Quantum-limited linewidth of a good-cavity laser: An analytical theory from near to far above threshold

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The problem of the quantum-limited or intrinsic linewidth of a good-cavity laser is revisited. Starting from the Scully-Lamb master equation, we present a fully analytical treatment to determine the correlation function and the spectrum of the cavity field at steady state. For this purpose, we develop an analytical approximation method that implicitly incorporates the microscopic fluctuations of both the phase and intensity of the field, and, in addition, takes full account of the saturation of the nonlinear gain. Our main result is a simple formula for the quantum-limited linewidth that is valid from near to far above threshold and also includes the presence of thermal photons. Close to the threshold, the linewidth is twice as large as predicted by the standard phase-diffusion treatment neglecting intensity fluctuations, and even 50% above threshold the increase is still considerable. In general, quantum fluctuations of the intensity are present and continue to influence the linewidth as long as the photon-number distribution is not strictly Poissonian. This inherent relationship is displayed by a formula relating the linewidth and the Mandel Q parameter. More than 100% above treshold the linewidth is found to be smaller than predicted by the standard treatment, since the simple phase-diffusion model increasingly overestimates the rate of phase fluctuations by neglecting gain saturation. In the limit of a very large mean photon number the expected perfectly coherent classical field is obtained.

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I. INTRODUCTION

When the radiation field is described classically and spontaneous emission is neglected in comparison to induced emission, the steady-state linewidth of an ideal single-mode laser, i.e., of a laser with a perfectly stabilized classical intensity, has the shape of a δ function. This is due to the fact that the cavity losses are exactly compensated for by the gain and, as a result, the field in the resonator remains constant. From a fully quantized description of the electromagnetic field, however, it follows that the laser linewidth cannot be smaller than a certain quantum limit, related to the well-known Schawlow-Townes linewidth, which is inversely proportional to the laser intensity [1]. This limit has been studied extensively in the past decades (see Refs. [2–5] and references therein). In view of the importance of stable coherent signals for various high-precision measurements and because of an ongoing interest in fundamental problems of quantum optics, renewed attention has been paid recently to the quantum limitation of the laser linewidth. The investigations have been extended to cover bad-cavity lasers [6], lasers without inversion [7], and a chaotic laser cavity [8]. Different systems have been devised in this context to reduce the quantum limit of the linewidth by means of correlated spontaneous emission schemes [9]. Recently, even theoretical models for light amplification without stimulated emission have been investigated in order to obtain a reduced ultimate quantum limit of the laser linewidth [10].

The quantum-limited laser linewidth is also called the intrinsic or natural linewidth of the laser. Its origin, like the origin of the microscopic intensity fluctuations, lies in the fact that in the stationary regime of operation the balance between the gain and loss processes sustains a constant average field only but, due to the quantum nature of these processes, fluctuations of the field around its mean are induced on a microscopic scale. We note that the resonator losses are caused by the outcoupling of the field through the output mirror as well as by any additional linear damping process such as absorption. For good-cavity lasers the combined effect of these losses can be described by a single cavity-damping constant γ that is the sum of the constants referring to the individual processes. The usual treatment [2–5] of the intrinsic laser linewidth rests on the approximation that the linewidth arises from fluctuations of the phase of the field described by phase diffusion.

In this paper, we rely on the Scully-Lamb model [3] of the laser since it is applicable for an arbitrary strong saturation of the lasing atomic transition. Neglecting intensity fluctuations, a simple analytic expression for the linewidth has been derived in this model by means of different methods [3–5]. On the other hand, the intrinsic laser linewidth can be determined exactly by numerically calculating the first-order correlation function of the field and performing the Fourier transform of the latter. Thus, both the effects of fluctuations of the phase of the field and of its amplitude, or intensity, respectively, are implicitly taken into account. Investigations of this kind have been performed by Lu [11] who started from the Scully-Lamb master equation [3] and found numerically that near threshold the intrinsic laser linewidth is up to twice as large as that given by the phase-diffusion coefficient at the mean photon number.

In the present paper, we derive an analytical expression for the quantum-limited laser linewidth by means of investigating the two-time first-order correlations of the field. The treatment is restricted to single-mode lasers in the good-cavity limit, i.e., we make the usual assumption that the cavity-damping time γ^{-1} is long in comparison to all other relevant time scales. This ensures that the time dependence of the polarization and population inversion of the gain medium can be adiabatically eliminated and the Scully-Lamb model of the laser can be applied. Our approximation scheme is an extension of an analytical method developed previously by one of the authors for calculating photon-number variances [12,13]. It makes use of the fact that the photon-number distribution of the laser radiation is strongly peaked at a large mean photon number. Therefore, it is not necessary to study the density-matrix elements of the field in detail but it suffices to directly evaluate the expectation values and correlation functions of interest, in the approximation of small fluctuations.

