Spin squeezing in two-level systems

L. Vernac, M. Pinard, and E. Giacobino
Laboratoire Kastler Brossel, Université Pierre et Marie Curie, 4 place Jussieu, F75252 Paris Cedex 05, France
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Using a quantum theory for an ensemble of two-level atoms driven by a field in an optical cavity, we show that the spin associated with the atomic ensemble can be squeezed. Two kinds of squeezing are obtained: on the one hand, self-squeezing of the spin when the input field is a coherent one and the atomic ensemble exhibits a large nonlinearity; and on the other hand, squeezing transfer from an incoming squeezed field when the atomic ensemble has a quasilinear behavior.

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I. INTRODUCTION

In high-precision atomic-physics experiments, accuracy is ultimately limited by the so-called quantum projection noise, due to the fact that the atoms are not in an eigenstate of the measured quantity. If an ensemble of N independent atoms is studied, fluctuations proportional to $N^{-1/2}$ result from this effect [1,2]. A few years ago, it was shown that using squeezed atomic states would allow one to reduce these fluctuations [3]. Squeezed atomic states are correlated states of the atomic ensemble that exhibit reduced fluctuations for the measurement of interest. Implementing such squeezed states would be of particular interest in atomic clocks, where the transition between the $F=3$ and 4 states of the ground level of cesium is detected.

As a model system we have studied the specific case of a two-level atom, and investigated the possibility of squeezing the associated spin $\frac{1}{2}$ via the interaction with an electromagnetic field. Already in the early 1980s, it was conjectured that atomic spin squeezing appeared as a counterpart of squeezing of the electromagnetic field [4]. However, the quantum noise reduction on atomic variables computed in several papers [4,5] can be obtained by a mere rotation of the atomic variables of a two-level system interacting with a coherent field. As shown in Ref. [6], in order to be useful for noise reduction in actual experiments, spin squeezing should be obtained in a plane orthogonal to the direction of the mean spin, where the mean value of the spin component is zero. Recently, it was proposed to use absorption of squeezed light by atoms in a single pass to produce spin squeezing [7].

In this paper we investigate an alternative method, in which atoms interact with light in an optical cavity. We show that collective atomic spin squeezing may appear in two different situations. On the one hand, one can achieve spin squeezing by letting the atoms interact with incoming squeezed light in an optical cavity. This configuration was already used in Refs. [8] and [9] to study the modification of the spontaneous emission due to the interaction of atoms with squeezed light. We find that the optimal conditions for the transfer of squeezing from the incident light to the atoms correspond to a nonsaturating intensity, in the strong-coupling regime. On the other hand, one can also achieve spin squeezing by letting the atoms interact with incident *coherent* light. In this case, one relies on the nonlinearity of the atomic ensemble itself to generate squeezing. This squeezing may then be called self-squeezing. For conditions in which the spin is squeezed, the outgoing field is also squeezed. Indeed, as demonstrated theoretically [10–12] and experimentally [13,14], under proper conditions the field going out of the cavity is squeezed, due to the nonlinear response of the atomic ensemble. We show that field squeezing and spin squeezing originate from the same physical process.

Spin squeezing is obtained by solving the full quantum Maxwell-Bloch equations. We derive the fluctuation spectra of the spin components in any direction in the X,Y plane, assuming that the mean spin is along the Z direction. We obtain the variances either by integrating the spectra or with a direct calculation. For a particular direction in the X,Y plane, the variance is found to be below the quantum limit, while the variance in the perpendicular direction exhibits excess noise. This result is conceptually very important, since it shows that it is possible to create quantum correlations within an ensemble of atoms interacting with a laser field in spite of the inevitable coupling to the vacuum fluctuations.

II. MODEL FOR ATOMIC FLUCTUATIONS

We consider an ensemble of motionless two-level atoms placed inside a single-ended optical cavity and interacting with a single-mode field. In order to obtain a coupling between atoms and light uniform in space, we deal with a ring cavity configuration as shown in Fig. 1, but the results obtained can be extended with a good approximation to a Fabry-Perot-type cavity. Actually in the linear cavity case and with motionless atoms, an averaging should be per-

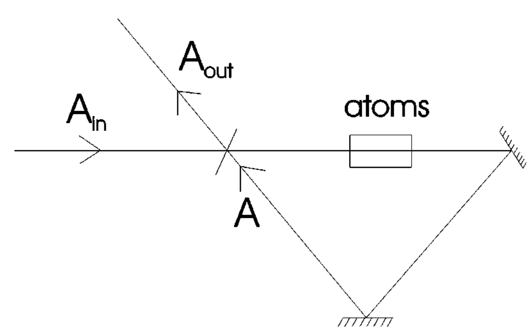

FIG. 1. Ring cavity configuration for the study of atom-field coupling.

formed over the values of the field ''seen'' by the atoms in the standing-wave structure. However, no major changes in the conclusion are expected from this effect. In most cases quantum optics experiments performed in linear cavities were found to be in good agreement with models assuming ring cavities [13,15,16].

The round-trip time in the cavity is τ, the amplitude transmission coefficient of the coupling mirror is t_{cav}, and the amplitude reflection coefficient is r_{cav}, with $r_{cav}^2+t_{cav}^2=1$. The cavity is assumed to have a high finesse ($t_{cav}\ll 1$). The decay rate of the field in the cavity is $\kappa=(1-r_{cav})/\tau=T/2\tau$, where $T=t_{cav}^2$. The atomic system has a ground state g and an excited state e, separated by the energy $\hbar\omega_0$. We call γ the decay rate of the atomic dipole, due to a purely radiative process. The atoms are driven by a field the frequency of which is ω_L. This field is represented by the operator $A(t)e^{-i\omega_L t}$. The mean-square value of the field will be expressed in number of photons per second. The cavity resonance closest to ω_L has a frequency ω_C. We define the atomic and cavity detuning parameters as $\Delta=(\omega_0-\omega_L)$, $\delta=\Delta/\gamma$ and $\Delta_C=(\omega_C-\omega_L)$, $\delta_C=\Delta_C/\kappa$. The atom-field coupling constant is $g=\mathcal{E}_0 d/\hbar$, where d is the atomic dipole, and $\mathcal{E}_0=\sqrt{\hbar\omega_L/2\epsilon_0\mathcal{S}c}$.

