Magnetoelectric birefringences of the quantum vacuum

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(Received 23 March 2000; published 8 December 2000)

We calculate the optical properties of a vacuum when a static magnetic field B_0 and electric field E_0 are applied perpendicular to the direction of light propagation. Apart from the known Cotton-Mouton and Kerr effects, with crossed fields we find magnetoelectric linear birefringence with optical axes parallel to the external fields. With parallel fields we find magnetoelectric Jones birefringence with optical axes under 45° with the external fields. Both birefringences are linear in E_0B_0, and have the same magnitude.

DOI: 10.1103/PhysRevA.63.012107 PACS number(s): 12.20.Ds, 42.25.Lc

I. INTRODUCTION

Nonlinear optical phenomena in vacuum have been predicted since 1935 [1] in the framework of quantum electrodynamics [2,3]. In particular, the existence in vacuum of a linear birefringence induced by a transverse static magnetic field (the Cotton-Mouton effect), or by a transverse static electric field (the Kerr effect), was predicted [4]. Due to their smallness, an experimental verification of these phenomena is still lacking. Most of the attention concentrated on the Cotton-Mouton effect (see, e.g., Ref. [5], and references therein) for practical reasons; the linear birefringence induced by a transverse magnetic field of 1 G (10^{-4} T) equals the one induced by an electric field of 1 statvolt/cm (3×10^4 V/m). Magnets providing fields of 3×10^5 G (30 T) over centimeter lengths are available, but the corresponding electric fields of 3×10^5 statvolt/cm (9 GV/m) are not easily accessible.

In this paper we will calculate the optical properties of a vacuum when a static magnetic field B_0 and an electric field E_0 are applied perpendicular to the direction of light propagation. Under these conditions, magnetoelectric linear birefringence [6–8] and magnetoelectric Jones birefringence [10,11] were recently observed in molecular liquids [9,12]. Here, we will demonstrate that these effects also exist in vacuum, and that both are linear in E_0B_0. We will also comment on the relevance of these results for the experimental observation of vacuum birefringence.

II. MAGNETOELECTRO-OPTICS OF VACUUM

The starting point of the calculation is the Heisenberg-Euler Lagrangian [1], which in Heaviside-Lorentz units, under neglect of the second-order fine-structure correction, reads

$$L=\frac{1}{2}(\mathbf{E}^2-\mathbf{B}^2)+\frac{1}{2}\xi[(\mathbf{E}^2-\mathbf{B}^2)^2+7(\mathbf{E}\cdot\mathbf{B})^2]\equiv L_0+L_{NL},$$

where $\xi=e^4\hbar/45\pi m^4c^7=2.67\times10^{-32}\ \mathrm{G}^{-2}$. This leads to the following constitutive relations:

$$\mathbf{D}=\mathbf{E}+\mathbf{D}_{NL},$$

$$\mathbf{D}_{NL}=\frac{\partial L_{NL}}{\partial\mathbf{E}}=\xi[2(\mathbf{E}^2-\mathbf{B}^2)\mathbf{E}+7(\mathbf{E}\cdot\mathbf{B})\mathbf{B}],$$

$$\mathbf{H}=\mathbf{B}+\mathbf{H}_{NL},$$

$$\mathbf{H}_{NL}=-\frac{\partial L_{NL}}{\partial\mathbf{B}}=\xi[2(\mathbf{E}^2-\mathbf{B}^2)\mathbf{B}-7(\mathbf{E}\cdot\mathbf{B})\mathbf{E}].$$

The fields have optical and static components $\mathbf{B}=\mathbf{B}_\omega+\mathbf{B}_0$ and $\mathbf{E}=\mathbf{E}_\omega+\mathbf{E}_0$, with $\mathbf{B}_\omega\ll\mathbf{B}_0$ and $\mathbf{E}_\omega\ll\mathbf{E}_0$. To simplify the algebra, we redefine the static fields as $E_0=\sqrt{\xi}E_0$ and $B_0=\sqrt{\xi}B_0$. The static fields are now dimensionless, and much smaller than unity for all practical cases. The components of $\mathbf{D}_{NL}$ and $\mathbf{H}_{NL}$ at ω are

