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Input-output relations in optical cavities: A simple point of view

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In this paper, we present a very simple approach to input-output relations in optical cavities, limiting ourselves to one- and two-photon states of the field. After field quantization, we derive the nonunitary transformation between *inside* and *output* annihilation and creation operators. Then we express the most general two-photon state generated by *inside* creation operators, through base states generated by *output* creation operators. After renormalization of coefficients of the inside two-photon state, we calculate the outside photon-number probability distribution in a general case; then we treat with some detail the single-mode and symmetrical cavity case. We found that the statistics of emitted photons may exhibit either quantum or classical behavior, depending both on source properties and on cavity characteristics.

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I. INTRODUCTION

The problem of interaction between a pair of atoms in free space and in a cavity has been the subject of several investigations in past years [1–6]. Recently, the spontaneous emission of a pair of two identical atoms or molecules in a planar Fabry-Perot microcavity has been the subject of theoretical and experimental research [7,8]. This paper starts from our attempt to give a simple interpretation to some recent experimental results [8]. Consider a process of spontaneous emission of a pair of photons by two distinct molecules inside a microcavity. If we measure the number of photons emitted outside the cavity, and if there are not dissipative phenomena, we will find only three possible results: two photons detected on the right and none on the left, two photons detected on the left and none on the right, one photon detected on the left and one on the right. In this paper, we look for a solution to this question: If we know the distribution of the number of photons outside the cavity, how can we obtain information about the process of emission that generated the photons without completely solving the problem? Our basic idea is to describe the electromagnetic field inside the cavity by means of a two-photon state as generally as possible; the coefficients of the expansion of this state in a proper basis set depend on the process which generated the state itself. If we project this state on the basis number states defined outside the cavity, we obtain directly the probability distribution we look for. Therefore, if we change the above-mentioned coefficients, we can study how the probability distribution changes, and by comparison with the measured distribution we can obtain the values of these coefficients, getting information about the process active inside the cavity. This procedure is not orthodox or fully justified, however the obtained results show good internal consistency. The relations of input-output and their connection with the traditional stochastic methods based on Langevin equations have been studied by Knöll *et al.* [9]. The aim of the present paper is to generalize the methods used to describe an optical element with two inputs and two outputs, as a beam splitter, to derive simple relations between fields inside and outside a planar Fabry-Perot cavity. In Sec. II, we quantize the electromagnetic field generalizing to three dimensions an approach presented by Barnett *et al.* [10] and derive the relations between operators defined in the space inside and outside the cavity. Then, in Sec. III, we build and study the states of the field generated by linear and bilinear forms of creation operators both inside and outside the cavity and, in Sec. IV, we calculate the outside probability distributions for these two-photon states. Finally, we summarize our results in Sec. V.

II. SPATIAL MODES AND FIELD QUANTIZATION

Now we calculate the appropriate normal modes for quantization of the electromagnetic field. In a traditional approach, one first determines the modes of the classical boundary value problem, then one quantizes the field in terms of these modes [11–17]. An alternative approach has been presented by Barnett *et al.* [10]. We have generalized this work, restricted to one-dimensional fields, to three-dimensional fields.

Consider a pair of lossless infinitesimally thin dielectric slabs at $z=\pm l/2$, placed to form a planar Fabry-Perot cavity, as shown in Fig. 1.

We impose appropriate boundary conditions [10] both on slab 1 and on slab 2, obtaining, respectively,

$$\begin{aligned}\hat{b}_{R\lambda}(\mathbf{k})e^{i\varphi_R(-l/2)} &= t_{1\lambda}(\mathbf{k})\hat{a}_{R\lambda}(\mathbf{k})e^{i\varphi_R(-l/2)} \\ &\quad + r_{1\lambda}(\mathbf{k})\hat{b}_{L\lambda}(\mathbf{k})e^{i\varphi_L(-l/2)}, \\ \hat{a}_{L\lambda}(\mathbf{k})e^{i\varphi_L(-l/2)} &= t_{1\lambda}(\mathbf{k})\hat{b}_{L\lambda}(\mathbf{k})e^{i\varphi_L(-l/2)} \\ &\quad + r_{1\lambda}(\mathbf{k})\hat{a}_{R\lambda}(\mathbf{k})e^{i\varphi_R(-l/2)},\end{aligned} \tag{1}$$

and

$$\begin{aligned}\hat{c}_{R\lambda}(\mathbf{k})e^{i\varphi_R(l/2)} &= t_{2\lambda}(\mathbf{k})\hat{b}_{R\lambda}(\mathbf{k})e^{i\varphi_R(l/2)} \\ &\quad + r_{2\lambda}(\mathbf{k})\hat{c}_{L\lambda}(\mathbf{k})e^{i\varphi_L(l/2)},\end{aligned} \tag{2}$$

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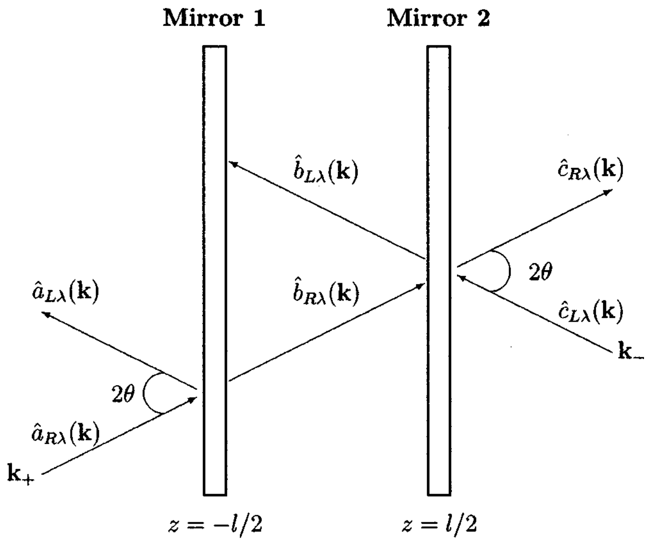

FIG. 1. Schematic representation of a planar Fabry-Perot cavity with notation for input operators $\hat{a}_{R\lambda}(\mathbf{k}),\hat{c}_{L\lambda}(\mathbf{k})$, inside operators $\hat{b}_{R\lambda}(\mathbf{k}),\hat{b}_{L\lambda}(\mathbf{k})$, and output operators $\hat{a}_{L\lambda}(\mathbf{k})$ and $\hat{c}_{R\lambda}(\mathbf{k})$.

$$\hat{b}_{L\lambda}(\mathbf{k})e^{i\varphi_L(l/2)}=t_{2\lambda}(\mathbf{k})\hat{c}_{L\lambda}(\mathbf{k})e^{i\varphi_L(l/2)} + r_{2\lambda}(\mathbf{k})\hat{b}_{R\lambda}(\mathbf{k})e^{i\varphi_R(l/2)},$$

where $r_{i\lambda}(\mathbf{k}),t_{i\lambda}(\mathbf{k})$ $(i=1,2)$ are the reflection and transmission coefficients of the i-dielectric slab and the annihilations operators are defined as in Fig. 1. $\varphi_R(z)$ and $\varphi_L(z)$ are the phases generated by field propagation inside the cavity. We freely choose the zero phases in the middle of the cavity, so that $\varphi_F(-z)=-\varphi_F(z)$, where $F=R,L$ and $\varphi_R(z)=-\varphi_L(z)$. Then we can put

$$\varphi_L(\pm l/2)=\mp\delta/2, \quad \varphi_R(\pm l/2)=\pm\delta/2, \tag{3}$$

where $\delta\equiv\omega l\cos\theta/c$ is half of the phase gained in a double traversal of the cavity [18]. From Eqs. (1) and (2), we can express the inside operators $\hat{b}_{R\lambda}(\mathbf{k})$ and $\hat{b}_{L\lambda}(\mathbf{k})$ and the output operators $\hat{a}_{L\lambda}(\mathbf{k})$ and $\hat{c}_{R\lambda}(\mathbf{k})$ in terms of the input operators $\hat{a}_{R\lambda}(\mathbf{k})$ and $\hat{c}_{L\lambda}(\mathbf{k})$. Assuming that input operators satisfy canonical commutation rules, it is not difficult to show that the output operators $\hat{c}_{R\lambda}(\mathbf{k})$ and $\hat{a}_{L\lambda}(\mathbf{k})$ satisfy canonical commutation rules too:

$$[\hat{a}_{L\lambda}(\mathbf{k}),\hat{a}^\dagger_{L\lambda'}(\mathbf{k}')]=[\hat{c}_{R\lambda}(\mathbf{k}),\hat{c}^\dagger_{R\lambda'}(\mathbf{k}')]=\delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}'), \tag{4}$$
$$[\hat{a}_{L\lambda}(\mathbf{k}),\hat{c}^\dagger_{R\lambda'}(\mathbf{k}')]=[\hat{c}_{R\lambda}(\mathbf{k}),\hat{a}^\dagger_{L\lambda'}(\mathbf{k}')]=0,$$

while the intracavity operators $\hat{b}_{R\lambda}(\mathbf{k})$ and $\hat{b}_{L\lambda}(\mathbf{k})$ satisfy anomalous commutation rules [10,19]

$$\begin{aligned}[\hat{b}_{R\lambda}(\mathbf{k}),\hat{b}^\dagger_{R\lambda'}(\mathbf{k}')]&=\delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}')\frac{1-|r_{1\lambda}(\mathbf{k})r_{2\lambda}(\mathbf{k})|^2}{|1-r_{1\lambda}(\mathbf{k})r_{2\lambda}(\mathbf{k})e^{2i\delta}|^2}\\&=[\hat{b}_{L\lambda}(\mathbf{k}),\hat{b}^\dagger_{L\lambda'}(\mathbf{k}')],\end{aligned} \tag{5}$$

$$\begin{aligned}[\hat{b}_{L\lambda}(\mathbf{k}),\hat{b}^\dagger_{R\lambda'}(\mathbf{k}')]&=\delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}')\frac{r_{2\lambda}(\mathbf{k})e^{i\delta}[1-|r_{1\lambda}(\mathbf{k})|^2]+r^*_{1\lambda}(\mathbf{k})e^{-i\delta}[1-|r_{2\lambda}(\mathbf{k})|^2]}{|1-r_{1\lambda}(\mathbf{k})r_{2\lambda}(\mathbf{k})e^{2i\delta}|^2}\\&=[\hat{b}_{R\lambda}(\mathbf{k}),\hat{b}^\dagger_{L\lambda'}(\mathbf{k}')]^*.\end{aligned} \tag{6}$$

These equations are the three-dimensional generalization of Eqs. (9) given in Ref. [10] for one-dimensional fields.