We define the collective polarization $P(t)$ and the collective population difference $S_z(t)$ as

$$P(t)=\sum_{i=1}^{N} S_i(t), \tag{1}$$

$$P^\dagger(t)=\sum_{i=1}^{N} S_i^\dagger(t), \tag{2}$$

$$S_z(t)=\sum_{i=1}^{N} S_{z\,i}(t), \tag{3}$$

where $S_i(t)$ and $S_i^\dagger(t)$ are the lowering and raising operators for individual atoms in the rotating frame,

$$S_i(t)=|g_i\rangle\langle e_i|e^{+i\omega_L t}, \tag{4}$$

$$S_i^\dagger(t)=|e_i\rangle\langle g_i|,e^{-i\omega_L t} \tag{5}$$

and $S_{z\,i}(t)$ is given by

$$S_{z\,i}(t)=\frac{1}{2}(|e_i\rangle\langle e_i|-|g_i\rangle\langle g_i|). \tag{6}$$

The field inside the cavity is related to the incident field and to the atomic polarization by

$$\frac{dA(t)}{dt}=-(\kappa+i\Delta_C)A(t)+ig/\tau P(t)+\sqrt{2\kappa/\tau}\,A^{in}(t), \tag{7}$$

and $A^\dagger(t)$ is given by an equation which is a Hermitian conjugate of Eq. (7). This equation expresses the derivative of the intracavity field as coming from the recycling of the field of the cavity and loss through the coupling mirror, from the field emitted by the atomic polarization, and from the incoming field A^{in}. The fluctuations of the incoming field can be seen as a Langevin force for this equation [17]. The atomic polarization and populations are given by quantum Langevin equations derived from the Bloch equations, by adding Langevin forces corresponding to the coupling with the vacuum field surrounding the system:

$$\frac{dP(t)}{dt}=-(\gamma+i\Delta)P(t)-2igA(t)S_z(t)+F_P(t), \tag{8}$$

$$\frac{dP^\dagger(t)}{dt}=-(\gamma-i\Delta)P^\dagger(t)+2igS_z(t)A^\dagger(t)+F_{P^\dagger}(t), \tag{9}$$

$$\frac{dS_z(t)}{dt}=-2\gamma(S_z(t)+N/2)-ig[A^\dagger(t)P(t)-A(t)P^\dagger(t)]+F_{S_z}(t). \tag{10}$$

The noise operators $F_P(t)$, $F_{P^\dagger}(t)$, and $F_{S_z}(t)$ are characterized by zero averages and by correlation functions that are given in the Appendix.

The field coming out of the cavity is given by

$$A^{out}(t)=t_{cav}A(t)-A^{in}(t). \tag{11}$$

We will be interested in the quantum fluctuations of the field operator A and of the atomic operators around their steady-state mean values

$$A^{in}(t)=a_{in}+\delta A^{in}(t), \tag{12}$$

$$A(t)=a_0+\delta A(t), \tag{13}$$

$$P(t)=p_0+\delta P(t), \tag{14}$$

$$P^\dagger(t)=p_{0c}+\delta P^\dagger(t), \tag{15}$$

$$S_z(t)=s_{z0}+\delta S_z(t), \tag{16}$$

where $a_{in}=\langle A^{in}(t)\rangle_{st}$, $a_0=\langle A(t)\rangle_{st}$, $p_0=\langle P(t)\rangle_{st}$, $p_{0c}=\langle P^\dagger(t)\rangle_{st}$, and $s_{z0}=\langle S_z(t)\rangle_{st}$ are the steady-state mean values. We choose the phases such that a_0 is real.

The mean values can easily be computed from Eqs. (7)–(10) without the fluctuating terms, which are the usual Maxwell-Bloch equations. The steady-state solutions of this system are given by

$$p_0=\frac{iN\beta_0(1-i\delta)}{1+\delta^2+2|\beta_0|^2}, \tag{17}$$

$$s_{z0}=-\frac{N(1+\delta^2)}{2(1+\delta^2+2|\beta_0|^2)}, \tag{18}$$

$$\sqrt{2/\kappa\tau}\beta_{in}=\beta_0\left[\left(1+\frac{2C}{1+\delta^2+2|\beta_0|^2}\right)+i\left(\delta_C-\frac{2C\delta}{1+\delta^2+2|\beta_0|^2}\right)\right], \tag{19}$$

where we have introduced the scaled fields

$$\beta_0=ga_0/\gamma, \tag{20}$$

$$\beta_{in}=ga_{in}/\gamma, \tag{21}$$

and the cooperativity parameter

$$C=\frac{Ng^2}{2\kappa\gamma\tau}. \tag{22}$$

For large enough values of β_0 and C, the solution exhibits the well-known bistable behavior of the intracavity field. It is in the vicinity of the bistable turning point that the squeezing of the outgoing field is maximum. On the other hand, for large g (or large C), one can reach, even for small β_0, a regime of vacuum Rabi splitting or a strong-coupling regime in which two peaks appear in the reflection spectrum when the atomic and cavity frequencies are equal.

To obtain equations for the field and atomic fluctuations, we linearize Eqs. (7)–(10):

$$\frac{d\,\delta A(t)}{dt}=-(\kappa+i\Delta_C)\,\delta A(t)+(ig/\tau)\,\delta P(t)+\sqrt{2\kappa/\tau}\;\delta A^{in}(t), \tag{23}$$

$$\frac{d\,\delta A^\dagger(t)}{dt}=-(\kappa-i\Delta_C)\;\delta A^\dagger(t)-(ig/\tau)\,\delta P^\dagger(t)+\sqrt{2\kappa/\tau}\delta A^{in\dagger}(t), \tag{24}$$

$$\frac{d\,\delta P(t)}{dt}=-(\gamma+i\Delta)\,\delta P(t)-2iga_0\delta S_z(t)-2ig\,\delta A(t)s_{z0}+F_P(t), \tag{25}$$

$$\frac{d\,\delta P^\dagger(t)}{dt}=-(\gamma-i\Delta)\,\delta P^\dagger(t)+2iga_0\delta S_z(t)+2ig\,\delta A^\dagger(t)s_{z0}+F_{P^\dagger}(t), \tag{26}$$

$$\frac{d\,\delta S_z(t)}{dt}=-2\gamma\delta S_z(t)-iga_0[\delta P(t)-\delta P^\dagger(t)]-ig[p_0\delta A^\dagger(t)-p_{0_c}\delta A(t)]+F_{S_z}(t). \tag{27}$$

