 + a\chi e^{-i\Phi} - \chi^2/2)dt + (ae^{-i\Phi} - \chi)dW]|\psi(t)\rangle, \tag{4.4}$$

where

$$\chi = \frac{1}{2}(\langle a\rangle e^{-i\Phi} + \langle a^\dagger\rangle e^{i\Phi}). \tag{4.5}$$

In order to determine the SDE's for the squeezing parameters, we use the method of Rigo *et al.* [10]. Squeezed states obey the relation

$$(a - B_t^S a^\dagger - A_t^S)|A_t^S, B_t^S\rangle. \tag{4.6}$$

The squeezing parameters A_t^S and B_t^S are related to the usual squeezing parameters in the same way as A and B are in Eq. (2.6) and Eq. (2.7). In the Stratonovich formalism

$$(a - B_t^S a^\dagger - A_t^S)d|\psi(t)\rangle = (dB_t^S a^\dagger + dA_t^S)|\psi(t)\rangle. \tag{4.7}$$

Converting the SDE for the state to the Stratonovich form in the usual way [5], we find

$$d|\psi(t)\rangle = \left[\left(-\frac{a^\dagger a}{2} - \frac{a^2 e^{-2i\Phi}}{2} + 2a\chi e^{-i\Phi} - \chi^2\right)dt + (ae^{-i\Phi} - \chi)dW - \frac{1}{2}[ad(e^{-i\Phi}) - d\chi]dW\right]|\psi(t)\rangle. \tag{4.8}$$

Here we have included the increments $d(e^{-i\Phi})$ and $d\chi$ because the phase of the local oscillator can vary stochastically. Using this form of the equation, the left hand side of Eq. (4.7) evaluates to

$$\left\{dt[-(a^\dagger B_t^S + A_t^S/2) - B_t^S(B_t^S a^\dagger + A_t^S)e^{-2i\Phi} + 2B_t^S\chi e^{-i\Phi}] + dW\left[B_t^S e^{-i\Phi} - \frac{1}{2}A_t^S d(e^{-i\Phi})\right]\right\}|\psi(t)\rangle.$$

This gives us the SDE's for the squeezing parameters,

$$dB_t^S = -B_t^S(1 + e^{-2i\Phi}B_t^S)dt, \tag{4.9}$$

$$dA_t^S = -\frac{1}{2}A_t^S dt + B_t^S\langle a^\dagger\rangle(1 + e^{-2i\Phi}B_t^S)dt + B_t^S e^{-i\Phi}dW - \frac{1}{2}B_t^S d(e^{-i\Phi})dW. \tag{4.10}$$

From these we find that the Stratonovich SDE for the standard (nonscaled) amplitude α_t^S is

$$d\alpha_t^S = -\frac{1}{2}\alpha_t^S dt + \frac{B_t^S dW}{1 - |B_t^S|^2}[(B_t^S)^* e^{i\Phi} + e^{-i\Phi}] - \frac{1}{2}\frac{B_t^S dW}{1 - |B_t^S|^2}[(B_t^S)^* d(e^{i\Phi}) + d(e^{-i\Phi})]. \tag{4.11}$$

Converting back to the Ito SDE, we get

$$d\alpha_t^S = -\frac{1}{2}\alpha_t^S dt + \frac{B_t^S dW}{1 - |B_t^S|^2}[(B_t^S)^* e^{i\Phi} + e^{-i\Phi}]. \tag{4.12}$$

The SDE for B_t^S is unchanged under the change to Ito form. If we take the signal to be $I(t)\delta t = (\delta N_+ - \delta N_-)/\beta$ (for consistency with Ref. [6]), then take the limit of large oscillator amplitude and small time intervals δt, we obtain

$$I(t)dt = 2\mathrm{Re}(\alpha_t^S e^{-i\Phi(t)})dt + dW(t). \tag{4.13}$$

The parameters A_t and B_t are then defined as in [6] by

$$A_t = \int_0^t e^{i\Phi}e^{-s/2}I(s)ds, \tag{4.14}$$

$$B_t = -\int_0^t e^{2i\Phi}e^{-s}ds. \tag{4.15}$$

In order to get rid of the exponential factors, we change the time variable to

$$v = 1 - e^{-t}, \tag{4.16}$$

and we redefine the amplitude to remove the systematic variation:

$$\alpha_v^S = \alpha_t^S e^{t/2}. \tag{4.17}$$

Here we use the v subscript to indicate the scaled amplitude, and the t subscript to indicate the original, unscaled amplitude. Since these are equal to each other at zero time, there is no ambiguity in the initial amplitude α^S. Reverting to our original definition of the signal (1.1), we find

$$I(v)dv = 2\mathrm{Re}(\alpha_v^S e^{-i\Phi(v)})dv + dW(v). \tag{4.18}$$

With these changes of variables, the definitions for A_v and B_v become

$$A_v = \int_0^v e^{i\Phi} I(u)\,du, \tag{4.19}$$

$$B_v = -\int_0^v e^{2i\Phi}\,du. \tag{4.20}$$

The differential equations for the squeezing parameters become

$$dB_v^S = -\frac{dv}{1-v} B_v^S (1 + e^{-2i\Phi} B_v^S), \tag{4.21}$$

$$d\alpha_v^S = \frac{1}{1-v} \frac{B_v^S dW(v)}{1-|B_v^S|^2} [(B_v^S)^* e^{i\Phi} + e^{-i\Phi}]. \tag{4.22}$$

Initial calculations were performed using these equations, but there is a further simplification that can be made. The solution for B_v^S is

$$B_v^S = \frac{1-v}{(B_0^S)^{-1} - B_v^*}. \tag{4.23}$$

For calculations with time-dependent ϵ this solution for B_v^S was used rather than solving a separate differential equation for B_v^S.

V. RESULTS

First we will describe the results for constant ϵ. For each mean photon number, ϵ was varied to find the value that gave the minimum phase variance. This method does not give results close to the theoretical limit for photon numbers above about 5000, but the phase variances continue to get smaller as compared to the phase variances for mark II measurements. This indicates that the results are following a different scaling law, and fitting techniques give the power for the introduced phase variance as 1.685 ± 0.007. The data and the fitted line along with the heterodyne and mark II cases and the theoretical limit are shown in Fig 2. These results are a significant improvement over the mark II case, but are still significantly above the theoretical limit.

In order to improve on this result we must vary ϵ during the measurement. The value of ϵ that we found to give the best result was

$$\epsilon(v) = \frac{v^2 - |B_v|^2}{|C_v|} \sqrt{\frac{v}{1-v}}. \tag{5.1}$$

The reason for the multiplying factor of $(v^2 - |B_v|^2)/|C_v|$ is that it is an estimator for $1/|\alpha^S|$. This means that the value of ϵ tends to be smaller for larger photon numbers, resulting in smaller values of $|A|$. The reason for the factor of $\sqrt{v/(1-v)}$ is that it makes the value of ϵ close to zero initially, and very large near the end of the measurement.

This second factor was found essentially by trial and error, and is thought to be related to the fact that the phase of α_v^S varies stochastically during the measurement. Recall that during the measurement we want the phase estimate to be between the phase of α_v^S and the phase of A_v. We only have an estimate of the phase of α^S (the initial phase), so if we use a phase estimate that is too close to the actual phase when the phase variance of α_v^S is large, the phase estimate is likely to be outside the interval between the phase of α_v^S and the phase of A_v. Since the phase variance of α_v^S increases with time, the value of ϵ is increased as well, to prevent this happening.

The results for this method are shown in Fig. 3 as a ratio to the theoretical limit. As this shows, the results are very close to the theoretical limit, and even for the largest photon