The paper is organized as follows. In Sec. II, we develop a general approximation method for the determination of the first-order correlation function of the field in the resonator of a micromaser or laser. The method is applied in Sec. III to study the laser linewidth. In order to reveal the influence of intensity fluctuations on the latter, an expression for the photon-number variance of the laser in the presence of thermal photons is also derived in this section. Our results are discussed in Sec. IV and compared to the standard linewidth formula ensuing from the phase-diffusion approximation. Finally, a summary is given in Sec. V.

II. GENERAL APPROXIMATION METHOD

The power spectrum of a radiation field is defined as the Fourier transform of its normalized first-order correlation function. When the field is a single-mode field with frequency ν, the steady-state spectrum is given by

$$S(\omega)=\frac{1}{\pi}\mathrm{Re}\int_0^\infty \frac{\langle a^\dagger(t)a(0)\rangle_s}{\langle a^\dagger a\rangle_s}e^{-i(\omega-\nu)t}dt, \qquad (2.1)$$

where a and $a^\dagger$ are the photon annihilation and creation operators in the interaction picture and the subscript s denotes the stationary state. We consider a single-mode radiation field contained in a leaky cavity and being sustained by a gain mechanism. The overall losses of the field mode are supposed to be due to the coupling of the cavity field to the environment, modeled by a reservoir in thermal equilibrium. Therefore the damping of the field can be described by the standard master equation $\dot{\rho}=L\rho$ for its reduced density operator ρ, where the action of the superoperator L is defined as

$$L\rho=-\frac{\gamma}{2}(1+n_b)(a^\dagger a\rho-2a\rho a^\dagger+\rho a^\dagger a)-\frac{\gamma}{2}n_b(aa^\dagger\rho-2a^\dagger\rho a+\rho aa^\dagger). \qquad (2.2)$$

Here, γ is the cavity-damping constant, and n_b denotes the mean number of thermal photons in the cavity. We also note that for any field operator σ, the relations

$$\mathrm{Tr}(a^\dagger L\sigma)=-\frac{\gamma}{2}\mathrm{Tr}(a^\dagger\sigma), \qquad (2.3)$$

$$\mathrm{Tr}(a^\dagger aL\sigma)=-\gamma\,\mathrm{Tr}(a^\dagger a\sigma)+\gamma n_b\,\mathrm{Tr}\,\sigma, \qquad (2.4)$$

follow from Eq. (2.2). To model the gain in a simple way, we assume that excited atoms are injected into the cavity with rate r. The atoms are supposed to interact with the field one after the other during a time τ that is negligibly short in comparison to both the cavity-damping time γ^{-1} and the mean time interval between successive atoms. The effect of a single atom on the field can be formally written as $\rho(t+\tau)=M\rho(t)$, where the superoperator M depends on the specific kind of interaction process. For a micromaser, the interaction time τ is given by the transit time of the atoms through the microwave cavity, and $M=M(\tau)$ has to be obtained in the usual way [14] from the Jaynes-Cummings Hamiltonian [15] for the atom-field interaction by performing the trace with respect to the atoms. In order to describe a laser, it is assumed that excited two-level atoms are injected into a resonant cavity and interact independently with the field during time intervals that are determined by the survival time Γ^{-1} of the atoms as effective two-level systems, before they decay into different energy states [4]. The superoperator M that has to be used for the laser is found by averaging $M(\tau)$ with respect to the interaction time τ, according to $M=\Gamma\int_0^\infty M(\tau)e^{-\Gamma\tau}d\tau$ [4].

When the injection times of the atoms are uncorrelated, i.e., for Poissonian pumping, the evolution of the field due to the combined action of the gain and loss mechanisms obeys the master equation

$$\dot{\rho}=r(M-1)\rho+L\rho, \qquad (2.5)$$

which has the formal solution $\rho(t)=V(t)\rho(0)$ with

$$V(t)=\exp\{[r(M-1)+L]t\}. \qquad (2.6)$$

Due to the Markovian character of the master equation, the two-time correlation function necessary to determine the spectrum can be easily calculated. For the stationary state we find

$$\langle a^\dagger(t)a(0)\rangle_s=\mathrm{Tr}[a^\dagger V(t)(a\bar{\rho})], \qquad (2.7)$$

with $\bar{\rho}=\lim_{t\to\infty}\rho(t)$ denoting the steady-state density operator. Making use of Eq. (2.6) and taking into account the relation (2.3), we obtain from Eq. (2.7) the differential equation

$$\frac{d}{dt}\langle a^\dagger(t)a(0)\rangle_s=\left\{r[b(t)-1]-\frac{\gamma}{2}\right\}\langle a^\dagger(t)a(0)\rangle_s, \qquad (2.8)$$

where we have introduced the abbreviation

$$b(t)=\frac{\mathrm{Tr}[a^\dagger M\sigma(t)]}{\langle a^\dagger(t)a(0)\rangle_s}=\frac{\sum_{n=1}^\infty\sqrt{n}[M\sigma(t)]_{n-1,n}}{\sum_{n=1}^\infty\sqrt{n}\sigma_{n-1,n}(t)} \qquad (2.9)$$

with

$$\sigma(t)=V(t)(a\overline{\rho}). \tag{2.10}$$

Under steady-state conditions, it is possible to eliminate the injection rate r from Eq. (2.8) by expressing it in terms of field expectation values and the cavity decay rate. To this end we start from the steady-state relation

$$\frac{d}{dt}\overline{n}=\sum_{n=0}^{\infty} n\dot{\overline{\rho}}_{n,n}=\mathrm{Tr}[a^{\dagger}a\dot{V}\overline{\rho}]=0. \tag{2.11}$$