We will compute the fluctuations in the Fourier space. For any operator $O(t)$ in the rotating frame, we define the Fourier transform as

$$O(\omega)=\int O(t)e^{i\omega t}dt, \tag{28}$$

$$O^\dagger(\omega)=\int O^\dagger(t)e^{i\omega t}dt. \tag{29}$$

To calculate the atomic fluctuations we are going to write Eqs. (25)–(27) in a matrix form in Fourier space. We introduce $|\delta S(\omega)]$ which is a three-dimensional vector,

$$|\delta S(\omega)]=\begin{bmatrix}\delta P(\omega)\\ \delta P^\dagger(\omega)\\ \delta S_z(\omega)\end{bmatrix}, \tag{30}$$

and its adjoint $[\delta S(\omega)|=|\delta S(\omega)]^\dagger$.

In the same way the atomic polarization and the fields will be represented by two-dimensional vectors:

$$|\delta P(\omega)]=\begin{bmatrix}\delta P(\omega)\\ \delta P^\dagger(\omega)\end{bmatrix}, \tag{31}$$

$$|\delta A(\omega)]=\begin{bmatrix}\delta A(\omega)\\ \delta A^\dagger(\omega)\end{bmatrix}. \tag{32}$$

In order to obtain physical insight into the spin squeezing and its connection with the field squeezing, we will solve the system of equations (23)–(27) in two steps. First, in atomic fluctuations, we separate the contributions of the fluctuations of the driving field present in the cavity from the contributions of all the empty modes. Second, we calculate the fluctuations of the cavity field to obtain the final expressions of the collective spin fluctuations.

The atomic fluctuations have two different origins, which are due to the fluctuations of the cavity field $\delta A(\omega)$ and $\delta A^\dagger(\omega)$ and to the fluctuations of the empty modes, $F_P(\omega)$, $F_{P^\dagger}(\omega)$, and $F_{S_z}(\omega)$, respectively. Thus the atomic fluctuations $|\delta S(\omega)]$ can be written as the sum of two contributions:

$$|\delta S(\omega)]=|\delta S_1(\omega)]+|\delta S_2(\omega)]. \tag{33}$$

$|\delta S_1(\omega)]$ is the linear response of the atomic spin to the fluctuations of the driving field, while $|\delta S_2(\omega)]$ is the response to the vacuum field fluctuations coming from all the directions of space, i.e., the spin fluctuations associated with spontaneous emission. We assume that spontaneous emission is not modified by the presence of the cavity, which is valid for cavities that subtend a small solid angle in space. We write $|\delta S_1(\omega)]$ as

$$|\delta S_1(\omega)]=gN[\chi(\omega)]|\delta A(\omega)], \tag{34}$$

where $[\chi(\omega)]$ is a 3×2 matrix, which can be deduced directly from Eqs. (25)–(27), or calculated by using the linear response theory [12]. We obtain the same equation for the polarization,

$$|\delta P_1(\omega)]=gN[\chi(\omega)]_{22}|\delta A(\omega)], \tag{35}$$

where $[\chi(\omega)]_{22}$ is the restriction of $[\chi(\omega)]$ to the two-dimensional subspace generated by $P(\omega)$ and $P^\dagger(\omega)$ only.

The second contribution $|\delta S_2(\omega)]$ is independent of the fluctuations of the driving field, and is linked to spontaneous emission. It is also computed from the Bloch equations. Its correlation functions can be calculated from Eqs. (25)–(27) without the terms in δA and $\delta A^\dagger$, or by using the quantum regression theorem of (see Ref. [18]).

Using Eq. (34), we have

$$|\delta S(\omega)]=gN[\chi(\omega)]|\delta A(\omega)]+|\delta S_2(\omega)]. \tag{36}$$

We now calculate the cavity field fluctuations $\delta A(\omega)$. The field seen by the atoms inside the cavity is obtained by writing the Fourier transform of Eq. (23):

$$(\kappa+i\Delta_C-i\omega)\delta A(\omega)=\sqrt{2\kappa/\tau}\delta A^{in}(\omega)+(ig/\tau)\delta P(\omega). \tag{37}$$

The dipole fluctuations generate a fluctuating field, $(ig/\tau)|\delta P]$, which is recycled by the cavity and contributes to the intracavity field fluctuations. As a result, the fluctuations of the field going out of the cavity are modified. This is at the origin of the well-known squeezing effect produced by two-level atoms in a cavity [13].

Equations for $\delta A(\omega)$ and $\delta A^\dagger(\omega)$ can be written in matrix form as

$$(\kappa+i[\varepsilon]\Delta_C-i\omega)|\delta A(\omega)]=\sqrt{2\kappa/\tau}|\delta A^{in}(\omega)]+(ig/\tau)\times[\varepsilon]|\delta P(\omega)], \tag{38}$$

where $[\varepsilon]$ is a 2×2 matrix:

$$[\varepsilon]=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{39}$$

With Eq. (36), Eq. (38) can be rewritten as

$$(\kappa+i[\varepsilon]\Delta_C-i\omega)|\delta A(\omega)]=\sqrt{2\kappa/\tau}|\delta A^{in}(\omega)]+ig^2N/\tau[\varepsilon]\times[\chi(\omega)]_{22}|\delta A(\omega)]+(ig/\tau)\times[\varepsilon]|\delta P_2(\omega)]. \tag{40}$$

We define the transfer matrix $[\mu(\omega)]$ from the incoming field to the intracavity field as

$$2\kappa[\mu(\omega)]^{-1}=(\kappa+i[\varepsilon]\Delta_C-i\omega)-(ig^2N/\tau)[\varepsilon][\chi(\omega)]_{22}. \tag{41}$$

The first term in Eq. (41) is due to the empty cavity while the second term accounts for the complex atomic susceptibility, evaluated at the operating point. Using Eq. (41) in Eq. (40), we have

$$|\delta A(\omega)]=\frac{1}{\sqrt{2\kappa\tau}}[\mu(\omega)]|\delta A^{in}(\omega)]+\frac{ig}{2\kappa\tau}[\mu(\omega)]\times[\varepsilon]|\delta P_2(\omega)]. \tag{42}$$

Indeed, each atom sees a quantum field that is modified by all the other atoms. This contributes to create a quantum correlation among the atomic ensemble.