$$\mathbf{D}_{NL\omega}=4(\mathbf{E}_\omega\cdot\mathbf{E}_0-\mathbf{B}_\omega\cdot\mathbf{B}_0)\mathbf{E}_0+2(\mathbf{E}_0^2-\mathbf{B}_0^2)\mathbf{E}_\omega+7\{(\mathbf{E}_\omega\cdot\mathbf{B}_0+\mathbf{E}_0\cdot\mathbf{B}_\omega)\mathbf{B}_0+(\mathbf{E}_0\cdot\mathbf{B}_0)\mathbf{B}_\omega\},$$

$$\mathbf{H}_{NL\omega}=4(\mathbf{E}_\omega\cdot\mathbf{E}_0-\mathbf{B}_\omega\cdot\mathbf{B}_0)\mathbf{B}_0+2(\mathbf{E}_0^2-\mathbf{B}_0^2)\mathbf{B}_\omega-7\{(\mathbf{E}_\omega\cdot\mathbf{B}_0+\mathbf{E}_0\cdot\mathbf{B}_\omega)\mathbf{E}_0+(\mathbf{E}_0\cdot\mathbf{B}_0)\mathbf{E}_\omega\}.$$

We can write the constitutive relations as

$$\mathbf{D}_\omega=\boldsymbol{\epsilon}(\mathbf{E}_0,\mathbf{B}_0)\cdot\mathbf{E}_\omega+\boldsymbol{\Psi}_{DB}(\mathbf{E}_0,\mathbf{B}_0)\cdot\mathbf{B}_\omega, \tag{1}$$

$$\mathbf{H}_\omega=\mu^{-1}(\mathbf{E}_0,\mathbf{B}_0)\cdot\mathbf{B}_\omega+\boldsymbol{\Psi}_{HE}(\mathbf{E}_0,\mathbf{B}_0)\cdot\mathbf{E}_\omega. \tag{2}$$

We use Maxwell's equations in the classical limit

$$\boldsymbol{\nabla}\cdot\mathbf{D}=0,\quad \boldsymbol{\nabla}\cdot\mathbf{B}=0, \tag{3}$$

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad \boldsymbol{\nabla}\times\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}. \tag{4}$$

To solve this problem, we assume that plane wave eigenmodes with refractive index n exist: $\mathbf{E}_\omega(\mathbf{r},t)=\mathbf{E}_{\omega0}e^{i\omega[(n/c)\hat{\mathbf{k}}\cdot\mathbf{r}-t]}$. With this ansatz, Eqs. (1) and (2) give, together with Eq. (4), the wave equation

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$$\{n^2\mathbf{\Phi}\cdot\mu^{-1}\cdot\mathbf{\Phi}+n(\mathbf{\Psi}_{DB}\cdot\mathbf{\Phi}+\mathbf{\Phi}\cdot\mathbf{\Psi}_{HE})+\boldsymbol{\epsilon}\}\cdot\mathbf{E}_{\omega0}=0, \quad (5)$$

where $\mathbf{\Phi}\cdot\mathbf{A}\equiv\hat{\mathbf{k}}\times\mathbf{A}$. If we assume $\hat{\mathbf{k}}=\hat{\mathbf{z}}$, the operator $\mathbf{\Phi}$ takes the form

$$\begin{pmatrix} 1 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It can be easily seen that the solutions of Eqs. (4) [or, equivalently, Eq. (5)] automatically fulfill Eq. (3). The problem is therefore reduced to finding the solutions of Eq. (5).

III. MAGNETOELECTRIC LINEAR BIREFRINGENCE

In crossed electric and magnetic fields, a linear birefringence with optical axes parallel and perpendicular to the external fields is predicted [6–8] and observed [9] in molecular liquids. For the vacuum, the effect of crossed fields on the propagation of electromagnetic waves was considered [4,13], but no clear magnetoelectric birefringence was reported. In our calculation, we choose $\hat{\mathbf{k}}=\hat{\mathbf{z}}$, $\mathbf{B}_0=B_0\hat{\mathbf{x}}$, and $\mathbf{E}_0=E_0\hat{\mathbf{y}}$. For the tensors in the constitutive relations, we find

$$\boldsymbol{\epsilon}=\begin{pmatrix} 1+5B_0^2+2E_0^2 & 0 & 0\\ 0 & 1+6E_0^2-2B_0^2 & 0\\ 0 & 0 & 1-2B_0^2+2E_0^2\end{pmatrix},$$

$$\mu^{-1}=\begin{pmatrix} 1-6B_0^2+2E_0^2 & 0 & 0\\ 0 & 1-2B_0^2-5E_0^2 & 0\\ 0 & 0 & 1-2B_0^2+2E_0^2\end{pmatrix},$$

and

$$\mathbf{\Psi}_{DB}=\begin{pmatrix} 0 & 7E_0B_0 & 0\\ -4E_0B_0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=-\mathbf{\Psi}_{HE}^T.$$