By inserting Eq. (2.6) and utilizing Eq. (2.4), we arrive at the photon-number balance equation

$$r[\mathrm{Tr}(a^{\dagger}aM\overline{\rho})-\overline{n}]=\gamma(\overline{n}-n_b), \tag{2.12}$$

where $\overline{n}=\langle a^{\dagger}a\rangle_s=\mathrm{Tr}(a^{\dagger}a\overline{\rho})$ is the steady-state mean photon number $\langle a^{\dagger}(0)a(0)\rangle$.

So far, all equations hold exactly and can be applied to lasers as well as to micromasers with Poissonian pumping. In order to obtain an analytical expression for the spectrum, it is necessary to calculate the quantity b in Eq. (2.8). For this purpose, we use an approximation method that rests on the assumption that the steady-state photon-number distribution in the cavity is strongly peaked at a large mean photon number $\overline{n}$. Since the order of magnitude of the width of the photon-number distribution is determined by $\sqrt{\Delta n^2}$, we assume that the relations

$$\overline{n}\gg 1, \quad \sqrt{\Delta n^2}=(\overline{n^2}-\overline{n}^2)^{1/2}\ll\overline{n} \tag{2.13}$$

are fulfilled for the mean photon number and its variance Δn^2. By means of performing an expansion with respect to suitable parameters in Eq. (2.9) and replacing $\sigma(t)$ by its initial value $\sigma(0)$ in small terms in this expansion, it is possible to obtain an approximate expression for b. It will not depend on time, provided that the leading term in the expansion proves to be time-independent. In this case the value of b depends only on $\overline{\rho}$ and on the specific form of M, i.e., the specific kind of the atom-field interaction process, and we can easily integrate Eq. (2.8) to obtain

$$\langle a^{\dagger}(t)a(0)\rangle_s=\overline{n}e^{[r(b-1)-\gamma/2]t}. \tag{2.14}$$

Fourier transformation according to Eq. (2.1) yields a Lorentzian steady-state spectrum that is centered at the frequency $\omega_0=\nu+(r/2)\mathrm{Im}\,b$. The linewidth (full width at half maximum) is given by

$$\Delta\omega=\gamma+2r(1-\mathrm{Re}\,b). \tag{2.15}$$

As required, the linewidth reduces to the empty-cavity linewidth γ when $r=0$ or when $b=1$, i.e., when either no atoms are injected at all, or when the atoms do not interact with the field and M is equal to the unit operator. Under these conditions the steady state is, of course, the thermal state with mean photon number n_b, or, for $n_b=0$, the vacuum.

We assume that the superoperator M in Eq. (2.5) is due to resonant one-photon interaction of the field with a two-level atom, initially in its excited state. In the photon-number representation, M has the general form

$$(M\rho)_{n,m}=A_{n,m}\rho_{n,m}+B_{n-1,m-1}\rho_{n-1,m-1}, \tag{2.16}$$

where the coefficients $A_{n,m}$ and $B_{n,m}$ are different for the cases of a micromaser or a laser, respectively. It then follows that the steady-state density operator is diagonal in the photon-number representation, as can be easily shown with the help of Eqs. (2.2) and (2.5). Therefore the expectation value of the field (which in the steady state corresponds to its time average) vanishes,

$$\langle a\rangle_s=\mathrm{Tr}(a\overline{\rho})=0. \tag{2.17}$$

In terms of the amplitude and phase of the expectation value of the field, $\langle a\rangle_s$, Eq. (2.17) requires that for parameter values for which the amplitude is fixed, the phase is uniformly distributed between 0 and 2π, i.e., all phase values are equally probable at steady state. We note that this holds for lasers as well as for micromasers unless the atoms are injected in a definite superposition of their energy states.

In general, a correlation exists between the values of the field at different times that decays with increasing time difference. It is this decay that determines the spectrum and the linewidth according to Eqs. (2.1) and (2.14).