Finally we replace the field fluctuations [Eq. (42)] in Eq. (36) to obtain the collective atomic spin fluctuations:

$$|\delta S(\omega)]=[J(\omega)]|\delta A^{in}(\omega)]+|\delta S_2(\omega)]+[K(\omega)]|\delta P_2(\omega)]. \tag{43}$$

The matrices $[J(\omega)]$ and $[K(\omega)]$ are given by

$$[J(\omega)]=\frac{Ng}{\sqrt{2\kappa\tau}}[\chi(\omega)][\mu(\omega)], \tag{44}$$

$$[K(\omega)]=\frac{iNg^2}{2\kappa\tau}[\chi(\omega)][\mu(\omega)][\varepsilon]. \tag{45}$$

We see that the spin fluctuations contain a contribution from the incoming field, modified by the atomic susceptibility [the first term in Eq. (43)] and a contribution from the resonance fluorescence [the last two terms in Eq. (43)]. These two contributions are not correlated, since they correspond to independent quantum fluctuations.

The second term in Eq. (43) is atomic excess noise due to spontaneous emission emitted into the cavity mode. The third contribution is also due to spontaneous emission. It comes from the fluorescence light emitted by the atoms into the cavity mode, and processed by the atomic medium in the cavity.

In order to evaluate the spin squeezing, we will use the variances of the spin components, which are equal to the noise spectrum integrated over the whole frequency range. Let us point out that the variances of the atomic spin ensemble can be calculated directly without calculating the spectra. We can write Eqs. (23)–(27) in matrix form as

$$\frac{d|\delta\xi(t)]}{dt}=-[B]|\delta\xi(t)]+|F_\xi], \tag{46}$$

where $|\delta\xi(t)]$ is a column vector:

$$|\delta\xi(t)]=[\delta A(t),\delta A^\dagger(t),\delta P(t),\delta P^\dagger(t),\delta S_z(t)|^T. \tag{47}$$

$[B]$ is the evolution matrix of the system of equations (23)–(27), and $|F_\xi]$ is the column vector:

$$|F_\xi(t)]=\left[\sqrt{\frac{2\kappa}{\tau}}\delta A^{in}(t),\sqrt{\frac{2\kappa}{\tau}}\times\delta A^{in\dagger}(t),F_P(t),F_{P^\dagger}(t),F_{S_z}(t)\right|^T. \tag{48}$$

We define the covariance matrix $[G(t)]$ by

$$[G(t)]=|\delta\xi(t)][\delta\xi(0)|, \tag{49}$$

and the diffusion matrix by

$$|F_\xi(t)][F_\xi(t')|=[D]\delta(t-t'). \tag{50}$$

The value of the diffusion matrix $[D]$ is given in the Appendix. We will study the two cases of a coherent input field and of a broadband squeezed input field. A single-mode squeezed input field can be written as [19]

$$\hat{a}_s=\hat{a}\cosh(r)-\hat{a}^{\dagger}e^{i\theta}\sinh(r), \tag{51}$$

$$\hat{a}_s^{\dagger}=\hat{a}^{\dagger}\cosh(r)-\hat{a}e^{-i\theta}\sinh(r), \tag{52}$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are the operators describing a coherent field, θ gives the phase of the squeezing, and r is the squeezing parameter, so that the amount of squeezing is e^{-2r}. The case of a coherent input field is obtained by taking $r=0$. This formula can be generalized to a broadband squeezed input field, for example, in the case of a squeezed vacuum produced in a parametric oscillator above threshold [20]. The correlation functions of the broadband squeezed field are given in the Appendix.

The variances of the spin components and their correlation functions are elements of $[G(0)]$, which verifies [5]

$$[B][G(0)]+[G(0)][B^{\dagger}]=[D]. \tag{53}$$

Inverting Eq. (53) gives $[G(0)]$, and consequently the spin variances. We have checked that the variances obtained by the two methods are the same. However, we will see in the following that the first method, which consists of evaluating the noise spectra, allows one to better understand the origin of the noise on the spin components, and to relate without ambiguity the occurrence of spin squeezing with the squeezing of a field inside the cavity.

III. SPIN SQUEEZING

Having calculated the quantum fluctuations of the spin-1/2 ensemble interacting with a field in a cavity, we are in a position to discuss squeezing. In the same way as a squeezed state of the electromagnetic field is defined by comparison with the coherent state, a squeezed spin state will be defined as having fluctuations in one component lower than that of a coherent spin state [3]. Since the noise spectrum of a coherent spin state is not white, contrary to the case of a light field, one has to compare the variances of the considered spin components to the variances of the components of a coherent spin state. We define the variance ΔO of any operator O as $\langle \delta O^2\rangle$, with $\delta O=O-\langle O\rangle$.

We introduce σ_x and σ_y,

$$\sigma_{xi}=(S_i+S_i^{\dagger})/2, \quad \sigma_{yi}=(S_i-S_i^{\dagger})/2i, \tag{54}$$

and S_x and S_y

$$S_x=\sum_{i=1}^{N}\sigma_{xi}, \quad S_y=\sum_{i=1}^{N}\sigma_{yi}. \tag{55}$$

Using the commutation relation of an angular momentum $[S_j,S_k]=iS_l$, where $j,k,l=x,y,z$, spin systems are often considered as squeezed if one of the spin component S_u in the x,y plane has a variance below the value given by the Heisenberg inequality:

$$\Delta S_u\leqslant\langle S_z\rangle/2. \tag{56}$$

For example, such an effect was found in the interaction of free atoms with a strong field [4]. However, this case has to be compared more carefully with the one of a coherent spin state.