Inserting these into Eq. (5) gives

$$\begin{pmatrix} 2-n^2+E_0^2(5n^2+2)+B_0^2(2n^2+5)+14nE_0B_0 & 0 & 0\\ 0 & 2-n^2+E_0^2(6-2n^2)+B_0^2(6n^2-2)+8nE_0B_0 & 0\\ 0 & 0 & 2-2B_0^2+2E_0^2\end{pmatrix}\times\mathbf{E}_{\omega0}=\mathbf{E}_{\omega0}.$$

The solutions for $\mathbf{E}_{\omega0}$ are the eigenvectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ with eigenvalues $\rho_x=2-n^2+E_0^2(5n^2+2)+B_0^2(2n^2+5)+14nE_0B_0$ and $\rho_y=2-n^2+E_0^2(6-2n^2)+B_0^2(6n^2-2)+8nE_0B_0$, which should equal unity. From $\rho_{x,y}=1$, we find $n_x\approx1+7/2B_0^2+7/2E_0^2+7E_0B_0$ and $n_y\approx1+2B_0^2+2E_0^2+4E_0B_0$. We therefore find a birefringence $\Delta n\equiv n_x-n_y=3/2B_0^2+3/2E_0^2+3E_0B_0$ in which we can recognize a Cotton-Mouton birefringence $\Delta n_{\mathrm{CM}}\equiv n_\parallel-n_\perp=3/2B_0^2$, a Kerr birefringence $\Delta n_K\equiv n_\parallel-n_\perp=-3/2E_0^2$, and a magnetoelectric linear birefringence $\Delta n_{\mathrm{ME}}\equiv n_B-n_E=3E_0B_0$. The Cotton-Mouton and Kerr results agree with literature reports [4]. In a magnetic field of 3×10^5 G (30 T) and an electric field of 3×10^3 statvolt/cm (9×10^7 V/m) this results in $\Delta n_{\mathrm{CM}}=3.5\times10^{-21}$, $\Delta n_K=-3.4\times10^{-25}$, and $\Delta n_{\mathrm{ME}}=7.2\times10^{-23}$.

IV. MAGNETOELECTRIC JONES BIREFRINGENCE

In parallel electric and magnetic fields, a linear birefringence with optical axes that bissect the axes defined by the fields is predicted [10,11] and observed [12] in molecular liquids. To our knowledge, the existence of this so-called Jones birefringence has never been considered before in vacuum. In our calculation, we choose $\mathbf{B}_0=B_0\hat{\mathbf{x}}$ and $\mathbf{E}_0=E_0\hat{\mathbf{x}}$. For the tensors in the constitutive relations, we find that

$$\boldsymbol{\epsilon}=\begin{pmatrix} 1+5B_0^2+6E_0^2 & 0 & 0\\ 0 & 1-2B_0^2+2E_0^2 & 0\\ 0 & 0 & 1-2B_0^2+2E_0^2\end{pmatrix},$$

$$\mu^{-1}=\begin{pmatrix} 1-5E_0^2-6B_0^2 & 0 & 0\\ 0 & 1-2B_0^2+2E_0^2 & 0\\ 0 & 0 & 1-2B_0^2+2E_0^2\end{pmatrix},$$

and

$$\mathbf{\Psi}_{DB}=\begin{pmatrix} 10E_0B_0 & 0 & 0\\ 0 & 7E_0B_0 & 0\\ 0 & 0 & 7E_0B_0\end{pmatrix}=-\Psi_{HE}.$$

Equation (5) gives

$$\begin{pmatrix} -n^2(1-2B_0^2+2E_0^2)+2+5B_0^2+6E_0^2 & -3nE_0B_0 & 0 \\ -3nE_0B_0 & -n^2(1-6B_0^2-5E_0^2)+2-2B_0^2+2E_0^2 & 0 \\ 0 & 0 & 2-2B_0^2+2E_0^2 \end{pmatrix} \mathbf{E}_{\omega 0} = \mathbf{E}_{\omega 0}.$$