Because of the presence of the cavity, the positive frequency part of the vector potential is now written as

$$\hat{\mathbf{A}}^+(\mathbf{r},t)=\int d\mathbf{k}\left(\frac{\hbar}{16\pi^3\varepsilon_0\omega}\right)^{1/2}\sum_{\lambda=1,2}\mathbf{F}_\lambda(\mathbf{k},\mathbf{r})\exp(-i\omega t), \tag{7}$$

where

$$\mathbf{F}_\lambda(\mathbf{k},\mathbf{r})=\begin{cases}\hat{a}_{R\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_+)e^{i\mathbf{k}_+\cdot\mathbf{r}}+\hat{a}_{L\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_-)e^{i\mathbf{k}_-\cdot\mathbf{r}}, & -\infty<z<l/2\\ \hat{b}_{R\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_+)e^{i\mathbf{k}_+\cdot\mathbf{r}}+\hat{b}_{L\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_-)e^{i\mathbf{k}_-\cdot\mathbf{r}}, & -l/2<z<l/2\\ \hat{c}_{R\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_+)e^{i\mathbf{k}_+\cdot\mathbf{r}}+\hat{c}_{L\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_-)e^{i\mathbf{k}_-\cdot\mathbf{r}}, & l/2<z<\infty.\end{cases} \tag{8}$$

We note that using different pairs of annihilation operators for each region of space delimited by the cavity, as in Eqs. (7) and (8), we obtain a free-field like representation for all space inside and outside the cavity, but the canonical commutation rules are lost for intracavity operators, as shown by Eqs. (5) and (6). Conversely, if we choose the mode functions, for example, as in Ref. [12], we lose free-field-like representation, but we obtain canonical commutation rules

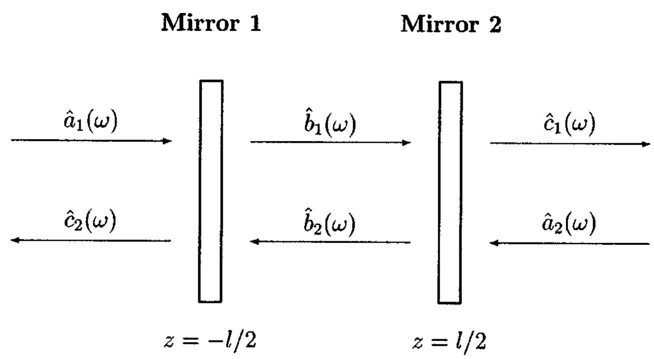

FIG. 2. Schematic representation of a one-dimensional Fabry-Perot cavity with notation for input operators $\hat{a}_i(\omega), i=1,2$, inside operators $\hat{b}_i(\omega), i=1,2$, and output operators $\hat{c}_i(\omega), i=1,2$.

for annihilation and creation operators in whole space. It is easy to show that our result agrees with that of De Martini *et al.* [12], indeed, defining

$$\hat{a}_{R\lambda}(\mathbf{k}) \equiv \hat{a}_{\mathbf{k}\lambda}, \qquad \hat{c}_{L\lambda}(\mathbf{k}) \equiv \hat{a}'_{\mathbf{k}\lambda}, \tag{9}$$

by a straightforward calculation, we obtain their result

$$\mathbf{F}_\lambda(\mathbf{k},\mathbf{r}) = \boldsymbol{\epsilon}(\mathbf{k},\lambda)(U_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} + U'_{\mathbf{k}\lambda}\hat{a}'_{\mathbf{k}\lambda}), \quad -\infty < z < +\infty, \tag{10}$$

where now the mode functions $U_{\mathbf{k}\lambda}$ are defined differently on the three regions of the space, as shown in Tables (2.6) and (2.7) of Ref. [12].

A. One-dimensional formulation

In this paper, we are interested in deriving input-output relations for a single transverse mode of the cavity, having finite cross-section area $\mathcal{A}$ orthogonal to the z axis, which, in fact, depend upon the geometrical and transmitting properties of the cavity itself [20,21]. Then, following Ref. [22], we impose periodic boundary conditions on both directions x and y, fix a linear polarization parallel to the x axis, and consider only field excitations transverse to the z axis. Then, defining

$$\left.\begin{aligned} \hat{a}_{R\lambda}(0,0,k_z) &\equiv c^{1/2}\hat{a}_1(\omega) \\ \hat{c}_{L\lambda}(0,0,k_z) &\equiv c^{1/2}\hat{a}_2(\omega) \end{aligned}\right\} \quad k_z = |\mathbf{k}| \equiv \omega/c, \tag{11}$$

and similarly for the other operators, we obtain a one-dimensional representation of the field, exactly like that in Ref. [10], and as shown in Fig. 2. It is not difficult to show with a straightforward calculation that the inside operators $\hat{b}_i(\omega)$, $i=1,2$ are related to input operators $\hat{a}_i(\omega)$ and $i=1,2$ by the relations

$$\hat{\mathbf{b}}(\omega) = \mathcal{B}(\omega)\hat{\mathbf{a}}(\omega), \qquad \mathbf{f}(\omega) \equiv \begin{pmatrix} f_1(\omega) \\ f_2(\omega) \end{pmatrix} \quad (f=a,b), \tag{12}$$

where the matrix elements $B_{ij}(\omega)$ of $\mathcal{B}(\omega)$ are given by

$$B_{ii}(\omega) = \frac{t_i(\omega)}{1 - r_1(\omega)r_2(\omega)\exp(2i\omega l/c)},$$
$$B_{ij}(\omega)|_{i\neq j} = r_i(\omega)\exp(i\omega l/c)B_{jj}(\omega) \tag{13}$$

$(i,j=1,2)$, where we used the same notation as in Ref. [10]. Similarly, the output operators $\hat{c}_i(\omega)$ and $i=1,2$ are related to the input operators by

$$\hat{\mathbf{c}}(\omega) = \mathcal{C}(\omega)\hat{\mathbf{a}}(\omega), \tag{14}$$

where the matrix elements $C_{ij}(\omega)(i,j=1,2)$ of the unitary matrix $\mathcal{C}(\omega)$ are given by

$$C_{ii}(\omega) = \frac{t_1(\omega)t_2(\omega)}{1 - r_1(\omega)r_2(\omega)\exp(2i\omega l/c)},$$
$$C_{ij}(\omega)|_{i\neq j} = \frac{r_j(\omega)\exp(-i\omega l/c) + r_i(\omega)\exp[i\omega l/c + 2i\arg t_j(\omega)]}{1 - r_1(\omega)r_2(\omega)\exp(2i\omega l/c)}. \tag{15}$$

This leads to the result

$$[\hat{a}_i(\omega),\hat{a}_j^\dagger(\omega')] = [\hat{c}_i(\omega),\hat{c}_j^\dagger(\omega')] = \delta_{ij}\delta(\omega-\omega') \quad (i,j=1,2), \tag{16}$$

while for inside operators the commutation rules are given by Eqs. (9) of Ref. [10]:

$$[\hat{b}_i(\omega),\hat{b}_j^\dagger(\omega')] = [\mathcal{B}(\omega)\mathcal{B}^\dagger(\omega')]_{ij}\delta(\omega-\omega') \equiv G_{ij}(\omega)\delta(\omega-\omega'), \tag{17}$$

where, for construction, $\mathcal{G}(\omega) = \mathcal{G}^\dagger(\omega)$, being $G_{ij}(\omega) \equiv [\mathcal{G}(\omega)]_{ij}$. Noting that $\mathrm{Det}[\mathcal{B}(\omega)] \neq 0$ for $t_1(\omega)\neq 0$ and $t_2(\omega)\neq 0$, we can invert Eq. (12) to express the relation between inside and output operators as

$$\hat{\mathbf{c}}(\omega) = \mathcal{C}(\omega)\mathcal{B}^{-1}(\omega)\hat{\mathbf{b}}(\omega) \equiv \mathcal{M}(\omega)\hat{\mathbf{b}}(\omega), \tag{18}$$

where

$$\mathcal{M}(\omega) = \begin{pmatrix} \dfrac{1}{t_2^*(\omega)} & \dfrac{r_2(\omega)}{t_2(\omega)}\exp(-i\omega l/c) \\ \dfrac{r_1(\omega)}{t_1(\omega)}\exp(-i\omega l/c) & \dfrac{1}{t_1^*(\omega)} \end{pmatrix}. \tag{19}$$

It should be noted that the generic 2×2 matrix that represents a beam splitter must be *unitary* to preserve the canonical bosonic relations of commutation for both input and output operators; in our case $\mathcal{M}(\omega)$ is not unitary at all. In fact, while output operators $\hat{c}_i(\omega)$ satisfy the canonical relations (16), inside operators $\hat{b}_i(\omega)$ satisfy the relations (17), that is the so-called anomalous relations of commutation. In this case, using Eqs. (18) and (19), we obtain

$$\mathcal{M}^{\dagger}(\omega)\mathcal{M}(\omega)=\mathcal{G}^{-1}(\omega). \tag{20}$$

Because of the nonunitarity of $\mathcal{M}(\omega)$, the ''photon-number'' operator is not conserved on a single mode. In fact, from Eqs. (18)–(20) we obtain

$$\hat{\mathbf{c}}^{\dagger}(\omega)\cdot\hat{\mathbf{c}}(\omega)=\hat{\mathbf{b}}^{\dagger}(\omega)\cdot\mathcal{G}^{-1}(\omega)\cdot\hat{\mathbf{b}}(\omega)\neq\hat{\mathbf{b}}^{\dagger}(\omega)\cdot\hat{\mathbf{b}}(\omega). \tag{21}$$

Anyway, because we are working with linear transformations, the most general bilinear form in inside creation operators $\hat{b}_i^{\dagger}(\omega)$ is still bilinear in output creation operators $\hat{c}_i^{\dagger}(\omega)$:

$$\begin{aligned}&\gamma_{11}\hat{c}_1^{\dagger}\hat{c}_1^{\dagger}+\gamma_{12}\hat{c}_1^{\dagger}\hat{c}_2^{\dagger}+\gamma_{21}\hat{c}_2^{\dagger}\hat{c}_1^{\dagger}+\gamma_{22}\hat{c}_2^{\dagger}\hat{c}_2^{\dagger}\\&\quad=\beta_{11}\hat{b}_1^{\dagger}\hat{b}_1^{\dagger}+\beta_{12}\hat{b}_1^{\dagger}\hat{b}_2^{\dagger}+\beta_{21}\hat{b}_2^{\dagger}\hat{b}_1^{\dagger}+\beta_{22}\hat{b}_2^{\dagger}\hat{b}_2^{\dagger}.\end{aligned} \tag{22}$$

Because the first of the two forms applied to a vacuum generates a two-photon state of the electromagnetic field, the equality ensures the same for the second one. Therefore, if it is possible to associate with the most general two-photon state generated by inside operators a state with two photons physically generated *inside* the cavity, then, from the relations between *input* and *output* operators, we can obtain information about the field outside the cavity, that is the actual object of measurement. In the following section, we study how we can do this.