We define a coherent spin state for N atoms as an ensemble of N uncorrelated spins, each of them being an eigenstate of the individual spin operator in the (θ,ϕ) direction,

$$\sigma_{\theta,\phi}=\sigma_x\sin\theta\cos\phi+\sigma_y\sin\theta\sin\phi+\sigma_z\cos\theta, \tag{57}$$

with eigenvalue $+1/2$. This coherent spin state is an eigenvalue of the collective spin operator $S_{\theta,\phi}=\Sigma_{j=1,N}\sigma_{j\theta,\phi}$, with eigenvalue $S=N/2$ [21]. It satisfies the minimum uncertainty relationship with fluctuations equally distributed over any two orthogonal components normal to the (θ,ϕ) direction, the variances of which are equal to $S/2=N/4$.

The previous effect can be obtained by just rotating the coherent spin state. In addition, it should be noted that this kind of definition for squeezing would not provide noise reduction as far as spin measurements are concerned. The criterion for spin squeezing should be that the noise is reduced in actual measurements on spin ensembles. These typically involve starting from the mean spin in some (θ,ϕ) direction, and applying a $\pi/2$ pulse, which brings the mean spin in a direction perpendicular to (θ,ϕ). One then measures the probability to find the spin along the (θ,ϕ) direction. For a coherent spin state, since the collective spin has fluctuations equal to the Heisenberg limit for all components in the plane normal to the new mean spin direction, the fluctuations in the measurement are proportional to $\sqrt{N}/2$. If, however, one can squeeze the fluctuations of the total spin within the plane orthogonal to the mean value, it will result in noise reduction in the above-mentioned measurement.

The condition for spin squeezing is then

$$\Delta S\varphi\leqslant\langle S_Z\rangle/2, \tag{58}$$

where the axes have been rotated in such a way that the Z axis is in the direction of the mean spin and φ represents a direction in the X,Y plane. $\langle S_Z\rangle$ is then the mean value of the spin, and S_X and S_Y have zero mean values. This can only occur for a spin ensemble with $N>1$, because it implies the emergence of quantum correlations within the spin ensemble.

We now calculate the variances ΔS_X and ΔS_Y of the spin variables in our reference frame. For this, we perform a rotation defined by angles θ and ϕ, such that

$$\cos\theta=\frac{\langle S_z\rangle}{\|\langle\vec{S}\rangle\|}, \tag{59}$$

$$\cos\theta\sin\phi=\frac{\langle S_x\rangle}{\|\langle\vec{S}\rangle\|}. \tag{60}$$

First, we calculate the spectra of the components of the spin in the X,Y plane, using Eq. (43). The correlation spectrum $V_{kl}(\omega)$ $(k=X,Y;l=X,Y)$ for the spin fluctuations in the X,Y plane is given by

$$\langle \delta S_k(\omega)\,\delta S_l(\omega')\rangle = 2\pi V_{kl}(\omega)\,\delta(\omega+\omega'). \tag{61}$$

A spin component S_α in the X,Y plane, making an angle α with the X axis, has a noise spectrum $V_\alpha(\omega)$ defined by

$$\langle \delta S_\alpha(\omega)\,\delta S_\alpha(\omega')\rangle = 2\pi V_\alpha(\omega)\,\delta(\omega+\omega'), \tag{62}$$

which verifies

$$V_\alpha(\omega) = V_{XX}(\omega)\cos^2\alpha + V_{YY}(\omega)\sin^2\alpha + \sin 2\alpha\,\mathrm{Re}(V_{XY}(\omega)). \tag{63}$$

The variance of this spin component, ΔS_α, is obtained by integration of its noise spectrum:

$$\Delta S_\alpha = \int V_\alpha(\omega)\frac{d\omega}{2\pi}. \tag{64}$$

Using Eq. (63), we obtain

$$\Delta S_\alpha = \cos^2\alpha\,\Delta S_X + \sin^2\alpha\,\Delta S_Y + \sin 2\alpha\,\mathrm{Re}(\Delta S_{XY}), \tag{65}$$

where ΔS_X and ΔS_Y are the variances of the spin components along the X and Y axes, respectively, and $\Delta S_{XY} = \int V_{XY}(\omega)\,(d\omega/2\pi)$.

Then the values of α for spin components having maximal and minimal variances verify

$$\tan 2\alpha = \frac{2\,\mathrm{Re}(\Delta S_{XY})}{\Delta S_X - \Delta S_Y}. \tag{66}$$

In order to investigate squeezing, we compare the minimal variance to $\langle S_Z\rangle/2$, which is slightly different from $N/4$ in our case because spontaneous emission tends to destroy the coherence of individual spins. The corresponding normalized variance thus obtained is called $\Delta S_{\min}$. Spin squeezing is achieved when $\Delta S_{\min}<1$.

A. Coherent input field: self-spin squeezing

Let us first consider the case of a coherent input field. We have explored various sets of parameters. Spin squeezing has been found when the nonlinearity of the atomic ensemble is high, which also corresponds to conditions where the output field is squeezed. Values for a few physical parameters are given in Table I, together with the contribution of the various terms of Eq. (43), which will be discussed further below. Let use emphasize that the parameters chosen in Table I correspond to feasible experiments. For example, in the experiments described in Refs. [13,14], using cesium atoms cooled in a magneto-optical trap, the dipole linewidth $\gamma/2\pi$ is equal to 2.6 MHz, and the optical cavity is such that $\kappa=2\gamma$ and $T=0.1$; the value of the incident field is then about 10 μW, and the cooperativity can reach 120. The value of the cooperativity could be increased to 1000 and more using other trapping configurations [22].

It can be seen that spin squeezing increases with the cooperativity parameter, and that squeezing values as high as 46% can be obtained. Let us mention that, for each value of the cooperativity C, the values of the input laser intensity $(\beta_{in})^2$ of the atomic detuning δ and of the cavity detuning δ_C have been optimized to obtain maximum squeezing. The optimal values of δ and of the intracavity intensity $I_0 = (\beta_0)^2$ are given in the second and third lines of Table I. This squeezing may be called "self-squeezing," inasmuch it is due to the nonlinear action of the atomic ensemble on the light fluctuations inside the cavity, which yields squeezing in the collective atomic spin.