III. THE LASER LINEWIDTH

For the case of the laser, the coefficients $A_{n,m}$ and $B_{n,m}$ in Eq. (2.16) can be written as [3–5]

$$A_{n,m}=1-\frac{\chi\left[1+\frac{1}{2}(n+m)\right]+\frac{1}{8}\chi^2(n-m)^2}{2+\chi(n+m+2)+\frac{1}{8}\chi^2(n-m)^2} \tag{3.1}$$

and

$$B_{n,m}=\frac{\chi\sqrt{(n+1)(m+1)}}{2+\chi(n+m+2)+\frac{1}{8}\chi^2(n-m)^2}, \tag{3.2}$$

where we introduced the saturation parameter $\chi=4g^2/\Gamma^2$ with g and Γ^{-1} denoting the atom-field coupling constant and the average lifetime of the atom as a two-level system, respectively. After changing the index of summation appropriately, from Eqs. (2.9) and (2.16) we obtain

$$b(t)=1+\frac{1}{4}\frac{\sum_{n=1}^{\infty}[\chi/(1+\chi(n+1/2))]\sqrt{n}\sigma_{n-1,n}(t)}{\sum_{n=1}^{\infty}\sqrt{n}\sigma_{n-1,n}(t)}. \tag{3.3}$$

Here, use has been made of the fact that the terms proportional to χ^2 in Eqs. (3.1) and (3.2) can be neglected for $|n-m|=1$ since under normal conditions the relation $\chi^2\ll 1$ [4] is fulfilled. To evaluate b, we now apply the approximation

$$\frac{1}{1+\chi\left(n+\frac{1}{2}\right)} \approx \frac{1}{1+\chi\bar{n}}\left[1-\frac{\chi\left(\frac{1}{2}+n-\bar{n}\right)}{1+\chi\bar{n}}\right]. \tag{3.4}$$

Because of the condition (2.13) the above approximation is justified for all terms of the sum in the nominator of Eq. (3.3), since in these terms $|n-\bar{n}|$ is of the order of magnitude of $\sqrt{\Delta n^2}$ or smaller, and since $\chi/(1+\chi\bar{n})<1/\bar{n}$ for any value of χ. The second term in the square brackets of Eq. (3.4) therefore leads to a contribution to b that is small in comparison to the time-independent contribution of the first term. Replacing $\sigma_{n-1,n}(t)$ by its initial value $\sigma_{n-1,n}(0)=\sqrt{n}\bar{\rho}_{n,n}$ in this small contribution and taking into account that $\Sigma_{n=0}^{\infty}n^2\bar{\rho}_{n,n}=\bar{n}^2+\Delta n^2$, we obtain after minor algebra a time-independent expression for b. The latter can be substituted into Eq. (2.15) to yield the approximative expression

$$\Delta\omega=\gamma-\frac{r\chi}{2(1+\chi\bar{n})}\left[1-\frac{\chi}{1+\chi\bar{n}}\left(\frac{1}{2}+\frac{\Delta n^2}{\bar{n}}\right)\right] \tag{3.5}$$

for the laser linewidth.

In the next step, we eliminate the injection rate r by making use of the photon-number balance equation (2.12). For the case of the laser, the latter takes the form

$$\frac{r\chi}{2}\sum_{n=0}^{\infty}\frac{n+1}{1+\chi(n+1)}\bar{\rho}_{n,n}=\gamma(\bar{n}-n_b). \tag{3.6}$$

Here again Eqs. (3.1) and (3.2) have been used together with Eq. (2.16), and the index of summation has been changed appropriately. We proceed by applying the same approximation scheme that led to Eq. (3.5) and perform the expansion

$$\frac{1}{1+\chi(n+1)} \approx \frac{1}{1+\chi\bar{n}}\left[1-\frac{\chi(1+n-\bar{n})}{1+\chi\bar{n}}\right]. \tag{3.7}$$

Using the condition (2.13) and the relation [16]

$$n_b \ll \bar{n}, \tag{3.8}$$

we obtain from Eq. (3.6) after simple transformations the approximation

$$\frac{r\chi}{2(1+\chi\bar{n})}=\gamma\left[1+\frac{\chi}{1+\chi\bar{n}}\left(1+\frac{\Delta n^2}{\bar{n}}\right)-\frac{1+n_b}{\bar{n}}\right]. \tag{3.9}$$

When only the first term in the square brackets is kept, Eq. (3.9) yields the familiar relation

$$\bar{n}=\frac{r}{2\gamma}-\frac{1}{\chi}=\frac{1}{\chi}\left(\frac{\alpha}{\gamma}-1\right), \tag{3.10}$$

where we have introduced the linear gain $\alpha=r\chi/2$. However, in order to study the quantum limit of the linewidth, also the terms that are small in comparison to the leading term have to be taken into account. When we substitute the expression (3.9) for the factor in front of the square brackets in Eq. (3.5), keeping only the terms of lowest order in $1/\bar{n}$ and $\chi/(1+\chi\bar{n})$, respectively, the contributions containing the variance cancel and we finally arrive at the laser linewidth

$$\Delta\omega=\frac{\gamma}{2\bar{n}}\left(\frac{2+\chi\bar{n}}{1+\chi\bar{n}}+2n_b\right). \tag{3.11}$$

The quantum origin of the intrinsic laser linewidth is revealed by noticing that along our lines we would have obtained the result $\Delta\omega=0$ if the terms of the order $1/n$ arising from the commutation relation for the field operators had been neglected in Eqs. (3.3) and (3.6). The two limiting cases $\chi\bar{n}\gg 1$ and $\chi\bar{n}\ll 1$ correspond to a laser that is operated far above threshold or near threshold, respectively, as can be seen from Eq. (3.10). Therefore it is apparent that the linewidth becomes inversely proportional to the mean photon number $\bar{n}$ only near threshold and far above threshold. In the intermediate regions the dependence on $\bar{n}$ is more complicated.