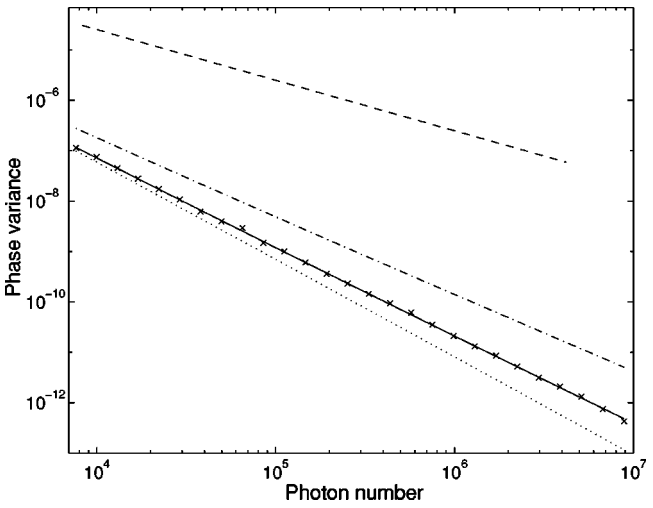

FIG. 2. Phase variance for phase measurements with a constant value of ϵ plotted as a function of the photon number of the input state. The crosses are the values obtained by stochastic integration and the continuous line is the fitted line. For comparison we have also plotted, in order from top to bottom, the variance for heterodyne measurements (dashed line), for mark II measurements (dash-dotted line), and the theoretical limit (dotted line).

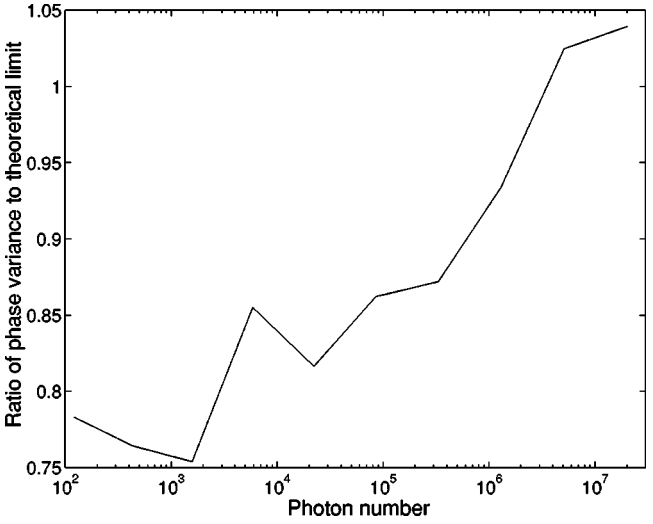

FIG. 3. Phase variance for phase measurements with a time-dependent ϵ plotted as a function of the photon number of the input state. The phase variance is plotted as a ratio to the theoretical minimum phase variance (i.e., twice the intrinsic phase variance).

number for which calculations have been performed the phase uncertainty is only about 4% above the theoretical limit. For these calculations the time steps used were approximately

$$\Delta v = \frac{\overline{n^S}\langle \Delta\phi^2\rangle_{\rm th}}{25}, \tag{5.2}$$

where $\langle \Delta\phi^2\rangle_{\rm th}$ is the theoretical limit to the phase uncertainty. With these time steps the uncertainty due to the finite step size is approximately 1%.

If the integration time step is reduced, while keeping the time interval at which the phase estimates are updated constant, the phase variance converges. If, however, the phase estimates are updated at smaller and smaller time intervals then the phase variance does not converge. For example, the phase uncertainty for measurements on an optimized squeezed state with a photon number of 1577 is 1.54×10^{-6} if we use the time steps given above. If, however, we use time steps that are 100 times smaller, then the phase variance is 1.93×10^{-6}, and if the time steps are 1000 times smaller the phase variance is 2.13×10^{-6}. These results indicate that the phase estimates must be incremented in finite time intervals for this method to give good results, and the size of the time steps that should be used depends on the photon number. The phase variance is not strongly dependent on these time steps, however, and only an order of magnitude estimate of the photon number is required.

VI. EVALUATION OF METHOD

A problem with determining the phase variance by the method above is that, for highly squeezed states (that are close to optimized for minimum phase variance), a significant contribution to the phase variance is from low probability results around π. In obtaining numerical results the actual phase variance for the measurement will tend to be underestimated because the results from around π are obtained too rarely for good statistics. It would require an extremely large number of samples to estimate this contribution. However, we can estimate it nonstatistically as follows.

Recall that in order to have a measurement that is close to optimum the multiplying factor $Q'(\overline{n},\zeta)$ should give values of ζ for each $\overline{n}$ that are close to optimized for minimum phase uncertainty. To test this for the phase measurement scheme described above, the $\overline{n}$ and ζ were determined from the values of A and B from the samples. The resulting data along with the line for optimized ζ are plotted in Fig. 4. The imaginary part of ζ should be zero for optimum measurements, and is small for these results. Therefore in Fig. 4 we have plotted the real part ζ_R. As can be seen, the vast majority of the data points are below the line, indicating greater squeezing than optimum. This means that if the low probability results around π are taken into account the phase variance for these measurements will be above the theoretical limit.