TABLE I. Evolution with the cooperativity C of the minimum spin variance, $\Delta S_{\min}$, of the field contribution ΔS_f and the atomic contribution $\Delta S_v+\Delta S_d+\Delta S_c$, for $\kappa=2\gamma$ and $T=0.1$. For each value of the cooperativity, the optimal values of β_{in}, δ, and δ_C have been determined to maximize the spin squeezing. The corresponding values of the atomic detuning δ and the intracavity intensity I_0 are given on the second and third lines.

C	10	100	1000	10000
δ	0,8	4	15	42
I_0	0,9	2,6	8,9	23
% of spin squeezing	20%	39%	44%	46%
$\Delta S_{\min}$	0,8	0,61	0,56	0,54
ΔS_f	0,34	0,17	0,12	0,08
$\Delta S_v+\Delta S_d+\Delta S_c$	0,46	0,44	0,44	0.46

The noise spectrum of the minimum spin component, $S_{\min}(\omega)$, is represented in Fig. 2, for parameters $C=100$, $\delta=20$, and $I_0=6.5$. It exhibits a peak close to zero frequency and a peak at a frequency Ω_1, which is an eigenfrequency of the cavity-atom system in a strong field (frequencies 0 and Ω_1 are roots of the determinant of matrix $[\mu(\omega)]$ defined in Eq. (41)). Squeezing in the output field occurs around frequencies 0 and Ω_1 [12]. The corresponding normalized spin variance $\Delta S_{\min}$ is 0.83, and the spin squeezing is 17%.

To obtain further insight into the self-squeezing effect, we can examine the contributions of the various terms in Eq. (43). The first term, containing $\delta A^{in}(\omega)$, is not correlated with the others and makes only one contribution to the normalized variance. The contribution of this term, ΔS_f, shown in Table I, decreases in the same way as $\Delta S_{\min}$ when the

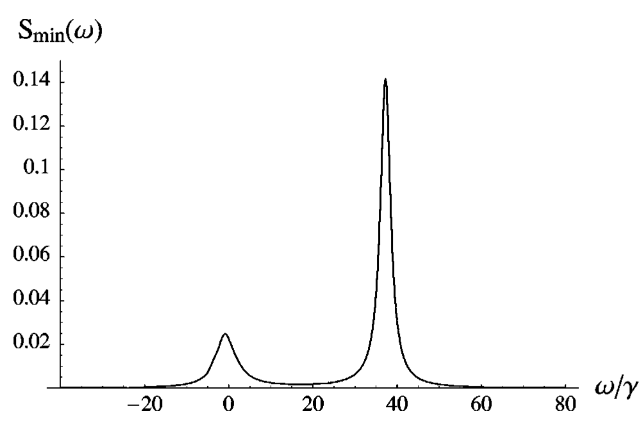

FIG. 2. Spectrum of the minimum spin component $S_{\min}(\omega)$ as a function of the normalized frequency. This spectrum has been obtained for $\kappa=2\gamma$, $I_0=6.5$, $\delta_C=6.25$, $\delta=20$, $T=0.1$, and $C=100$. These values correspond to a state close to the lower turning point of the bistability curve.

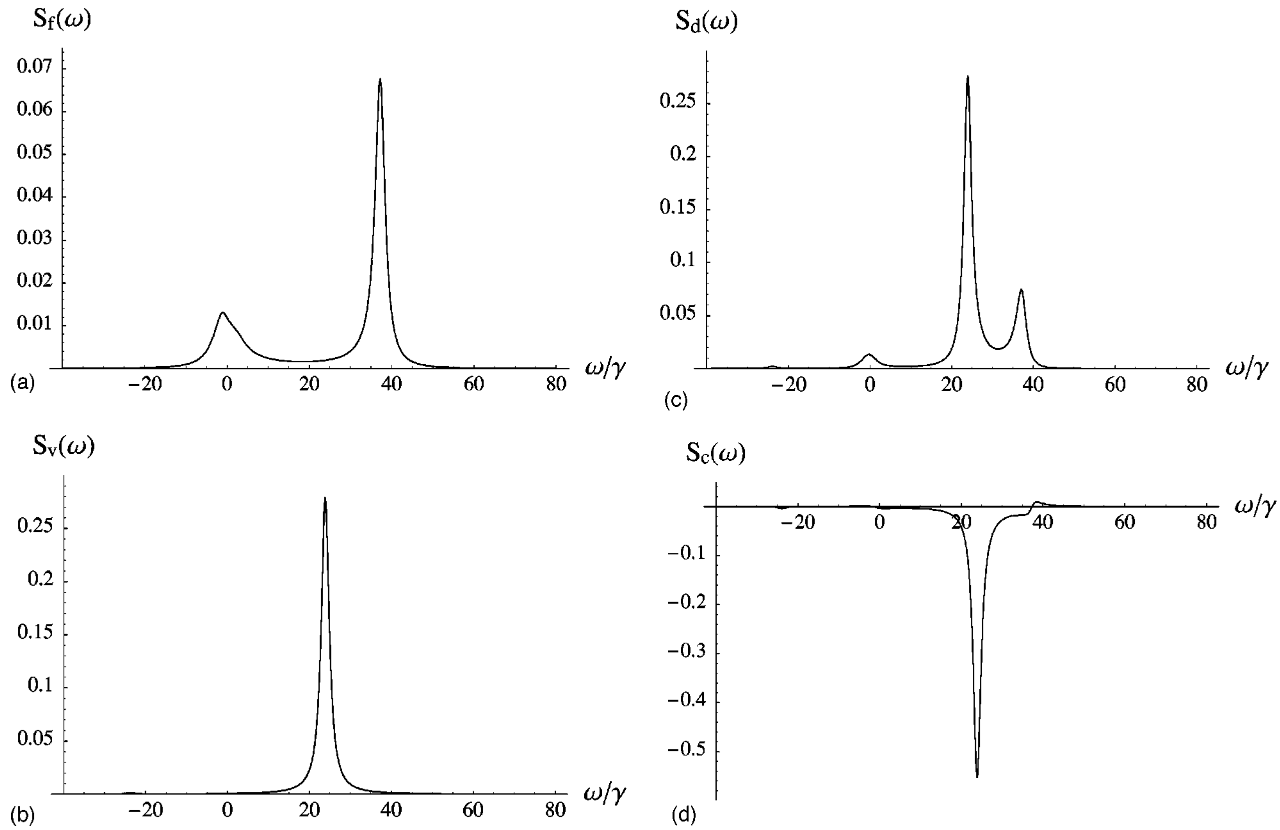

FIG. 3. Contributions of the various terms to the spectrum of Fig. 2. (a) $S_f(\omega)$ is the contribution of the incoming field modified by the nonlinear medium. (b) $S_v(\omega)$ is the atomic excess noise associated with spontaneous emission in the cavity mode. (c) $S_d(\omega)$ comes from the spontaneous emission processed by the nonlinear medium. (d) $S_c(\omega)$ is due to interference between the terms responsible for contributions $S_v(\omega)$ and $S_d(\omega)$.