The solutions for $\mathbf{E}_{\omega 0}$ are transverse eigenvectors $\mathbf{e}_+$ and $\mathbf{e}_-$, with the corresponding eigenvalues ρ_+ and ρ_-, which can be found by standard matrix manipulation techniques. The values for n follow from the condition that $\rho_\pm = 1$. The expressions for $\mathbf{e}_\pm$ and $\rho_\pm$ are very large and not very instructive. We therefore approximate them by neglecting all terms higher than quadratic or bilinear in the external fields, keeping in mind that $n \approx 1 + O(B_0^2)$. In practice we will have $E_0 \ll B_0$, so we neglect E_0^2 with respect to B_0^2. This leads to

$$\mathbf{e}_+ \approx \begin{pmatrix} -B_0^2 \\ E_0B_0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_- \approx \begin{pmatrix} E_0^2 \\ E_0B_0 \\ 0 \end{pmatrix}.$$

This result describes two orthogonal eigenvectors that are rotated around $\hat{\mathbf{k}}$ over E_0/B_0, as compared to the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axes. This is due to a combination of magnetoelectric Jones and Cotton-Mouton birefringences. We can disentangle these two birefringences by applying the Jones formalism [14]: The Jones N matrices for isotropic refraction, standard linear birefringence, and Jones birefringence are given by $N_i = \eta \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $N_{LB} = g_0 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $N_J = g_J \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, respectively where η, g_0, and g_J are real numbers. From this we calculate the Jones M matrix to be

$$M = e^{i\eta z} \begin{pmatrix} B + g_0 A & A g_J \\ A g_J & B - g_0 A \end{pmatrix},$$

where

$$A = i\,\frac{\sin\sqrt{g_0^2+g_J^2}z}{\sqrt{g_0^2+g_J^2}}$$

and $B = \cos\sqrt{g_0^2+g_J^2}z$. This matrix describes the transformation of the electric field vector of the light upon propagation. Its eigenvectors are

$$\mathbf{g}_- = \begin{pmatrix} g_0 - \sqrt{(g_0^2+g_J^2)} \\ g_J \end{pmatrix}$$

and

$$\mathbf{g}_+ = \begin{pmatrix} g_0 + \sqrt{(g_0^2+g_J^2)} \\ g_J \end{pmatrix}.$$

These eigenvectors represent electric-field vectors that propagate unchanged. Therefore, the eigenmodes of the Maxwell equations, $\mathbf{e}_+$ and $\mathbf{e}_-$, and the eigenvectors of the Jones M matrix, $\mathbf{g}_+$ and $\mathbf{g}_-$, are parallel. From this it easily follows that $g_J = -2g_0(E_0/B_0)$.

We can determine g_0 by considering only the Cotton-Mouton effect, i.e., by putting $g_J = 0$ in the Jones formalism. The M matrix reduces to $M_{\mathrm{CM}} = \begin{pmatrix} e^{ig_0 z} & 0 \\ 0 & e^{-ig_0 z} \end{pmatrix}$. The eigenvectors are $\hat{\mathbf{x}}$ with eigenvalue $e^{ig_0 z}$, and $\hat{\mathbf{y}}$ with eigenvalue $e^{-ig_0 z}$. The resulting phase difference upon propagation between the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ eigenmodes is $2 \arctan g_0 z \approx 2 g_0 z$. The refractive indices we have found above for the Cotton-Mouton birefringence are $n_x = 1 + 7/2B_0^2$ and $n_y = 1 + 2B_0^2$, which lead to a phase difference between the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ eigenmodes of

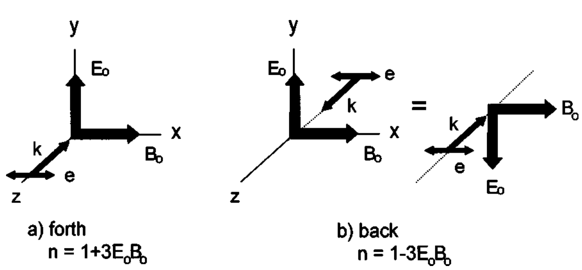

FIG. 1. Schematic view of the properties of the magnetoelectric linear birefringence in multipass cavities. The magnetoelectric effects on the refractive index on forth and back passages are shown to be opposite. A similar conclusion holds for magnetoelectric Jones birefringence. We have omitted the Cotton-Mouton and Kerr contributions for clarity.