III. STATES OF THE FIELD

We define the states generated by the linear and bilinear forms of inside operators as

$$\begin{aligned}\hat{b}_i^{\dagger}(\omega)|0\rangle&\equiv|F_i(\omega);\mathrm{in}\rangle,\\\hat{b}_i^{\dagger}(\omega)\hat{b}_j^{\dagger}(\omega')|0\rangle&\equiv|F_i(\omega),F_j(\omega');\mathrm{in}\rangle,\end{aligned} \tag{23}$$

where $F_i(\omega)$ and $F_j(\omega')$ are labels that depends, on continuous variables ω,ω' and on discrete variables $i,j=1,2$. Because of Eq. (17), these states are not orthogonal:

$$\langle F_i(\omega);\mathrm{in}|F_j(\omega');\mathrm{in}\rangle=G_{ij}(\omega)\,\delta(\omega-\omega'), \tag{24}$$

and

$$\begin{aligned}&\langle F_i(\omega_1),F_j(\omega_2);\mathrm{in}|F_k(\omega_3),F_l(\omega_4);\mathrm{in}\rangle\\&\quad=G_{ik}(\omega_1)G_{jl}(\omega_2)\,\delta(\omega_1-\omega_3)\,\delta(\omega_2-\omega_4)\\&\qquad+G_{il}(\omega_1)G_{jk}(\omega_2)\,\delta(\omega_2-\omega_3)\,\delta(\omega_1-\omega_4).\end{aligned} \tag{25}$$

From Eqs. (24) and (25) we note that anomalous commutation rules, represented by the 2×2 Hermitian matrix $\mathcal{G}(\omega)$, form a metric in the two-dimensional Hilbert space generated by inside operators $\hat{b}_i(\omega)$. For example, if we write the most general one-photon state created by *inside* operators as

$$|\phi\rangle=\sum_{i=1}^{2}\int d\omega\,K_i(\omega)|F_i(\omega);\mathrm{in}\rangle, \tag{26}$$

where $K_i(\omega)\in\mathbf{C}$, then its norm is

$$\begin{aligned}\langle\phi|\phi\rangle&=\sum_{i,j}^{1,2}\int d\omega\,K_i^{*}(\omega)G_{ij}(\omega)K_j(\omega)\\&=\int d\omega\,\mathbf{K}^{\dagger}(\omega)\cdot\mathcal{G}(\omega)\cdot\mathbf{K}(\omega),\end{aligned} \tag{27}$$

where the metriclike role of matrix $\mathcal{G}(\omega)$ is clear.

As before, we can define the states generated by the linear and bilinear forms of output operators as

$$\begin{aligned}\hat{c}_i^{\dagger}(\omega)|0\rangle&\equiv|F_i(\omega);\mathrm{out}\rangle,\\\hat{c}_i^{\dagger}(\omega)\hat{c}_j^{\dagger}(\omega')|0\rangle&\equiv|F_i(\omega),F_j(\omega');\mathrm{out}\rangle,\end{aligned} \tag{28}$$

which, using Eq. (18), can be written in terms of inside operators:

$$\begin{aligned}&|F_i(\omega);\mathrm{out}\rangle\equiv\sum_{k}^{1,2}M_{ik}^{*}(\omega)|F_i(\omega);\mathrm{in}\rangle,\\&|F_i(\omega),F_j(\omega');\mathrm{out}\rangle\\&\quad\equiv\sum_{k,l}^{1,2}M_{ik}^{*}(\omega)M_{jl}^{*}(\omega')|F_k(\omega),F_l(\omega');\mathrm{in}\rangle,\end{aligned} \tag{29}$$

where $M_{ij}(\omega)\equiv[\mathcal{M}]_{ij}$. Of course these states are orthonormal.

It is possible to make number states also for a continuous distribution of modes, following the method of Blow *et al.* [22]. For this purpose we define two operators $\hat{C}_i(\eta)$ as

$$\hat{C}_i(\eta)=\int d\omega\;\eta_i^{*}(\omega)\hat{c}_i(\omega),\quad i=1,2, \tag{30}$$

where $\eta_i(\omega)$ are two arbitrary complex functions that satisfy the normalization condition

$$\int d\omega|\eta_i(\omega)|^2=1,\quad i=1,2. \tag{31}$$

The construction of number states is more problematic for inside operators. As before, we define

$$\hat{B}_i(\xi)=\int d\omega\;\xi_i^{*}(\omega)\hat{b}_i(\omega),\quad i=1,2, \tag{32}$$

where $\xi_i(\omega)$ are two complex arbitrary functions that can be chosen to satisfy the four conditions

$$[\hat{B}_i(\xi),\hat{B}_j^{\dagger}(\xi)]=\int d\omega\;\xi_i^{*}(\omega)G_{ij}(\omega)\xi_j(\omega)\equiv\Gamma_{ij}(\xi), \tag{33}$$

where $\Gamma_{ij}(\xi)$ is a given matrix. Suppose we set $\Gamma_{ij}(\xi)=\delta_{ij}$. Because $\Gamma(\xi)$ is Hermitian by construction, Eq. (33) corresponds to $2\oplus 2$ conditions, two real and a complex one, which the two complex arbitrary functions $\xi_i(\omega)(i=1,2)$

must satisfy. The ξ_i modules can be determined imposing $\Gamma_{ii}(\xi)=1$. In fact, given an arbitrary function $\overline{\xi(\omega)}$ such as

$$\int d\omega|\overline{\xi(\omega)}|^2=1, \tag{34}$$

if $G_{ii}(\omega)\neq 0$, that is if $|r_1(\omega)r_2(\omega)|^2\neq 1$, we can write $\xi_i(\omega)$ as

$$\xi_i(\omega)=\frac{|\overline{\xi(\omega)}|}{\sqrt{G_{ii}(\omega)}}e^{i\phi_i(\omega)}, \quad i=1,2, \tag{35}$$

where $\phi_i(\omega)$ is an arbitrary phase, and we can obtain $\Gamma_{ii}(\xi)=1$. Phases are still arbitrary, but in off-diagonal elements of $\Gamma(\xi)$ there is only the phase difference between $\xi_1(\omega)$ and $\xi_2(\omega)$ that is not sufficient, by itself, to satisfy the two requested conditions. In fact, you can see that only proper linear combinations of inside operators can generate canonical commutation relations. Consider the unitary matrix $\mathcal{U}(\omega)$ that makes $\mathcal{G}(\omega)$ diagonal

$$\mathcal{U}^\dagger(\omega)\cdot\mathcal{G}(\omega)\cdot\mathcal{U}(\omega)=\mathcal{D}(\omega), \tag{36}$$

where the diagonal matrix $\mathcal{D}(\omega)$ has elements $D_{ij}(\omega)=\lambda_i(\omega)\delta_{ij}$, $\lambda_i(\omega)$, $i=1,2$ being the two $\mathcal{G}(\omega)$'s eigenvalues

$$\lambda_i(\omega)=G_{11}(\omega)-(-1)^i|G_{12}(\omega)|, \quad i=1,2. \tag{37}$$

Then if we define the operators $\hat{d}_i(\omega)$ as

$$\hat{\mathbf{d}}(\omega)\equiv\mathcal{U}^\dagger(\omega)\hat{\mathbf{b}}(\omega), \tag{38}$$

you can see that they satisfy the following ''quasicanonical'' relations:

$$[\hat{d}_i(\omega),\hat{d}_j^\dagger(\omega')]=\lambda_i(\omega)\delta_{ij}\delta(\omega-\omega'). \tag{39}$$

If we want to obtain fully canonical relations it is necessary to break the unitarity of the relation between operators $\hat{d}_i(\omega)$ and $\hat{b}_i(\omega)$ introducing the matrix $\mathcal{E}(\omega)$ with elements $E_{ij}=(\lambda_i)^{-1/2}\delta_{ij}$ and to define the operators $\hat{b}_i'(\omega)$ as

$$\hat{\mathbf{b}}'(\omega)\equiv\mathcal{E}(\omega)\hat{\mathbf{d}}(\omega). \tag{40}$$

In fact an easy calculation shows that

$$[\hat{b}_i'(\omega),\hat{b}_j'^\dagger(\omega')]=\delta_{ij}\delta(\omega-\omega'), \tag{41}$$

where, using Eq. (40),

$$\hat{b}_j'(\omega)\equiv\frac{1}{\sqrt{2\lambda_j(\omega)}}[e^{i\phi(\omega)/2}\hat{b}_1(\omega) -(-1)^j e^{-i\phi(\omega)/2}\hat{b}_2(\omega)] \quad j=1,2, \tag{42}$$

and $\phi(\omega)=\arg[G_{12}(\omega)]$ is the relative phase of the two components of $\mathcal{G}(\omega)$ eigenvectors. Of course by means of these operators we could make orthonormal input number states but this would result in their being associated with the functions $\sin(\omega z/c)$ and $\cos(\omega z/c)$ and, as a consequence, the free-field-like representation will be lost.