First we consider the effect of variations in the modulus of ζ, leaving consideration of error in the phase till later. In order to estimate how far above the theoretical limit the actual phase variance is, we make a quadratic approximation to the expression for the phase variance. From [9] the expression for the phase variance of a squeezed state is, for real ζ,

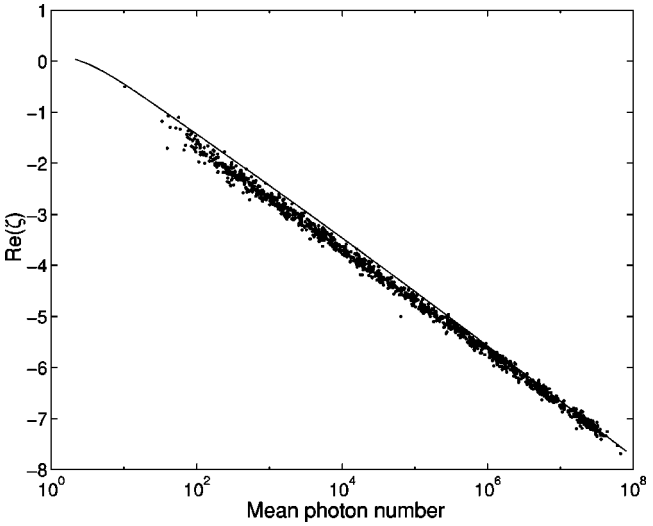

FIG. 4. Values of ζ_R and $\overline{n}$ (calculated from A and B) resulting from measurements on squeezed states of various mean photon numbers. The variation of ζ with $\overline{n}$ for optimum squeezed states is also plotted (continuous line).

$$\langle \delta\phi^2\rangle \approx \frac{e^{2\zeta}}{4\overline{n}} + \frac{1}{4\overline{n}^2}2\,\mathrm{erfc}(\sqrt{2\overline{n}}e^{\zeta}). \tag{6.1}$$

Taking the derivative with respect to ζ gives

$$\frac{d}{d\zeta}\langle \delta\phi^2\rangle \approx \frac{e^{2\zeta}}{2\overline{n}} - 4e^{\zeta}\sqrt{\frac{2\overline{n}}{\pi}}e^{-2\overline{n}e^{2\zeta}}. \tag{6.2}$$

Taking the second derivative and using the fact that the expression above is zero for minimum phase variance gives

$$\frac{d^2}{d\zeta^2}\langle \delta\phi^2\rangle \approx \frac{n_0}{2\overline{n}^2}(1+4n_0). \tag{6.3}$$

This means that for values of ζ close to optimum the increase in the phase variance over the optimum value is

$$\Delta\langle \delta\phi^2\rangle \approx (\Delta|\zeta|)^2\frac{n_0}{4\overline{n}^2}(1+4n_0). \tag{6.4}$$

The main contribution to the phase uncertainty is $n_0/(4\overline{n}^2)$, so the increase in the phase uncertainty as a ratio to the minimum phase uncertainty is

$$\frac{\Delta\langle \delta\phi^2\rangle}{\langle \delta\phi^2\rangle_{min}} \approx (\Delta|\zeta|)^2(1+4n_0). \tag{6.5}$$

This estimate indicates that the actual phase variance for the measurement scheme described above can be significantly larger than the intrinsic phase variance. For example, for a

mean photon number of about 332 000 the rms deviation of $|\zeta|$ from the optimum value is only about 0.16, but a squeezed state with $|\zeta|$ differing this much from optimum will have a phase variance more than twice the optimum value. This indicates that if the low probability results around π are taken into account the introduced phase variance is actually more than twice the theoretical limit.