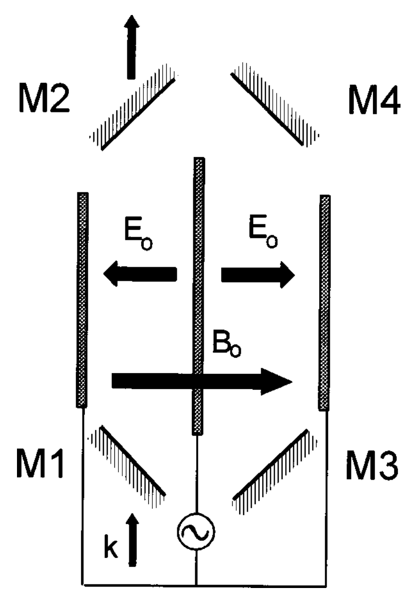

FIG. 2. Proposed geometry to increase the effective optical path length for a determination of the magnetoelectric Jones birefringence. $M1$ and $M2$ should be highly reflecting, slightly transmitting mirrors, and $M3$ and $M4$ should be 100% reflecting mirrors.

$3/2B_0^2kz$. The two phase differences have to be identical, and therefore $g_0=3/4B_0^2k$, which leads to $g_J=-3/2E_0B_0k$. From this it follows [14] that $n_{+45°}-n_{-45°}=2g_J/k=-3E_0B_0$.

V. DISCUSSION

Our calculations clearly show that both the magnetoelectric linear birefringence and the magnetoelectric Jones birefringence, predicted and observed for molecular liquids, also exist in vacuum, and are relatively strong:

$$\eta_{\mathrm{ME}}\equiv\left|\frac{\Delta n_{\mathrm{ME}}}{\sqrt{\Delta n_K\Delta n_{\mathrm{CM}}}}\right|=\left|\frac{\Delta n_J}{\sqrt{\Delta n_K\Delta n_{\mathrm{CM}}}}\right|\equiv\eta_J=2.$$

The molecular predictions also give $\Delta n_{\mathrm{ME}}=\Delta n_J$. Symmetry arguments suggest that such a relation always holds true for weak fields in media that are isotropic in the absence of external fields [8]. Calculations for atomic hydrogen give $\eta_{\mathrm{ME}}\approx0.016$ [8], whereas recent experiments on molecular liquids show that $\eta_J\leqslant3.6\times10^{-3}$ [9,12]. From this point of view it would be advantageous to search for the magnetoelectric birefringences of the vacuum, instead of the Cotton-Mouton effect, as unwanted contributions from residual gas molecules, windows, mirrors, etc. are relatively much weaker for the magnetoelectric case.

The most sensitive experiments on the propagation of light in vacuum in the presence of a transverse magnetic field [15] were realized by using a multipass optical cavity and by modulating the magnetic field. In contrast to the Cotton-Mouton effect, magnetoelectric birefringences do not give rise to an accumulation of the phase differences on the forth and back paths inside an optical cavity, but to a cancellation (see Fig. 1). One therefore has to use a ring cavity to increase the effective path length of the light (see Fig. 2). The modulation frequency in Ref. [15] was limited to less than 0.1 Hz. In an upgraded version of this experiment [5], a modulation frequency of a few Hz is envisaged. Modulation of a strong magnetic field at an even higher frequency is not practical, although it is very desirable, as it would allow one to reach the shot noise limit in the detection of the signal. In the magnetoelectric case, one could use a very strong static magnetic field (3×10^5 G is available) and a comparatively small electric field ($\geqslant3\times10^3$ statvolt/cm) alternated at relatively high frequency ($\geqslant$1 kHz). One gains a factor of 4 in signal strength for the magnetoelectric case as compared to the Cotton-Mouton case by reversing the field directions, as $\Delta n_{\mathrm{ME}}(E_0)=-\Delta n_{\mathrm{ME}}(-E_0)$ and $\Delta n_{\mathrm{ME}}(B_0)=-\Delta n_{\mathrm{ME}}(-B_0)$. With these figures, for the same total integration time one could obtain a shot-noise-limited sensitivity that is higher than the sensitivity found in Ref. [5].

In conclusion, we have presented calculations of the optical properties of a vacuum under transverse electric and magnetic fields. In addition to the known effects, we find magnetoelectric linear birefringence and magnetoelectric Jones birefringence, both linear in E_0B_0. These effects may be more favorable to the experimental observation of vacuum birefringence than the Cotton-Mouton effect.

ACKNOWLEDGMENTS

We gratefully acknowledge B. A. van Tiggelen and P. Wyder for helpful discussions. The Grenoble High Magnetic Field Laboratory is a ''laboratoire conventionné aux universités UJF et INP de Grenoble.''

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