However, it is still possible to write the most general two-photon state generated by inside operators $\hat{b}_i(\omega)$ as

$$|\psi\rangle=\sum_{i,j}^{1,2}\int d\omega\int d\omega' K_{ij}(\omega,\omega')|F_i(\omega),F_j(\omega');\text{in}\rangle, \tag{43}$$

where, by construction, the matrix $\mathcal{K}(\omega,\omega')$ of elements $K_{ij}(\omega,\omega')$, satisfy

$$\mathcal{K}(\omega,\omega')=\mathcal{K}^{\mathrm{T}}(\omega',\omega), \tag{44}$$

where ''T'' indicates transposition. In fact, the matrix $\mathcal{K}(\omega,\omega')$ is fixed by the emission process inside the cavity.

In a simpler way, a two-photon state generated by *output* operators (30) can be written as

$$\begin{aligned}|F_a(\eta),F_b(\eta);\text{out}\rangle&=(2^{-1/2})^{\delta_{ab}}\hat{C}_a^\dagger(\eta)\hat{C}_b^\dagger(\eta)|0\rangle\\&=(2^{-1/2})^{\delta_{ab}}\int d\omega\int d\omega'\,\eta_a(\omega)\eta_b(\omega')\\&\quad\times|F_a(\omega),F_b(\omega');\text{out}\rangle,\end{aligned} \tag{45}$$

where $(a,b=1,2)$. In the next section, we will show how to express this state by inside states.

IV. TWO-PHOTON STATES PROBABILITY DISTRIBUTIONS

It is well known that the inverse of photon mean flight time in a planar Fabry-Perot cavity is given by [9]

$$\gamma_{\text{cav}}\cong\frac{c}{l}\frac{1-|r_1(\omega)r_2(\omega)|}{2|r_1(\omega)r_2(\omega)|^{1/2}}. \tag{46}$$

Now we consider the spontaneous emission of a pair of two identical atoms or molecules within the microcavity [8]. Let γ_{atom} be the single atomic decay rate. In the atom-dominate decay regime (that is, when $\gamma_{\text{cav}}\ll\gamma_{\text{atom}}$ [23]), for $1/\gamma_{\text{atom}}\ll t\ll 1/\gamma_{\text{cav}}$, the electromagnetic field can be found in a state similar to $|\psi\rangle$. If the matrix $\mathcal{M}(\omega)$ should be unitary, we should calculate easily, as in the quantum theory of a lossless beamsplitter [24], the probability distribution of photon number states outside the cavity. This is not our case, however, we will see that after renormalization of the state $|\psi\rangle$ coefficients $K_{ij}(\omega,\omega')$, it is possible to obtain significant results. For this, we calculate the probability amplitude to find the electromagnetic field, represented by the state $|\psi\rangle$ within the cavity, in the outside state $|F_a(\eta),F_b(\eta);\text{out}\rangle$, $(a,b=1,2)$. It is simple to show with the use of Eqs. (29) and (43)–(45), whose result is

$$\langle F_a(\eta),F_b(\eta);\text{out}|\psi\rangle$$
$$=2(2^{-1/2})^{\delta_{ab}}\int d\omega\int d\omega'\,\eta_a(\omega)\,\eta_b(\omega')$$
$$\times[\mathcal{M}(\omega)\cdot\mathcal{G}(\omega)\cdot\mathcal{K}(\omega,\omega')\cdot\mathcal{G}^{\mathrm{T}}(\omega')\cdot\mathcal{M}^{\mathrm{T}}(\omega')]_{ab}. \tag{47}$$

By a lengthy but straightforward calculation, it is simple to show that

$$[\mathcal{M}(\omega)\cdot\mathcal{G}(\omega)\cdot\mathcal{K}(\omega,\omega')\cdot\mathcal{G}^{\mathrm{T}}(\omega')\cdot\mathcal{M}^{\mathrm{T}}(\omega')]_{ab}$$
$$\equiv P_{ab}(\omega,\omega'), \tag{48}$$

where we have defined the 2×2 matrix elements $P_{ab}(\omega,\omega')$ as

$$\begin{aligned}
P_{11}(\omega,\omega')&=\mathcal{L}_2(\omega)\mathcal{L}_1(\omega')[K_{11}+K_{22}\alpha_1(\omega)\alpha_1(\omega')\\
&\quad+K_{12}\alpha_1(\omega')+K_{21}\alpha_1(\omega)],\\
P_{12}(\omega,\omega')&=\mathcal{L}_2(\omega)\mathcal{L}_1(\omega')[K_{11}\alpha_2(\omega')+K_{22}\alpha_1(\omega)\\
&\quad+K_{12}+K_{21}\alpha_1(\omega)\alpha_2(\omega')],\\
P_{21}(\omega,\omega')&=\mathcal{L}_1(\omega)\mathcal{L}_2(\omega')[K_{22}\alpha_1(\omega')+K_{11}\alpha_2(\omega)\\
&\quad+K_{21}+K_{12}\alpha_2(\omega)\alpha_1(\omega')],\\
P_{22}(\omega,\omega')&=\mathcal{L}_1(\omega)\mathcal{L}_2(\omega')[K_{22}+K_{11}\alpha_2(\omega)\alpha_2(\omega')\\
&\quad+K_{21}\alpha_2(\omega')+K_{12}\alpha_2(\omega)],
\end{aligned}\tag{49}$$

being

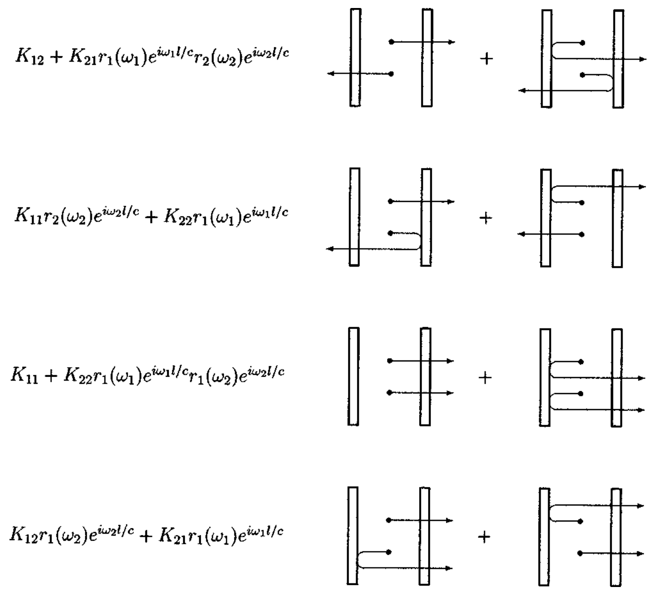

FIG. 3. Diagrams illustrating the probability amplitudes (reported in the left column), relative to Eq. (52). Here, $r_1(\omega)$ $[r_2(\omega)]$ is the reflection coefficient of mirror 1 (at the left) [2 (at the right)]. The photon of angular frequency ω_1 is always plotted higher than the photon of angular frequency ω_2.

$$\mathcal{L}_i(\omega)\equiv\frac{t_i(\omega)}{D(\omega)},\quad \alpha_i(\omega)\equiv r_i(\omega)e^{i\omega l/c},\quad i=1,2. \tag{50}$$

Now we evaluate the ratio $\mathcal{R}_{\text{out}}(R,L|R,R)$ between the probability $P_{\text{out}}(R,L)$ of observing one photon behind mirror 2 and one photon behind mirror 1 (coincidence), and the probability $P_{\text{out}}(R,R)$ of observing two photons behind mirror 2. Using Eqs. (47) we obtain

$$\mathcal{R}_{\text{out}}(R,L|R,R)=\frac{P_{\text{out}}(R,L)}{P_{\text{out}}(R,R)}=\left|\frac{\langle F_1(\eta),F_2(\eta);\text{out}|\psi\rangle}{\langle F_1^2(\eta);\text{out}|\psi\rangle}\right|^2=\left|\frac{\int d\omega_1\int d\omega_2\,\eta_1^*(\omega_1)\,\eta_2^*(\omega_2)P_{12}(\omega_1,\omega_2)}{2^{-1/2}\int d\omega_1\int d\omega_2\,\eta_1^*(\omega_1)\,\eta_1^*(\omega_2)P_{11}(\omega_1,\omega_2)}\right|^2. \tag{51}$$

We now illustrate the meaning of this formula. We start writing explicitly the value of the ratio P_{12}/P_{11}:

$$\frac{P_{12}(\omega_1,\omega_2)}{P_{11}(\omega_1,\omega_2)}=\frac{K_{11}r_2(\omega_2)e^{i\omega_2 l/c}+K_{22}r_1(\omega_1)e^{i\omega_1 l/c}+K_{12}+K_{21}r_1(\omega_1)e^{i\omega_1 l/c}r_2(\omega_2)e^{i\omega_2 l/c}}{K_{11}+K_{22}r_1(\omega_1)e^{i\omega_1 l/c}r_1(\omega_2)e^{i\omega_2 l/c}+K_{12}r_1(\omega_2)e^{i\omega_2 l/c}+K_{21}r_1(\omega_1)e^{i\omega_1 l/c}}. \tag{52}$$

This expression is only apparently complicated, but each term at numerator and denominator is susceptible to a clear physical interpretation. We assume that $K_{ij}(\omega_1,\omega_2)$ is proportional to the probability amplitude that a pair of excited molecules within the cavity emit spontaneously one photon with angular frequency ω_1 on mode *i*, and one photon with angular frequency ω_2 on mode *j*, the proportionality factor being the same for all coefficients $K_{ij}(\omega_1,\omega_2)$. More precisely, we assume that $|K_{ij}(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2$ is proportional to the emission probability of one photon on mode *i* with angular frequency between ω_1 and $\omega_1+d\omega_1$, and one photon on mode *j* with angular frequency between ω_2 and $\omega_2+d\omega_2$. At this point it is easy to see how each term that appears in the ratio (52) admits a clear physical interpretation.