cooperatively increases. This can be interpreted as originating from the effect of the atomic nonlinearity on the incoming field. The incoming coherent field is squeezed by the atomic nonlinear susceptibility, and this squeezing is in turn transferred to the atoms. It can be seen in Fig. 3(a) that the spectrum corresponding to this term, $S_f(\omega)$, is similar to the total spin noise spectrum.

The second and third terms of Eq. (43), associated with spontaneous atomic emission, are correlated, and make three contributions to the total spin noise power. The noise spectrum given by the second term alone in Eq. (43), $S_v(\omega)$, is shown in Fig. 3(b). It corresponds to the noise of free atoms, and exhibits peaks at the generalized Rabi frequency $\Omega_R = \sqrt{(4I_0+\delta^2)}$. Figure 3(c) shows that the noise spectrum of the dipole in the cavity, $S_d(\omega)$ [the third term in Eq. (43)], has three peaks, at zero, Ω_1, and Ω_R. Figure 3(d) shows $S_c(\omega)$, the interference term between the two previous terms. It has a negative peak at Ω_R which completely cancels the two previous contributions at Ω_R. This interference simply corresponds to the fact that the atoms interact strongly with the cavity in the direction of propagation, and that the emission peak at the natural frequency Ω_R of the atoms in free space disappears and is replaced by an emission at the frequency Ω_1 of the coupled system. The spontaneous atomic emission may also contribute to spin squeezing [23], but, in the considered situation, the sum of the three corresponding variances ΔS_v, ΔS_d, and ΔS_c brings a noise contribution that is virtually independent of the cooperativity, as can be seen in Table I. We can then conclude that spin squeezing occurs via the interaction of the atoms with the squeezed field present in the cavity.

B. Squeezed input field

We have also examined a different case in which the input light is broadband squeezed light. We have investigated the optimum conditions for this squeezing to be transferred to the atoms. The best conditions in this case correspond to strong coupling of the atomic ensemble with the cavity and to a very weak intracavity resonant field. In contrast to the previous case, the atomic ensemble behaves like a linear system. The noise spectrum $S_{\mathrm{spin}}(\omega)$ of a spin component for a coherent input field is shown in Fig. 4 (in this case all the components are the same). It consists of two identical peaks at the characteristic frequency of the linear strongly coupled cavity-atom system, $\Omega_{lin}=g\sqrt{N/\tau}$. When the input field is squeezed, the only difference between the spectrum of the minimum spin component and the previous spectrum is the height of the peaks: they become smaller. The amount of squeezing transferred to the atoms is always less than the one of the incoming light, because of the coupling of the atoms with the vacuum. In Figs. 5(a) and 5(b), we show the efficiency of the spin squeezing for the exact resonance of the field with the atoms and the cavity as a function of the mag-

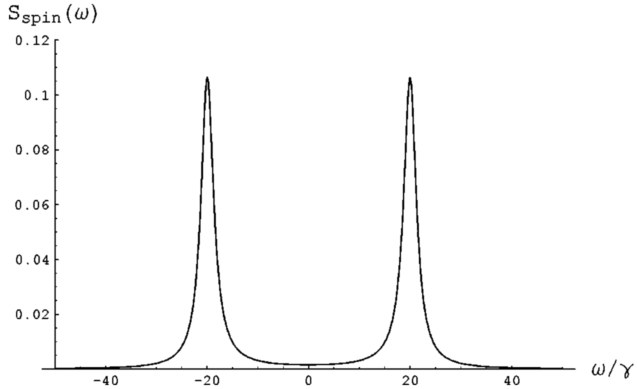

FIG. 4. Spectrum of the minimum spin component $S_{\rm min}(\omega)$ obtained for an incoming coherent field corresponding to a nonsaturating intracavity intensity. This spectrum has been obtained for $\kappa=2\gamma$, $I_0=4.10^{-6}$, $\delta_C=\delta=0$, $T=0.1$, and $C=100$.

nitude of the incoming squeezing. The spin squeezing is then highest for perfect squeezing of the incoming field. In Fig. 5(a) we see that the amount of spin squeezing increases with the cooperativity C, and tends to a limit. This limit depends on the value of the ratio γ/κ, and the maximum efficiency of the squeezing transfer defined as $(1-\Delta S_{\rm min}/1-e^{-2r})$ is equal to $\kappa/(\gamma+\kappa)$, as can be seen in Fig. 5(b). This shows that the coupling of the atoms with the surrounding vacuum field limits the degree of achievable squeezing in the collective spin.

When the incoming field is not exactly resonant with both the atoms and the cavity, the situation is quite different, as can be seen in Fig. 5(c). For a large squeezing of the incoming light, excess noise is obtained for the atomic spin. The excess noise goes to infinity for a perfect squeezing of the incoming light. This comes from the fact that nonresonant atoms cause a rotation of the noise ellipse. As a result, the squeezed and antisqueezed components are mixed inside the cavity, and induce excess noise on the spin.

IV. CONCLUSION

Using a full quantum model for an ensemble of two level atoms in a cavity, we derived the atomic spin fluctuation spectra and variances, and rigorously showed the occurrence of spin squeezing in such systems. This result is likely to be generalized to atoms interacting with two fields in a Raman-type configuration.

Spin squeezing may occur in two different cases. In the first one, the nonlinearity of the atomic ensemble is exploited to squeeze the intracavity field, which in turn imprints squeezing on the atomic ensemble. This effect may be called spin self-squeezing. In the second case, the atomic ensemble has a linear behavior. It cannot create squeezing in the intracavity field. However, if the incoming field is squeezed, the atom-field coupling in the cavity yields spin squeezing.