Next, we estimate the contribution from error in the phase (rather than the modulus) of ζ. For a squeezed state with real α the intrinsic uncertainty in the zero quadrature is

$$\langle X_0^2\rangle = e^{-2|\zeta|}\cos^2\frac{\mu}{2} + e^{2|\zeta|}\sin^2\frac{\mu}{2}, \tag{6.6}$$

where $\mu = \arg\zeta$. Since $X_0 = 2\alpha\sin(\phi) \approx 2\alpha\phi$, the intrinsic uncertainty in the phase is

$$\langle \delta\phi^2\rangle \approx \frac{e^{-2|\zeta|}\cos^2(\mu/2) + e^{2|\zeta|}\sin^2(\mu/2)}{4\bar{n}}. \tag{6.7}$$

If the phase of ζ is small, we can make the approximation

$$\langle \delta\phi^2\rangle \approx \frac{e^{-2|\zeta|} + e^{2|\zeta|}\mu^2/4}{4\bar{n}}. \tag{6.8}$$

Clearly the first term in the numerator is just the original phase variance, and the second term is the excess phase variance due to the error in the phase of ζ. Therefore the extra phase variance due to error in the phase of ζ is given by

$$\Delta\langle \delta\phi^2\rangle \approx \frac{(\Delta\arg\zeta)^2}{16n_0}. \tag{6.9}$$

Using this estimate on the previous example it can be seen that this is not so much of a problem, with the introduced phase uncertainty being increased by less than 3% by this factor.

VII. IMPROVED METHOD

The problem of the large contribution of the low probability results around π can be effectively eliminated in the following way. At each time step the photon number is estimated from the values of A_v and B_v, and the optimum value of ζ is estimated using the asymptotic formula in [9]. Then if ζ_R (the real part of ζ) is too far below the optimum value, rather than using the feedback phase above, we use

$$\Phi(v) = \frac{1}{2}\arg\left[\frac{B_v}{|B_v|} - \frac{C_v}{C_v^*}\right]. \tag{7.1}$$

Using this feedback phase both raises ζ_R and corrects slightly for error in the phase of ζ. To see why it corrects the phase of ζ, note that

$$e^{2i\Phi(v)} \propto \frac{B_v}{|B_v|} - \frac{C_v}{C_v^*}. \tag{7.2}$$

If, for example, C_v is real, then

$$e^{2i\Phi(v)} \propto \frac{B_v}{|B_v|} - 1. \tag{7.3}$$

This means that $e^{2i\Phi(v)}$ will be approximately imaginary and in the same direction as the imaginary part of B_v. Since the increment in B_v is given by $-e^{2i\Phi(v)}dv$, using this feedback corrects the phase of B_v slightly. In addition, since $e^{2i\Phi(v)}$ is approximately imaginary it does not increase the magnitude of B_v as would a feedback phase based on a phase estimate. This results in a raised value of ζ_R. The cases where C_v has a phase other than zero are identical (except rotated by the phase of C_v).

The details of exactly when ζ_R is considered too far below optimum can be varied endlessly, but for the results that will be presented here we use this alternate phase estimate when

$$|\zeta| > |\zeta_{\rm opt}|e^{|\alpha_v|^2(1-v)/2000}, \tag{7.4}$$

where $\zeta_{\rm opt}$ is the estimated optimum value of ζ and α_v is $C_v/(v^2 - |B_v|^2)$. Using the exponential multiplying factor means that the alternative feedback is used only toward the end of the measurement.

In addition, when the above condition was satisfied and the value of ζ was too far from optimum the feedback phase was chosen to take B_v directly toward the optimum value. Specifically, when

$$|\zeta - \zeta_{\rm opt}| > 1 - v, \tag{7.5}$$

the feedback phase used was

$$\Phi = \frac{1}{2}\arg\left[B_v - v\tanh|\zeta_{\rm opt}|\frac{C_v}{C_v^*}\right]. \tag{7.6}$$

The reason for using this additional scheme was to prevent occasional results that were a long way from optimum.

Another variation from the previous scheme is that the values of ϵ given by the original expression were divided by 1.1. The above corrections correct only for values of ζ_R that are below optimum, and for the larger photon numbers many of the uncorrected values of ζ_R tend to be above optimum (see Fig. 4). The corrections will still work well, however, if we use a dividing factor to bring the uncorrected values below the line. The best dividing factor to use increases very slowly with photon number, but we still obtain good results for the range tested if we use a constant dividing factor of 1.1.