With the help of Fig. 3 we can see, e.g., that the first term at numerator, corresponding to the first diagrams of the second row, give us the probability amplitude of simultaneous emission of a pair of photons toward the right, and that the photon of angular frequency ω_1 is detected behind the mirror

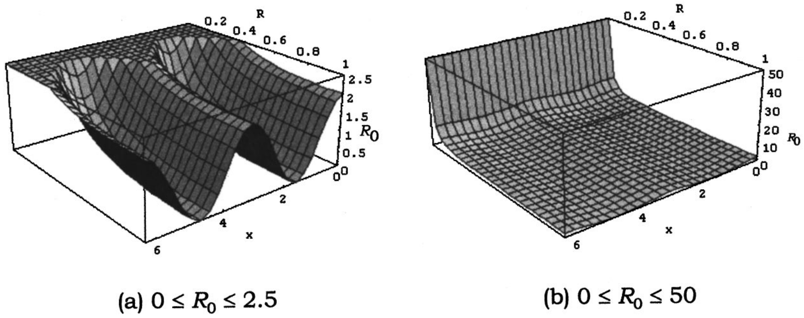

FIG. 4. Plots of $\mathcal{R}_0$ for different range of values. In (a) the plane part corresponds to values of $\mathcal{R}_0$ greater than 2.5.

2, while the photon of angular frequency ω_2 is detected behind the mirror 1 after reflection on the mirror 2. The transmission coefficients and all contributions generated by multiple reflections on the cavity mirrors are computed into terms $\mathcal{L}_2(\omega_1)\mathcal{L}_1(\omega_2)$, which we have simplified into Eq. (52). All the other terms in Eq. (52) admit analog interpretation shown by the remaining diagrams in Fig. 5.

Of course, for reasons of internal consistency of the theory, we need to renormalize the coefficients $K_{ij}(\omega_1,\omega_2)$ imposing

$$\sum_{i,j}^{1,2} \int d\omega_1 \int d\omega_2 |K_{ij}(\omega_1,\omega_2)|^2 = 1. \tag{53}$$

At this point it is easy to obtain the correct probability distributions. If we define

$$\begin{aligned} \mathcal{R}_{\text{out}}(R,L|R,R) &\equiv \mathcal{R}_1, \\ \mathcal{R}_{\text{out}}(R,L|L,L) &\equiv \mathcal{R}_2, \end{aligned} \tag{54}$$

and we impose the normalization condition

$$P_{\text{out}}(R,R) + P_{\text{out}}(R,L) + P_{\text{out}}(L,L) = 1, \tag{55}$$

the desired distributions are then obtained after some algebra in the form

$$P_{\text{out}}(R,R) = \frac{\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_1\mathcal{R}_2}, \tag{56}$$

$$P_{\text{out}}(R,L) = \frac{\mathcal{R}_1\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_1\mathcal{R}_2},$$

$$P_{\text{out}}(L,L) = \frac{\mathcal{R}_1}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_1\mathcal{R}_2},$$

where, for example, $P_{\text{out}}(R,R)$ is the normalized probability to find two photons outside the cavity behind mirror 2 of the cavity.

A. Single mode

Now we suppose that the field-mode spectrum is discretized by an appropriate procedure [22], furthermore we fix the attention on a single mode of assigned angular frequency ω. The commutation relations for creation and annihilation operators defined on this discrete set of modes are written as

$$[\hat{a}_i, \hat{a}_j^\dagger] = [\hat{c}_i, \hat{c}_j^\dagger] = \delta_{ij}, \tag{57}$$

$$[\hat{b}_i, \hat{b}_j^\dagger] = G_{ij}. \tag{58}$$

Now we can have two photons on a single discrete mode, therefore we define the states generated by bilinear and quadratic forms of inside and output operators as

$$|f_i, f_j; \text{in}\rangle \equiv (2^{-1/2})^{\delta_{ij}} \hat{b}_i^\dagger \hat{b}_j^\dagger |0\rangle, \tag{59}$$

and

$$\begin{aligned} |f_i, f_j; \text{out}\rangle &\equiv (2^{-1/2})^{\delta_{ij}} \hat{c}_i^\dagger \hat{c}_j^\dagger |0\rangle \\ &= (2^{-1/2})^{\delta_{ij}} \sum_{k,l}^{1,2} (2^{-1/2})^{-\delta_{kl}} M_{ik}^* M_{jl}^* |f_k, f_l; \text{in}\rangle, \end{aligned} \tag{60}$$

respectively. It is easy to see that

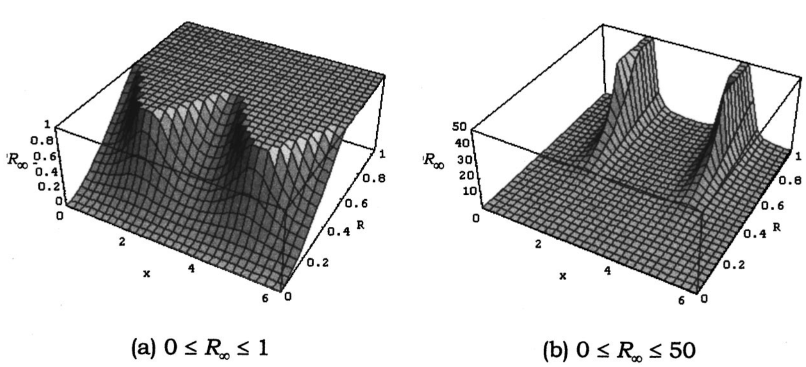

FIG. 5. Plots of $\mathcal{R}_\infty$ for different range of values. In (a) the plane part corresponds to values of $\mathcal{R}_\infty$ greater than 1.

$$\langle f_i, f_j; \text{in} | f_k, f_l; \text{in} \rangle = (2^{-1/2})^{\delta_{ij}+\delta_{kl}} (G_{ik}G_{jl} + G_{il}G_{jk}). \tag{61}$$

Exactly as before, we define the most general two-photon state created by inside operators as

$$|\psi\rangle = \sum_{i,j}^{1,2} K_{ij} |f_i, f_j; \text{in}\rangle, \tag{62}$$

where $\mathcal{K} = \mathcal{K}^{\mathrm{T}}$, and we calculate

$$\langle f_a, f_b; \text{out} | \psi \rangle = 2 (2^{-1/2})^{\delta_{ab}} (\mathcal{M} \cdot \mathcal{G} \cdot \overline{\mathcal{K}} \cdot \mathcal{G}^{\mathrm{T}} \cdot \mathcal{M}^{\mathrm{T}})_{ab}, \tag{63}$$

where we have defined the matrix $\overline{\mathcal{K}}$ as

$$[\overline{\mathcal{K}}]_{ij} \equiv \overline{K_{ij}} = (2^{-1/2})^{\delta_{ij}} K_{ij}. \tag{64}$$

Therefore, Eq. (49) is formally still valid if we make the substitution $K_{ii} \to K_{ii}/\sqrt{2}$.

From this point we consider the case of a symmetrical cavity, that is we assume $r_1(\omega) = r_2(\omega) \equiv r(\omega)$ and $t_1(\omega) = t_2(\omega) \equiv t(\omega)$. For simplicity, we choose the phase of transmission and reflection coefficients as

$$t(\omega) = i\sqrt{1-R} \quad \text{and} \quad r(\omega) = -\sqrt{R}, \tag{65}$$

and redefine, into a more expressive form:

$$K_{11} \equiv C_{RR}, \quad K_{22} \equiv C_{LL}, \quad K_{12} + K_{21} = 2K_{12} \equiv C_{RL}. \tag{66}$$

Then in discrete mode representation, the normalization conditions can be written as

$$|C_{RR}|^2 + |C_{RL}|^2 + |C_{LL}|^2 = 1. \tag{67}$$

Finally, we can write

$$\mathcal{R}_{\text{out}}(R,L|R,R) = \left| \frac{\sqrt{2}(C_{RR}+C_{RR}) r e^{i\omega l/c} + C_{RL}(1 + r^2 e^{2i\omega l/c})}{C_{RR} + C_{LL} r^2 e^{2i\omega l/c} + \sqrt{2} C_{RL} r e^{i\omega l/c}} \right|^2. \tag{68}$$

We note the presence of factors $\sqrt{2}$, which we have introduced because of different normalization required by the discrete mode spectrum. Because of them, there is no exact correspondence between the diagram of Fig. 3 and terms into Eq. (68). This factor arises since we are not using an orthonormal base and therefore a mixing between normalization of states with a photon on mode and two photons on mode is produced. Indeed, we will see in the next section that by working with single-photon states that admit only a single normalization factor, it is possible to obtain a direct association between diagrams and formulas.

Now we calculate Eq. (68) for some particular value of the state $|\psi\rangle$. Let $|\psi\rangle$ coincide with each of the three states of the orthonormal base defined in the Appendix. It can readily be shown that for (a) $|\psi\rangle = |n_\pm\rangle$

$$C_{RL} = \pm \frac{1}{\sqrt{2}}, \quad C_{RR} = C_{LL} = \frac{1}{2} \Rightarrow \mathcal{R}_{\text{out}}(R,L|R,R) = 2, \tag{69}$$

and for (b) $|\psi\rangle = |n_0\rangle$

$$C_{RL} = 0, \quad C_{RR} = -C_{LL} = -\frac{1}{\sqrt{2}} \Rightarrow \mathcal{R}_{\text{out}}(R,L|R,R) = 0. \tag{70}$$

It is remarkable that for these states having high symmetry, $\mathcal{R}_{\text{out}}$ does not depend either on mirrors reflectivity, nor on the phase $\omega l/c$. Following our interpretative scheme, Eq. (69) shows that when the emission probability of the pair of photons on the same way or on the opposite way is the same, that is $|C_{RL}|^2 = |C_{RR}|^2 + |C_{LL}|^2 = 1/2$, the probability of observing a coincidence is twice that of the probability of not observing. Instead when the two photons are emitted along a common way, but with pair emission probability amplitude toward right and left that differs for a sign, Eq. (70), the probability of observing a coincidence is zero. But within our interpretative scheme, which requires no distinction between left and right for emission of the pair of the photons in a symmetrical cavity, it is hard to think that this state really exists. Therefore a state $|n_0\rangle$ having $C_{RR} \neq C_{LL}$ is difficult to accept.