We note that because of the relation (3.10), the linewidth can be expressed in terms of any three of the four parameters $\bar{n},\chi,\gamma$, and α (or r, respectively). By inserting Eq. (3.10) into Eq. (3.11) and thus eliminating $\bar{n}$, we obtain another useful expression for the linewidth,

$$\Delta\omega=\frac{\gamma\chi}{2}\,\frac{\gamma}{\alpha-\gamma}\left(1+\frac{\gamma}{\alpha}+2n_b\right). \tag{3.12}$$

Equation (3.11) describes the dependence of the linewidth of a given laser on the mean photon number, while Eq. (3.12) describes the dependence on the gain or, alternatively, on the above-threshold ratio, defined simply as the normalized gain α/γ. Since the linear gain is easily measurable, this latter equation is the most important of the possible expressions for the intrinsic laser linewidth from the point of view of experimental accessibility.

In order to facilitate later comparison with the standard result delivered by the phase-diffusion model, we eliminate χ from the linewidth expression (3.11). Using Eq. (3.10) one more time, we arrive immediately at

$$\Delta\omega=\frac{\alpha+\gamma}{2\bar{n}}\,\frac{\gamma}{\alpha}+\frac{\gamma}{\bar{n}}n_b. \tag{3.13}$$

Although it might seem from this expression that $\Delta\omega$ is proportional to $\bar{n}^{-1}$ in the entire region of laser operation, this is not true since the gain α is not constant for a given laser but depends on the pumping rate and is connected with $\bar{n}$ via the relation (3.10). In contrast to this, the cavity decay constant

γ and the saturation parameter χ are fixed, χ^{-1} being the saturation photon number for the lasing transition between the atomic energy levels.

It is interesting to consider two important limiting cases. In the far-above-threshold limit, where $\chi\bar{n}\gg 1$ or $\alpha\gg\gamma$, respectively, Eqs. (3.11)–(3.13) yield the limiting value

$$\Delta\omega_{\rm lim}=\frac{\gamma}{2\bar{n}}(1+2n_b)=\frac{\chi\gamma^2}{2\alpha}(1+2n_b). \tag{3.14}$$

On the other hand, in the vicinity of the threshold, where $0<\alpha/\gamma-1\ll 1$ [17], the linewidth depends on $\bar{n}$ in a different way, described by $\Delta\omega_{\rm thr}=(\gamma/\bar{n})(1+n_b)$. This difference can be interpreted to be due to the fact that the contribution of intensity fluctuations to the linewidth increases when the above-threshold ratio decreases, as we shall show next.

In a quantized description of the radiation field, intensity fluctuations are revealed in an enhancement of the photon-number variance as compared to the Poissonian value of $\Delta n^2=\bar{n}$ that corresponds to a constant intensity. Therefore, we first calculate the steady-state photon-number variance Δn^2 of the laser, taking into account the presence of thermal photons. To do so we apply an approximation method that has been developed previously by one of the authors in order to investigate the photon statistics in saturated multiphoton atom-field interaction [12,13]. We start from the steady-state equation

$$\frac{d}{dt}(\Delta n^2)=\sum_{n=0}^{\infty}(n^2-2n\bar{n})\dot{\bar{\rho}}_{n,n}=0, \tag{3.15}$$

and use the right-hand side of the master equation (2.5), together with Eqs. (2.2) and (2.16), in order to express $\dot{\bar{\rho}}_{n,n}$. By taking into account Eqs. (3.1) and (3.2) and by appropriately changing the index of summation in the individual terms, the resulting equation can be transformed to yield

$$\sum_{n=0}^{\infty}\bar{\rho}_{n,n}\left\{\alpha\frac{(n+1)(2n+1-2\bar{n})}{1+\chi(n+1)}-2\bar{n}\gamma(n_b-n)\right.$$
$$\left.+\gamma[n_b(n+1)(2n+1)-(1+n_b)n(2n-1)]\right\}$$
$$=0, \tag{3.16}$$

with $\alpha=r\chi/2$. The following treatment again relies on the assumption of a strongly peaked photon-number distribution subject to the conditions (2.13) that imply that the approximation (3.7) is valid. When the latter is applied in Eq. (3.16), we obtain a term that is proportional to $\Sigma_{n=0}^{\infty}n^3\bar{\rho}_{n,n}=\overline{n^3}$. Applying the approximation

$$\overline{n^3}=\overline{[\bar{n}+(n-\bar{n})^3]}\approx\bar{n}^3+3\bar{n}\Delta n^2, \tag{3.17}$$

which is justified because of the condition (2.13), we arrive at the equation

$$\frac{\alpha}{1+\chi\bar{n}}\left[\frac{2\Delta n^2}{1+\chi\bar{n}}+\bar{n}\right]=\gamma[2\Delta n^2-\bar{n}(1+2n_b)]. \tag{3.18}$$