The estimated contributions to the phase variance due to error in the magnitude and phase of ζ are plotted in Fig. 5. As can be seen, the contribution due to error in the phase of ζ is very small, less than 3% for the larger photon numbers tested. The contribution due to the error in the magnitude of ζ is a bit larger, but it still does not rise above 5%. Thus we can see that the introduced phase variance can be made very close to the theoretical limit, within 7% for the largest photon number tested.

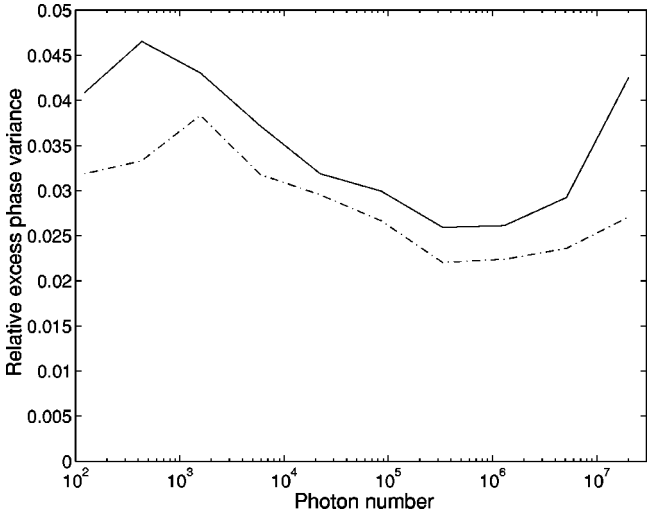

FIG. 5. Contributions to the phase uncertainty from error in the magnitude of ζ (continuous line) and the phase of ζ (dash-dotted line). These contributions are plotted as a ratio to the theoretical minimum introduced phase uncertainty.

With this modified technique the phase variance again does not converge as the feedback phase is updated in smaller and smaller time intervals. The phase variance is less dependent on the time step with this technique, however. For example, for a mean photon number of 1577 the total phase variance for measurements on an optimized squeezed state increases by only about 7% as the time steps are reduced by a factor of 1000. In contrast, the phase variance increases by a factor of 38% for the previous technique.

VIII. CONCLUSIONS

Any estimate of an initially unknown optical phase made using standard devices (linear optical and opto-electronic devices, a local oscillator, and photodetectors) must have an uncertainty above the intrinsic quantum uncertainty in the phase of the input state. The minimum magnitude of the added phase variance was determined in Ref. [3] to scale asymptotically as

$$\frac{\ln \overline{n^S}}{4(\overline{n^S})^2}, \tag{8.1}$$

where $\overline{n^S}$ is the mean photon number of the input state. Previous phase measurement schemes do not approach this theoretical limit. In this paper we have shown that an adaptive phase measurement scheme not previously considered can attain this theoretical limit. In other words, we have determined what is essentially the best possible phase measurement technique.

In practice, phase measurements are currently limited by detector inefficiency. For detector efficiency η the introduced phase variance cannot be reduced below [2]

$$\frac{1-\eta}{4\eta\overline{n^S}}. \tag{8.2}$$

When the mark II phase variance is less than this there is not likely to be any significant advantage to using a more advanced feedback scheme. For the best photodetectors available today, with around 98% efficiency [11], the mark II phase variance falls below this limit for photon numbers above 1000. Below this photon number the mark II phase variance is never more than about 27% above the limits determined using Eqs. (8.1) and (8.2), so only relatively small improvements can be obtained by using a more advanced feedback scheme.

Nevertheless, the technology is always improving, and there is no fundamental reason why photodetectors cannot be built with efficiencies extremely close to 1 [12]. When very efficient photodetectors are developed, the feedback techniques described here have the potential to give great improvements in the accuracy of phase measurements for applications where there is a limitation on the photon number that can be used. The other detrimental factors are relatively minor, although the time delay in the feedback loop will become significant for very short pulses.

The primary significance of the result obtained in this paper is theoretical, however, as it represents the culmination of the search for the best optical phase measurement schemes using standard devices. To do any better would require using nonlinear optical devices. For example, it is conceivable that down-converting some portion of the signal field, and then measuring the phase of the down-converted light, could enable the above theoretical limit to be surpassed. This is a question for future work.

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