From this point we consider only the case $C_{RR} = C_{LL}$ and we define

$$\frac{C_{RR}}{C_{RL}} \equiv \zeta = |\zeta| e^{i \arg \zeta}. \tag{71}$$

Then using Eq. (71), Eq. (68) can be written as

$$\mathcal{R}_{\text{out}}(R,L|R,R) = \frac{8R^2 z^2 - 4\sqrt{2R}[\cos(x+y) + R\cos(x-y)]z + 1 + 2R\cos 2x + R^2}{(1 + 2R\cos 2x + R^2)z^2 - 2\sqrt{2R}[\cos(x-y) + R\cos(x+y)]z + 2R}, \tag{72}$$

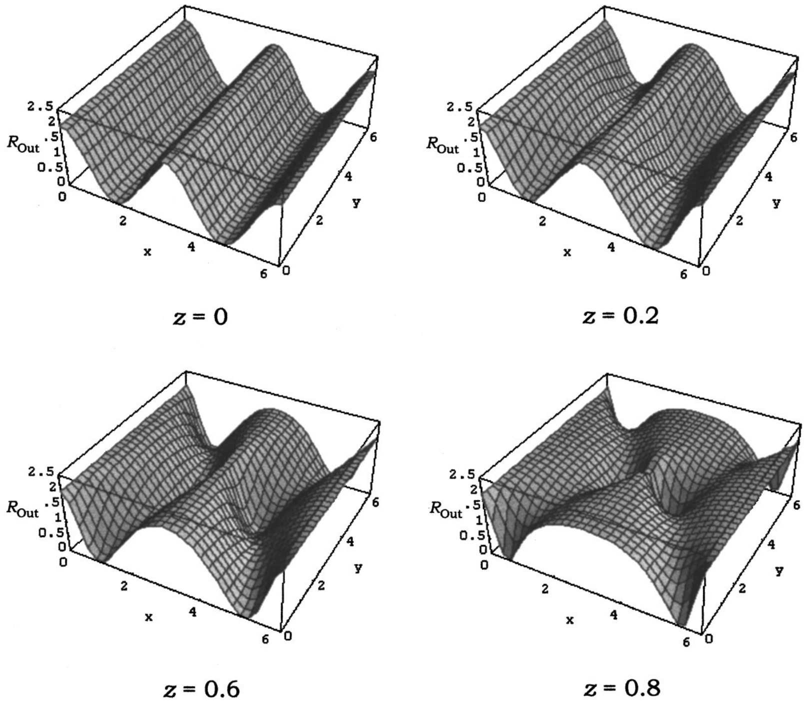

FIG. 6. Four plots of $\mathcal{R}_{\text{out}}$ calculated for a symmetrical cavity $R=0.999$ and several values of z. The dependence from R for $R\geqslant 0.8$ is negligible and not reported in the figure.

where we have used the following notation:

$$x\equiv \omega l/c,\quad y\equiv \arg \zeta,\quad z\equiv |\zeta|. \tag{73}$$

This expression seems still rather complicated, but we can learn something from it considering the limit cases $C_{RR}=0\Leftrightarrow z=0$ and $C_{RL}=0\Leftrightarrow z=\infty$:

$$\begin{aligned}\lim_{z\to 0}\mathcal{R}_{\text{out}}(R,L|R,R)&=\frac{1+2R\cos 2x+R^2}{2R}\equiv \mathcal{R}_0,\\ \lim_{z\to \infty}\mathcal{R}_{\text{out}}(R,L|R,R)&=\frac{8R^2}{1+2R\cos 2x+R^2}\equiv \mathcal{R}_\infty .\end{aligned} \tag{74}$$

The first of Eqs. (74) is shown in Fig. 4. From Fig. 4(b), we can see that when $R\to 0$, $\mathcal{R}_0\to\infty$ that is, from Eq. (68), $P_{\text{out}}(R,R)\to 0$. Indeed, if in the absence of the cavity the two photons are emitted, one toward the right and one toward the left, it is impossible to detect two from the same side. When $R\geqslant 0.8$, $\mathcal{R}_0$ is practically independent from R while it presents an oscillation of period π in x. Observing Fig. 4(a), it is evident that when $R\geqslant 0.5$, near the resonance $x\simeq\pi$ we have $\mathcal{R}_0\simeq 2$, while near $x=(2n+1)\pi/2$, n integer, there is a region for which $\mathcal{R}_0<1$. This loss of coincidence is due to the fact that only the first and the last pairs of diagrams in Fig. 3 contribute to $\mathcal{R}_0$, but in the first, which gives the probability amplitude of observing the coincidence, the two amplitudes C_{RL} and $C_{RL}Re^{2i\omega l/c}$ interfere destructively for $x\simeq(2n+1)\pi/2$ and $R\simeq 1$, causing zero total amplitude. The second of Eqs. (74) is shown in Fig. 5(a) for $0<\mathcal{R}_0<1$; the plane part of the graph corresponds to the larger than 1 values. First of all we observe an obvious fact: for $R=0$, we have $\mathcal{R}_\infty=0$, that is if the two photons are emitted both on the same way, it is impossible to observe a coincidence in absence of a cavity that mixes the directions. From the graph it is evident that when R increments, the reflections increment too, and the coincidence probability arises from the zero value. In Fig. 5(b), the behavior of $\mathcal{R}_\infty$ is shown for $\mathcal{R}_\infty\leqslant 50$. We note that when $x=(2n+1)\pi/2$ and $R\to 1$, we have $\mathcal{R}_\infty\to\infty$, that is the probability of observing a pair of photons from a side of the cavity goes to zero. Indeed the second and the third diagram of Fig. 3 contribute to $\mathcal{R}_\infty$ but the third diagram, giving the probability of observation of two parallel photons, goes to zero for $x=(2n+1)\pi/2$ and $R\simeq 1$ because of destructive interference between C_{RR} and $C_{LL}Re^{2i\omega l/c}$. Therefore for $z\gg 1$ and realistic reflectivity, the probability of observing coincidence is always bigger than the probability of observing two photons on the same side of the cavity.

On the other hand, we have also seen that for $z\to 0$, $\mathcal{R}_{\text{out}}$ can be less then 1 for $R\to 1$ and this is certainly the most interesting case to investigate. In Fig. 6, show the behavior of Eq. (72), as a function of x and y, for several values z and $R=0.999$. For $R\geqslant 0.8$, the dependence from R is negligible. In Fig. 7, we show the contour plot of $\mathcal{R}_{\text{out}}$, between 0 and 1. From this figure it is evident that the dark zones, corresponding to $\mathcal{R}_{\text{out}}<1$, have an extension gradually decreasing for z increasing, until they disappear for $z\geqslant 1$ (not shown in Fig. 7). It is interesting to note that the probability of observing two photons on one side of the cavity is bigger than coincidence probability, when within the cavity the two photons are emitted along the opposite way. Indeed only the third and fourth diagrams in Fig. 3 contribute to $P_{\text{out}}(R,R)$, but while in the fourth diagram the two amplitudes are always in phase, in the third diagram the two amplitudes C_{RR} and $C_{LL}Re^{2i\omega l/c}$ can have opposite phases and interfere destructively for $R\sim 1$. Then if $|C_{RR}|>|C_{RL}|$, that is if $z>1$, either the third or the fourth diagram give

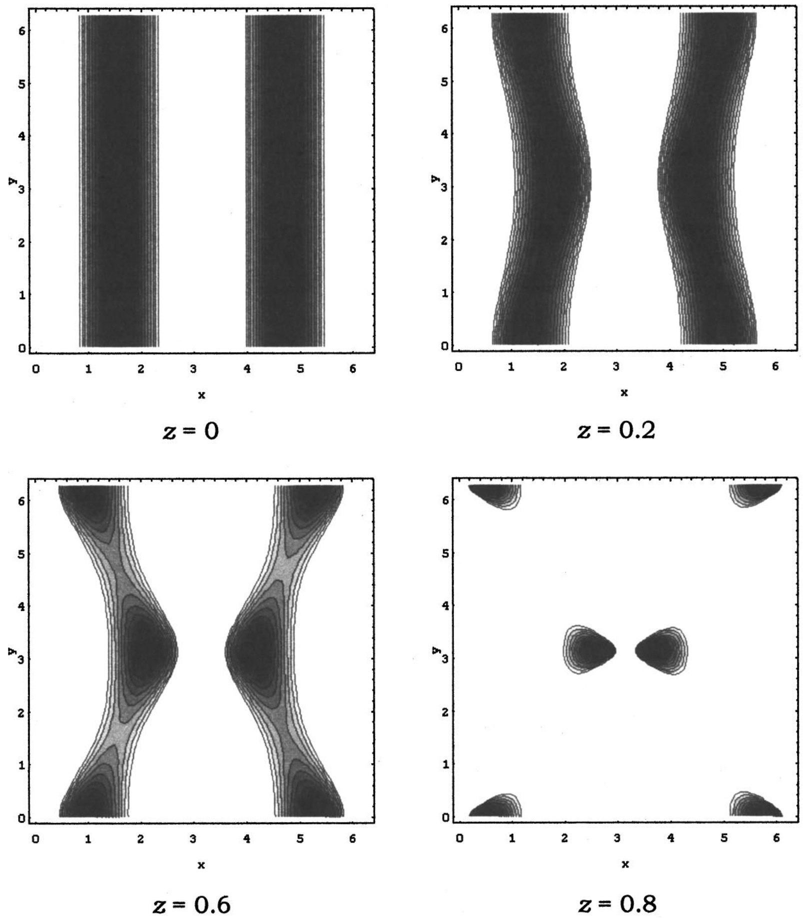

FIG. 7. Contour plot of corresponding plot in Fig. 6, shows for values $\mathcal{R}_{\text{out}}$ between 0 and 1. It is evident that for z increasing, the zone on the plane x-y in which $\mathcal{R}_{\text{out}}<1$ is decreasing.

negligible contributions and $P_{\text{out}}(R,R)\sim 0$. Instead if $|C_{RL}|>|C_{RR}|$ $(z<1)$, the fourth diagram gives a consistent contribution to $P_{\text{out}}(R,R)$ and at the same time the first diagram (proportional to C_{RL} but in the condition of destructive interference) gives a negligible contribution to $P_{\text{out}}(R,L)$.