Here again the relation $\overline{n^2}=\bar{n}^2+\Delta n^2$ has been taken into account, and small contributions have been neglected. Finally, we again make use of the lowest-order photon-number balance equation $\alpha/(1+\chi\bar{n})=\gamma$ [cf. Eq. (3.10)]. Thus, from Eq. (3.18), after simple transformations, we arrive at the relative photon-number variance of the laser

$$\frac{\Delta n^2}{\bar{n}}=\left(1+\frac{1}{\chi\bar{n}}\right)(1+n_b)=\frac{\alpha}{\alpha-\gamma}(1+n_b). \tag{3.19}$$

Clearly, when the thermal photon number n_b is not small in comparison to 1, its influence on the photon-number variance of the laser is crucial even for $n_b\ll\bar{n}$, as is its influence on the intrinsic laser linewidth. For $n_b=0$, Eq. (3.19) corresponds to the standard result that is known from the literature [3–5]. We note that in general the relative strength of the intensity fluctuations is characterized by the normalized quantity

$$\frac{\langle a^{\dagger 2}a^2\rangle-\langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle^2}=\frac{\overline{n(n-1)}}{\bar{n}^2}-1=\frac{Q}{\bar{n}}. \tag{3.20}$$

Here we introduced the Mandel Q parameter

$$Q\equiv\frac{\Delta n^2}{\bar{n}}-1=n_b+\frac{1+n_b}{\chi\bar{n}}=\frac{\gamma+n_b\,\alpha}{\alpha-\gamma}, \tag{3.21}$$

where Eq. (3.19) has been applied. Since the conditions (2.13) imply that $Q\ll\bar{n}$, the intensity fluctuations characterized by Eq. (3.20) are extremely small under these conditions, and they vanish in the limit $\bar{n}\to\infty$. Therefore, intensity fluctuations can be considered to be a true quantum effect in the above-threshold regime of the laser. Nevertheless their influence on the laser linewidth cannot be neglected, because the nonzero value of the latter itself is a small quantum effect only.

With the help of Eq. (3.21), the linewidth equations (3.11)–(3.13) can be finally cast into the form

$$\Delta\omega=\frac{\gamma}{2\bar{n}}\left(1+\frac{Q}{1+Q}\right)(1+n_b). \tag{3.22}$$

Obviously, close to the threshold, where $\alpha/\gamma-1\ll 1$ [17] and hence $Q\gg 1$, the contribution of the intensity fluctuations to the intrinsic laser linewidth is the same as that of the phase fluctuations. In this case the linewidth is twice as large as it would be without intensity fluctuations, i.e., for $Q=0$.

IV. DISCUSSION

Before discussing the analytic results in more detail, it seems appropriate to say a few words about their range of validity, determined by the applicability of our basic assumptions (2.13). Making use of Eq. (3.19), the second of the inequalities (2.13) can be transformed to yield the condition $[(1+n_b)\alpha/(\alpha-\gamma)]^{1/2} \ll \overline{n}^{1/2}$. When the laser is operated e.g., 10% above threshold, i.e., for $\alpha/\gamma=1.1$, this requirement can be assumed to be fulfilled if $\overline{n} \gtrsim 10^3$ which, because of Eq. (3.10), corresponds to $\chi \lesssim 10^{-4}$. At higher above-threshold ratios our approximation is valid for even smaller mean photon numbers, or larger values of χ, respectively. In general, because of Eqs. (3.10) and (3.19), we can combine the two inequalities of the condition (2.13) to yield the requirement

$$\sqrt{\chi(1+n_b)} \ll \frac{\alpha}{\gamma} - 1 \ll \frac{1}{\chi}, \tag{4.1}$$

which has to be fulfilled for the linewidth formulas (3.11)–(3.13) to be valid. Obviously, if $\chi \approx 10^{-6}$ and $n_b \ll 1$ as in typical continuous-wave gas lasers, our results are already approximately valid when the laser is operated only more than 1% above threshold, i.e., for $\alpha/\gamma \gtrsim 1.01$.

The linewidth formulas (3.11)–(3.13) constitute the main result of this paper. They differ from the standard laser linewidth formula

$$\Delta\omega_{\mathrm{pd}} = \frac{\gamma+\alpha}{4\overline{n}} = \frac{\gamma\chi}{4}\,\frac{\alpha+\gamma}{\alpha-\gamma} = \frac{\gamma\chi}{4} + \frac{\gamma}{2\overline{n}}, \tag{4.2}$$

which has been derived for $n_b=0$ in the so-called phase-diffusion model neglecting intensity fluctuations [18]. Here, for the last two steps, we used Eq. (3.10) and the equivalent relation $\alpha=\gamma(1+\chi\overline{n})$ in order to transform the standard result.