APPENDIX

Here we give the expression of the diffusion matrix appearing in Eq. (50). The matrix elements of the higher 2 $\times 2$ quadrant of $[D]$ are the correlation functions of a broadband squeezed field. Their values are given above for the case of a squeezing bandwidth much larger than all the characteristic frequencies of the studied system [20]:

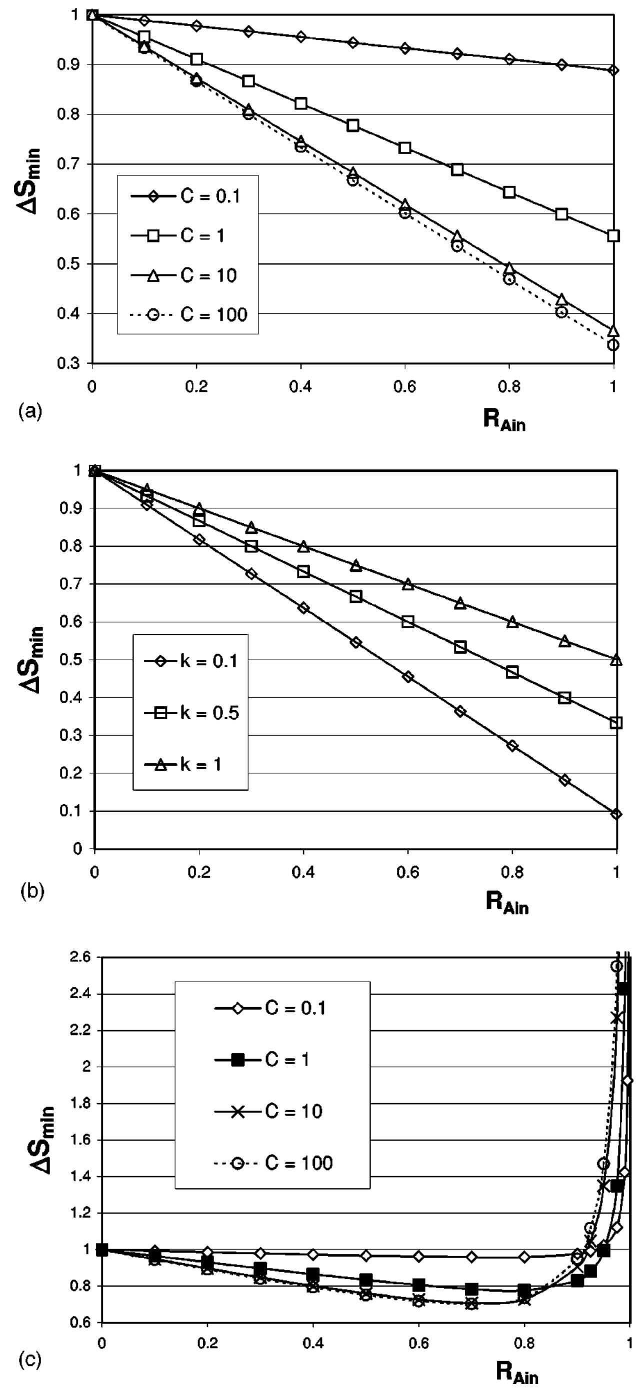

FIG. 5. Minimum spin variance $\Delta S_{\rm min}$ as a function of the noise reduction of the incoming field ($R_{\rm Ain}=1-e^{-2r}$) in linear conditions. (a) For values of the cooperatively $C=0.1$, 1, 10, and 100, with $\delta_C=\delta=0$, and $\kappa=2\gamma$. (b) for values of the parameter $k=\gamma/\kappa=0.1$, 0.5, and 1, with $C=1000$, and $\delta_C=\delta=0$. (c) For values of the cooperatively $C=0.1$, 1, 10, and 100, with an atomic detuning different from 0: $\delta=-1$, $\delta_C=0$, and $\kappa=2\gamma$.

$$\langle \delta A^{in}(t)\,\delta A^{in\dagger}(t')\rangle=\cosh^2(r)\,\delta(t-t'), \tag{A1}$$

$$\langle \delta A^{in}(t)\,\delta A^{in}(t')\rangle = \tfrac{1}{2}\sinh(r)e^{i\theta}\delta(t-t'), \tag{A2}$$

$$\langle \delta A^{in\dagger}(t)\,\delta A^{in\dagger}(t')\rangle = \tfrac{1}{2}\sinh(r)e^{-i\theta}\delta(t-t'), \tag{A3}$$

$$\langle \delta A^{in\dagger}(t)\,\delta A^{in}(t')\rangle = \sinh^2(r)\,\delta(t-t'). \tag{A4}$$

The matrix elements of the lower 3×3 quadrant of $[D]$ are the correlation functions of the atomic noise operators appearing in Eqs. (25)–(27). They were evaluated with the Einstein generalized relations [18]. The only nonzero elements are given below:

$$\langle F_P(t)F_{P^\dagger}(t')\rangle = 2\gamma N\,\delta(t-t'), \tag{A5}$$

$$\langle F_P(t)F_{S_z}(t')\rangle = 2\gamma p_0\,\delta(t-t'), \tag{A6}$$

$$\langle F_{S_z}(t)F_{P^\dagger}(t')\rangle = 2\gamma p_{0c}\,\delta(t-t'), \tag{A7}$$

$$\langle F_{S_z}(t)F_{S_z}(t'))\rangle = 2\gamma(N/2+s_{z0})\,\delta(t-t'). \tag{A8}$$

The other elements of $[D]$ are equal to zero since there are no correlations between atomic and field fluctuations at the same time. Thus we obtain

$$[D] = \begin{bmatrix} 2\kappa/\tau\cosh^2(r) & \kappa/\tau\sinh(2r)e^{i\theta} & 0 & 0 & 0 \\ \kappa/\tau\sinh(2r)e^{-i\theta} & 2\kappa/\tau\sinh^2(r) & 0 & 0 & 0 \\ 0 & 0 & 2\gamma N & 0 & 2\gamma p_0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\gamma p_{0c} & 0 & 2\gamma(N/2+s_{z0}) \end{bmatrix}.$$

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