Another interesting case is that of the resonance in a broad sense, that is $\omega l/c=\pi N$, with N integer: for N odd there is resonance in a strict sense while for N even there is anti-resonance. In this case Eq. (72) can be simplified and written as

$$\mathcal{R}_{\text{out}}(R,L|R,R)=\frac{4F^2z^2-(-1)^N 4Fz\cos y+1}{z^2-(-1)^N 2Fz\cos y+F^2}, \tag{75}$$

where $F\equiv\sqrt{2R}/(1+R)$. In Fig. 8(a), a plot of Eq. (75) is shown as a function of z and y, for $R=0.5$ and N odd. The analog plot for N even can be obtained translating the plot by an amount π along the y axis. It is evident that the ratio $\mathcal{R}_{\text{out}}$ is always greater than 1 except for a small region centered around $y=\pi$ and $z=1/2F$ that disappears for $R\rightarrow 1$. We can always write Eq. (75) as a ratio between two second degree polynomials in z:

$$\mathcal{R}_{\text{out}}(R,L|R,R)=4F^2\frac{(z-z_+^u)(z-z_-^u)}{(z-z_+^d)(z-z_-^d)}, \tag{76}$$

where we have defined the roots of the two polynomials as

$$z_\pm^u\equiv-\frac{1}{2F}e^{\pm iy}, \quad z_\pm^d\equiv-Fe^{\pm iy}. \tag{77}$$

Only for $y=\pi(\operatorname{mod}2\pi)$ we can have a real positive root:

$$\mathcal{R}_{\text{out}}(R,L|R,R)|_{y=\pi}=\left(\frac{2Fz-1}{z-F}\right)^2. \tag{78}$$

In Fig. 8(b), a plot of Eq. (78) is shown for values of z near to $1/2F$; from this plot we can see in detail the ''jump'' from the pole to zero. From Fig. 8(d), we can observe that for $R\rightarrow 1$ the pole in F and the zero in $1/2F$ tend to the common value $1/\sqrt{2}$ compensating each other, so that $\mathcal{R}_{\text{out}}=2$. The distance between the pole and zero decreases as $\sim(1-R)^2$ and already for $R=0.9$ is less than a part in a hundred. It is reasonable to think that for higher and more realistic reflectivity, it is not possible to generate really a state so well-defined to discriminate between the pole and zero. Furthermore the really physical situation is always described by a continuous superposition of modes, therefore we think that the effective value of $\mathcal{R}_{\text{out}}(R,L|R,R)|_{y=\pi}$ is ~ 2 in all of plane y-z. Lastly, we note that when $C_{RR}=0$ or $C_{RL}=0$, we have, respectively,

$$\lim_{z\rightarrow 0}\mathcal{R}_{\text{out}}(R,L|R,R)=\frac{1}{F^2}\overset{R\rightarrow 1}{\longrightarrow}2, \tag{79}$$

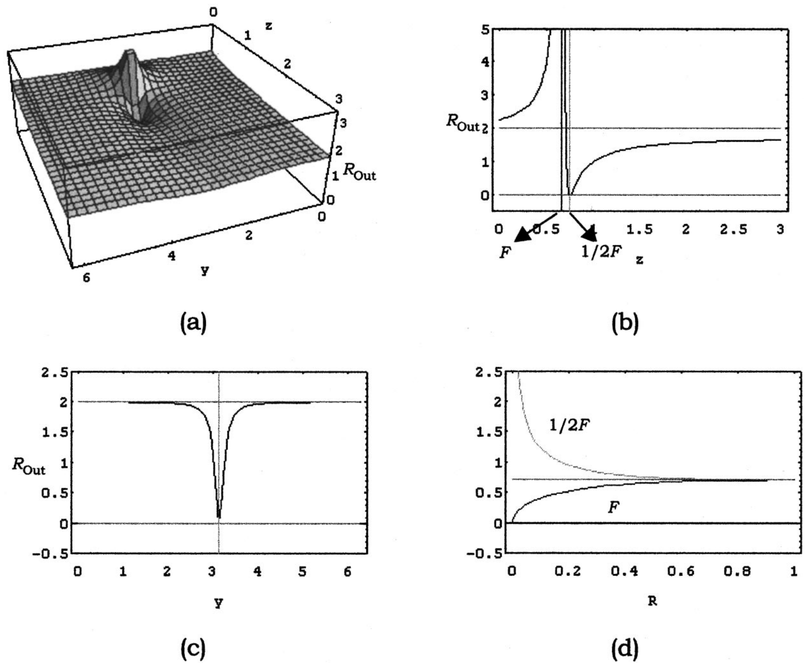

FIG. 8. (a) Plot of Eq. (75) for $R=0.5$. This is not a realistic value, but we choose it to show in a clear manner the various quantities. (b) The same as in (a) for $y=\pi$; the vertical straight lines passing through the pole in F and through zero in $1/2F$. (c) The same that in (a) for $z=1/2F$. (d) Plot of $1/2F$ (up) and F (down), as function of the reflectivity R.

$$\lim_{z\to\infty} \mathcal{R}_{\mathrm{out}}(R,L|R,R)=4F^2 \overset{R\to 1}{\to} 2. \tag{80}$$

From Eqs. (79) and (80) we deduce that in these limits it is physically indifferent if the two photons are emitted in the same or in the opposite way within the cavity. We have already obtained the result $\mathcal{R}_{\mathrm{out}}=2$, when $|\psi\rangle=|n_\pm\rangle$ independently from R, corresponding to equal probability of emission of a pair of photons along the same way or the opposite way. In the present case, we have the same result for $\omega l/c=\pi N$ and $R\sim 1$. This is consistent with the fact that in the limit of total reflectivity in which all frequencies satisfy the resonance (in a broad sense) conditions $\omega_n=n\pi c/l$, n integer, it is impossible to speak of direction (left or right) of emission of a photon, because the two counterpropagating waves that constitute a stationary wave within the cavity have exactly the same weight. Finally, we note that since $P_{\mathrm{out}}(R,R)=P_{\mathrm{out}}(L,L)$, from Eqs (55) and (56) follows,

$$P_{\mathrm{out}}(R,L)=\frac{\mathcal{R}_{\mathrm{out}}}{2+\mathcal{R}_{\mathrm{out}}},\quad P_{\mathrm{out}}(R,R)=\frac{1}{2+\mathcal{R}_{\mathrm{out}}}. \tag{81}$$

Then when $\mathcal{R}_{\mathrm{out}}=2$, we have

$$P_{\mathrm{out}}(R,L)=\frac{1}{2},\quad P_{\mathrm{out}}(R,R)=\frac{1}{4}. \tag{82}$$

B. Single-photon states

In this section, we started with investigation of two-photon states, because we were interested in the leak of symmetry in the photon-number probability distributions. Nevertheless the study of one-photon states is not void of interest. Indeed in the previous subsection we have shown that in discrete-mode representation, the interpretation of the results was ‘‘contaminated’’ from factors $\sqrt{2}$ generated by mixing between normalization of states with one photon per mode (e.g., $|1,1\rangle$) and two photons per mode (e.g., $|2,0\rangle$). Now we will see that when working with one-photon states this mixing never appears. In the most general form, the one-photon state $|\phi\rangle$ generated by inside operators can be written as

$$|\phi\rangle=\sum_{i=1}^{2}\int d\omega\, K_i(\omega)|F_i(\omega);\mathrm{in}\rangle, \tag{83}$$

while the analogous state generated by output operators is given by

$$|F_a(\eta);\mathrm{out}\rangle=\hat{C}_a^\dagger(\eta)|0\rangle=\int d\omega\; \eta_a(\omega)|F_a(\omega);\mathrm{out}\rangle. \tag{84}$$

The probability amplitude to find the electromagnetic field, represented by the state $|\phi\rangle$ within the cavity, in the state $|F_a(\eta);\mathrm{out}\rangle$, is

$$\langle F_a(\eta);\mathrm{out}|\phi\rangle=\sum_{i=1}^{2}\int d\omega\; \eta_a^*(\omega)[\mathcal{M}(\omega)\mathcal{G}(\omega)]_{ai}K_i(\omega). \tag{85}$$

The ratio between the probability $P_{\mathrm{out}}(R)$ of observing a photon behind mirror 2 and the probability $P_{\mathrm{out}}(L)$ of observing a photon behind mirror 1 is equal to

$$\mathcal{R}_{\rm out}(R|L)=\frac{P_{\rm out}(R)}{P_{\rm out}(L)}=\left|\frac{\langle F_1(\eta);{\rm out}|\phi\rangle}{\langle F_2(\eta);{\rm out}|\phi\rangle}\right|^2$$
$$=\left|\frac{\int d\omega\ \eta_1^*(\omega)\mathcal{L}_2(\omega)[C_R(\omega)+C_L(\omega)r_1(\omega)e^{i\omega l/c}]}{\int d\omega\ \eta_2^*(\omega)\mathcal{L}_1(\omega)[C_L(\omega)+C_R(\omega)r_2(\omega)e^{i\omega l/c}]}\right|^2, \tag{86}$$

where we have redefined $K_1(\omega)\equiv C_R(\omega)$ and $K_2(\omega)\equiv C_L(\omega)$. Exactly as in Sec. V A, if we assume $C_R(\omega)(C_L(\omega))$ proportional to the probability amplitude that an active medium within the cavity emits a photon of angular frequency ω toward the right (left), each of the terms into Eq. (86) admits a clear physical interpretation illustrated in Fig. 9.

Since in discrete mode representation $|f_i;{\rm in}\rangle\equiv\hat{b}_i^\dagger|0\rangle$, it is evident that passing to the discrete case, each term between the square brackets in Eq. (86) remain formally unchanged, without any $\sqrt{2}$ factor.