In Fig. 1, the linewidth is plotted for different operating regimes of the laser. We emphasize that the curves representing our result, (3.12), are in perfect agreement with the numerical results, found previously for the laser linewidth by computing the two-time correlation function of the field [11]. From a comparison of Eq. (4.2) to Eq. (3.13), it is obvious that the phase-diffusion result is only a good approximation for the linewidth when $\alpha/\gamma \approx 2$, whereas it underestimates the linewidth closer to the threshold. Higher above threshold, on the other hand, the linewidth is overestimated by Eq. (4.2) which, for $\overline{n} \to \infty$, yields the intensity-independent residual linewidth $\Delta\omega=\gamma\chi/4$, instead of $\Delta\omega=0$, to be expected in the classical limit. Moreover, we conclude from Fig. 1 that the linewidth can be approximated by $\Delta\omega_{\mathrm{lim}}$, given by Eq. (3.14), provided that $\alpha/\gamma \gtrsim 5$. The difference between our result and the standard one in the far-above-threshold region shows that the phase-diffusion assumption of the standard treatment overestimates the contribution of phase fluctuations to the linewidth the more the higher above threshold the laser is operated, i.e., the stronger the effect of the nonlinearity stemming from the gain saturation. On the other hand, since intensity fluctuations are neglected, the standard treatment underestimates the linewidth in the near-threshold regime and corrections are necessary to incorporate the effect of the super-Poissonian photon statistics, as has been emphasized already by Lu [11]. As the above-threshold ratio increases and the intensity becomes more stabilized, the effect

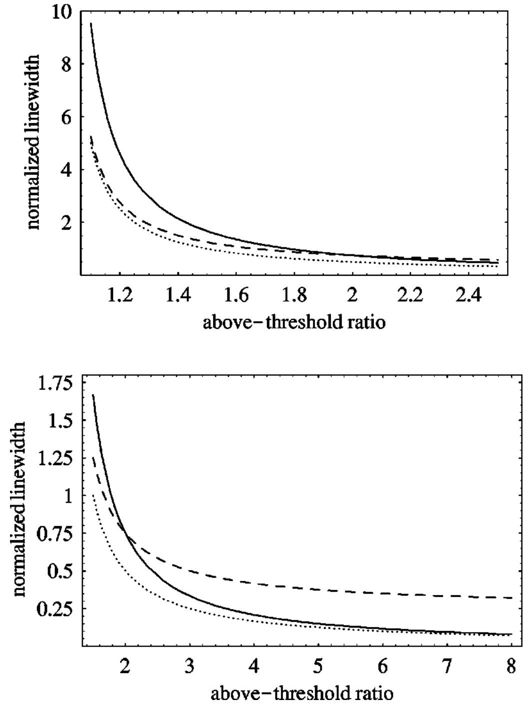

FIG. 1. Normalized laser linewidth $\Delta\omega/(\chi\gamma)$ versus the above-threshold ratio α/γ for $n_b=0$. The full line corresponds to our formula [Eq. (3.12)]. For comparison, the curves resulting from the standard expression $\Delta\omega_{\mathrm{pd}}$ [dashed line, corresponding to Eq. (4.2)] and from the approximation $\Delta\omega_{\mathrm{lim}}$ [dotted line, corresponding to Eq. (3.14)] are also displayed.

of this underestimation decreases. Figure 1 suggests that the effects of overestimating phase fluctuations, on the one hand, and underestimating intensity fluctuations, on the other, just compensate approximately when the laser is operated around 100% above threshold.

In the following, we shall discuss the reasons for the discrepancy between our result and the standard one in more detail. For this purpose, we first consider the different approximation methods that are employed in the literature for the derivation of Eq. (4.2). In the most common approach, the master equation of the density operator is transformed into an equation for its P representation. After changing to polar coordinates by writing the complex field amplitude as $\epsilon=r\exp(\mathrm{i}\phi)$, the exact evolution equation for the quasiprobability density $P(r,\phi)$ contains derivatives with respect to r and ϕ to all orders [4]. This is due to the nonlinearity of the underlying master equation, which is revealed by the denominators in Eqs. (3.1) and (3.2). In the standard treatment it is assumed that P does not change along the radial coordinate, corresponding to a neglect of intensity fluctuations, and that only derivatives up to second order have to be taken into account. The latter assumption is necessary to ensure phase diffusion, but it becomes less and less justifiable as gain saturation, hence nonlinearity of the laser equations, increases. In this case, the simple phase diffusion model can no longer be applied. In addition, mixed terms also become important in the evolution equation for P, which are products of a differential operator acting on the amplitude r and another one acting on the phase. These, in turn, lead to cross correlations between the fluctuations of the intensity and the phase.