For sake of consistency now we must impose

$$\int d\omega\{|C_R(\omega)|^2+|C_L(\omega)|^2\}=1. \tag{87}$$

In this case the normalized probabilities $P_{\rm out}(R)$ and $P_{\rm out}(L)$ are given by

$$P_{\rm out}(R)=\frac{\mathcal{R}_{\rm out}(R|L)}{1+\mathcal{R}_{\rm out}(R|L)},\quad P_{\rm out}(L)=\frac{1}{1+\mathcal{R}_{\rm out}(R|L)}. \tag{88}$$

V. SUMMARY

We have derived some simple relations for an electromagnetic field inside and outside an optical cavity, using a nonunitary transformation between inside and output operators. The convenience of this approach lies in the fact that we do not need to know any details of internal processes that generate the two photons, to calculate the photon-number probability distribution outside the cavity. Conversely, we can obtain information on internal processes, by comparing the calculated and measured probability distribution. The method is a natural extension to nonunitary transformation, of the usual method employed, e.g., in the quantum theory of the lossless beamsplitter. [24]. The key role of this analysis is played by the $\mathcal{R}_{\rm out}(R,L|R,R)$ function defined in Eq. (51), which is the ratio between the probability of detecting a single photon behind each mirror's cavity, and the probability of detecting two photons behind a single mirror's cavity. Each term of this expression has a straightforward physical counterpart, as shown in Fig. 3. Using the single-mode version of Eq. (51), we found both Bose-Einstein ($\mathcal{R}_{\rm out}=1$) and Maxwell-Boltzmann ($\mathcal{R}_{\rm out}=2$) partition statistics of photons emitted, varying source and cavity characteristics, as shown in Figs. 4–7.

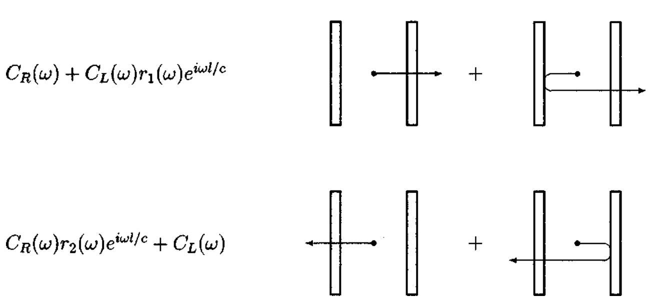

FIG. 9. Diagrams illustrating the probability amplitudes (reported in the left column), relative to Eq. (86). Here, $r_1(\omega)$ $[r_2(\omega)]$ is the reflection coefficient of mirror 1 (at the left) [2 (at the right)].

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APPENDIX

In Sec. IV, we introduced the three orthonormal states $|n_\pm\rangle$ and $|n_0\rangle$ without derivation. This will be done in this appendix.

We have rewritten the commutation relation for operators $\hat{b}_1(\omega)$ and $\hat{b}_2(\omega)$ in discrete-mode representation and the symmetrical cavity as

$$[\hat{b}_2(\omega),\hat{b}_2^\dagger(\omega)]=[\hat{b}_1(\omega),\hat{b}_1^\dagger(\omega)]\equiv\Delta(\omega),$$
$$[\hat{b}_2(\omega),\hat{b}_1^\dagger(\omega)]=[\hat{b}_1(\omega),\hat{b}_2^\dagger(\omega)]^*\equiv\rho(\omega)\Delta(\omega), \tag{A1}$$

where we have defined

$$\Delta(\omega)\equiv\frac{1-R^2}{1-2R\cos(2\omega l/c)+R^2},$$
$$\rho(\omega)\equiv-2\frac{\sqrt{R}}{1+R}\cos(\omega l/c). \tag{A2}$$

Using a slightly different notation with respect to Sec. V A, we define

$$|\mathcal{R},\mathcal{L};\text{in}\rangle \equiv \frac{(\hat{b}_1^\dagger)^{\mathcal{R}}}{\sqrt{\mathcal{R}!}}\frac{(\hat{b}_2^\dagger)^{\mathcal{L}}}{\sqrt{\mathcal{L}!}}|0\rangle, \quad \mathcal{R}+\mathcal{L}=2, \tag{A3}$$

where again the factor $(\mathcal{R}!\mathcal{L}!)^{-1/2}$ is due to the possible presence of two photons on a single mode. Of course, since $\hat{b}_1$ and $\hat{b}_2^\dagger$ does not commute, these kets do not form an orthonormal base, but they are however linearly independent. Indeed if we define

$$\langle \mathcal{R},\mathcal{L};\text{in}|\mathcal{R}',\mathcal{L}';\text{in}\rangle \equiv \tilde{G}(\mathcal{R},\mathcal{L};\mathcal{R}',\mathcal{L}'), \tag{A4}$$

we can calculate, using Eqs. (A1) and (A2),

$$\tilde{G}(\mathcal{R},\mathcal{L};\mathcal{R}',\mathcal{L}') = \Delta^2 \begin{pmatrix} 1 & \sqrt{2}\rho & \rho^2 \\ \sqrt{2}\rho & 1+\rho^2 & \sqrt{2}\rho \\ \rho^2 & \sqrt{2}\rho & 1 \end{pmatrix}. \tag{A5}$$

Therefore the kets defined in Eq. (A3) are linearly independent, their Gram being determinant positive [25]:

$$\text{Det}[\tilde{G}] = \Delta^6(1-\rho^2)^3 \geqslant 0. \tag{A6}$$

By diagonalization of $\tilde{G}$, after some algebra we obtain the orthonormal base we look for

$$\begin{aligned} |n_+\rangle &= \frac{1}{2}|2,0;\text{in}\rangle + \frac{1}{\sqrt{2}}|1,1;\text{in}\rangle + \frac{1}{2}|0,2;\text{in}\rangle, \\ |n_0\rangle &= -\frac{1}{\sqrt{2}}|2,0;\text{in}\rangle + \frac{1}{\sqrt{2}}|0,2;\text{in}\rangle, \\ |n_-\rangle &= \frac{1}{2}|2,0;\text{in}\rangle - \frac{1}{\sqrt{2}}|1,1;\text{in}\rangle + \frac{1}{2}|0,2;\text{in}\rangle. \end{aligned} \tag{A7}$$

Note on matrix $\tilde{G}$

The form of the matrix $\tilde{G}$ is particular and justifies this little note. Let $\mathbf{S}_0$, $\mathbf{S}_1$, and $\mathbf{S}_2$ from the following 3×3 matrix:

$$\mathbf{S}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{S}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{S}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{A8}$$

It can be readily shown that they satisfy the following multiplication table:

$\mathbf{S}_\mu \cdot \mathbf{S}_\nu$	$\mathbf{S}_0$	$\mathbf{S}_1$	$\mathbf{S}_2$
$\mathbf{S}_0$	$\mathbf{S}_0$	$\mathbf{S}_1$	$\mathbf{S}_2$
$\mathbf{S}_1$	$\mathbf{S}_1$	$\mathbf{S}_0$	$\mathbf{S}_2$
$\mathbf{S}_2$	$\mathbf{S}_2$	$\mathbf{S}_2$	$\mathbf{S}_0+\mathbf{S}_1$

(A9)

Now consider the generic matrix $\mathcal{N}(\alpha)$, $(\alpha \in \mathbf{R})$ given by

$$\mathcal{N}(\alpha) = \mathbf{S}_0 + \alpha^2\mathbf{S}_1 + \sqrt{2}\alpha\mathbf{S}_2. \tag{A10}$$

It is characterized by

$$\text{Det}[\mathcal{N}(\alpha)] = (1-\alpha^2)^3, \quad \text{Tr}[\mathcal{N}(\alpha)] = 3+\alpha^2. \tag{A11}$$

If we indicate with $\lambda_0, \lambda_\pm$ $N(\alpha)$'s eigenvalues and with $\mathbf{n}_0, \mathbf{n}_\pm$ the corresponding eigenvectors, we can write

$$\begin{aligned} \mathbf{n}_+ &= \frac{1}{2}(1,\sqrt{2},1) \qquad \lambda_+ = (1+\alpha)^2, \\ \mathbf{n}_0 &= \frac{1}{\sqrt{2}}(-1,0,1) \qquad \lambda_0 = (1-\alpha^2), \\ \mathbf{n}_- &= \frac{1}{2}(1,-\sqrt{2},1) \qquad \lambda_- = (1-\alpha)^2. \end{aligned} \tag{A12}$$

Using Eq. (A9), it is easy to see that

$$\mathcal{N}(\alpha)\mathcal{N}(\beta) = (1+\alpha\beta)^2\mathcal{N}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right). \tag{A13}$$

Since $\mathcal{N}(0) = \mathbf{S}_0$ is the identity matrix, it is clear that the inverse of $\mathcal{N}(\alpha)$ is still a matrix of the form (A10). Indeed putting $\beta = -\alpha$ into Eq. (A13), we obtain

$$\mathcal{N}(\alpha)\mathcal{N}(-\alpha) = (1-\alpha^2)^2\mathbf{S}_0, \tag{A14}$$

that is

$$\mathcal{N}^{-1}(\alpha) = \frac{1}{(1-\alpha^2)^2}\mathcal{N}(-\alpha). \tag{A15}$$

Finally, from Eq. (A5) we get

$$\tilde{G} = \Delta^2\mathcal{N}(\rho). \tag{A16}$$

As a curiosity, we note that in two dimensions the matrices $\mathcal{N}(\alpha)$ that satisfy the algebra (A13) are given by

$$\mathcal{N}(\alpha) \equiv \sigma_0 + \alpha\sigma_1, \tag{A17}$$

where $\sigma_0 = \mathbf{I}$ and σ_1 is the first of the Pauli matrices ([26], p. 160). It is easy to show that

$$\text{Det}[\mathcal{N}(\alpha)] = 1-\alpha^2, \quad \text{Tr}[\mathcal{N}(\alpha)] = 2. \tag{A18}$$

The eigenvalues $\lambda_\pm$ and the corresponding eigenvectors $\mathbf{n}_\pm$ are given by

$$\begin{aligned} \mathbf{n}_+ &= \frac{1}{2}(1,1) \qquad \lambda_+ = 1+\alpha, \\ \mathbf{n}_- &= \frac{1}{2}(-1,1) \qquad \lambda_- = 1-\alpha. \end{aligned} \tag{A19}$$

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