In the frame of the photon-number representation of the density operator, the standard result has been derived by applying the quantum fluctuation-regression theorem and approximately investigating the decay of an initial value of the field instead of the two-time correlation function. This is done by means of determining the lowest eigenvalue that characterizes a single decay rate for all nondiagonal density-matrix elements $\rho_{n,n-1}$ [4]. In a more precise treatment, a single decay rate would have to be determined for the quantities $\sqrt{n}\rho_{n,n-1}$ [19], since the average field follows from performing the sum over these quantities. Moreover, with increasing above-threshold ratio the influence of the nonlinearity also increases and therefore the quantum fluctuation-regression theorem cannot be applied anymore, in general [20].

With respect to the Heisenberg-Langevin approach, we mention that a nonlinear c-number Langevin equation can be derived for the complex field amplitude $\epsilon = r\exp(i\phi)$. The coupling of the fluctuations of the real amplitude r and the phase ϕ is clearly obvious from this Langevin equation. The derivation of the phase diffusion result, Eq. (4.2), rests on implicitly making a factorization assumption for expectation values containing the complex field amplitudes in the denominator and their noise operators in the nominator [4]. Because of the intensity fluctuations near threshold, and because of the nonlinearity due to gain saturation far above threshold, it would be extremely difficult to go beyond this approximation.

We finally conclude that the laser linewidth cannot be explained satisfactorily with the help of the simple assumption that the intensity is constant and the electric-field phasor executes a random walk in the complex plane as described by phase diffusion. In the linear approximation, valid near threshold, it is true that the behavior of the phase fluctuations alone can be described by phase diffusion, but intensity fluctuations contribute to the linewidth as well. Farther above threshold, on the other hand, the simple model breaks down because the fluctuations of the intensity and of the phase are coupled due to the nonlinearity of the gain, and the behavior of the phase fluctuations cannot be characterized as simple phase diffusion. The frequently used procedure of disregarding the microscopic intensity fluctuations and assuming that the photon-number distribution is strictly Poissonian, on the one hand, and not properly taking into account or even completely neglecting gain saturation, on the other, in general does not yield sufficiently accurate results for the quantum-limited linewidth.

V. CONCLUSIONS

In this paper, we have studied the quantum-limited linewidth of a good-cavity laser by determining the first-order correlation function of the laser field at steady state. It is the decay of this correlation function and not the phase fluctuations alone that determines coherence properties such as, e.g., the visibility of interference fringes. By taking the Fourier transform, we obtained the power spectrum of the laser field and derived an analytical expression for the quantum-limited linewidth as a function of the mean photon number [see Eq. (3.11)] or of the above-threshold ratio of the laser [see Eq. (3.12)]. Our analytical result is in perfect agreement with the results of earlier numerical studies [11]. We explicitly demonstrated the effect of a super-Poissonian photon statistics [see Eq. (3.22)] and showed that near threshold the linewidth is considerably larger than the standard phase-diffusion result [cf. Eq. (4.2)], where the intensity is assumed to be constant, or the photon statistics to be Poissonian, respectively. Although in most practical cases the laser linewidth is limited by the much larger technical noise and the intrinsic quantum limit cannot be reached, there exists a variety of proposals to reach or even go beyond the quantum limit with the help of sophisticated methods [9,10]. Our results show that for a precise quantum-mechanical description of the laser linewidth it is necessary to directly calculate the first-order correlation function of the laser field, thus implicitly incorporating intensity fluctuations as well as phase fluctuations, and to properly take into account the nonlinearity of the gain.

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[17] Note, however, that because of the assumption (2.13) our approximation procedure does not hold directly at the threshold, where the photon statistics does not differ appreciably from a thermal one and the photon-number variance is still in the order of $\overline{n}^2$. (See also the discussion in Sec. V.)

[18] The standard linewidth formula (4.2) has been derived using different methods (see [3–5]). It has been stated in Refs. [3] and [5] that the result refers to the near-threshold region where $\alpha \approx \gamma$.

[19] For $n_b=0$ the correct far-above threshold limit $\Delta\omega_{\mathrm{lim}} = \gamma/(2\overline{n})$ has been alternatively derived recently [10] from the two-time correlation function by applying a method that is an extension of the standard density-matrix investigation [4]. The procedure in Ref. [10] consists in determining a single decay rate for the quantities $\sqrt{n}\rho_{n,n-1}$ rather than for the nondiagonal density-matrix elements $\rho_{n,n-1}$. It is applied to a so-called ideal standard laser master equation that is equivalent to the master equation used in our model provided that in the coefficients (3.1) and (3.2) the limit $\chi n \gg 1$ is performed. This idealized formal equation guarantees that the photon-number distribution is strictly Poissonian.

[20] In the quantum case the fluctuation-regression theorem does not hold for an arbitrary Markovian quantum system but can be only applied when the equations of motion for the mean values are linear [see, e.g., C. W. Gardiner, *Quantum Noise* (Springer, Berlin, 1991) and D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994)]. Instead of studying the decay of the electric field, one therefore has to investigate the decay of its correlation function itself as soon as a nonlinearity is involved.