Quantized mode of a leaky cavity

S. M. Dutra and G. Nienhuis
Huygens Laboratory, University of Leiden, P.O. Box 9504, 2300 RA Leiden, The Netherlands
(Received 14 July 2000; published 6 November 2000)

We use Thomson's classical concept of mode of a leaky cavity to develop a quantum theory of cavity damping. This theory generalizes the conventional system-reservoir theory of high-Q cavity damping to arbitrary Q. The small system now consists of *damped* oscillators corresponding to the natural modes of the leaky cavity rather than *undamped* oscillators associated with the normal modes of a fictitious perfect cavity. The formalism unifies semiclassical Fox-Li modes and the normal modes traditionally used for quantization. It also lays the foundations for a full quantum description of excess noise. The connection with Siegman's semiclassical work is straightforward. In a wider context, this theory constitutes a radical departure from present models of dissipation in quantum mechanics: unlike conventional models, system and reservoir operators no longer commute with each other. This noncommutability is an unavoidable consequence of having to use natural cavity modes rather than normal modes of a fictitious perfect cavity.

PACS number(s): 03.65.Bz, 42.50.−p, 42.50.Lc

I. INTRODUCTION

Cavity modes are a powerful ubiquitous concept in semiclassical laser physics. Modes are also popular in quantum optics because they simplify the quantum description of light [1]. Yet, the concepts of mode used in quantum optics and semiclassical laser physics are intrinsically different [2,3]. Whereas quantum optics has traditionally restricted itself to normal modes of closed systems, the Fox-Li cavity modes [4,5] adopted in laser physics are modes of an open system and do not even have to be orthogonal [3,6]. From the point of view of a laser physicist, the quantum optics notion of mode is rather limited. The aim of this paper is to set down an exact framework to describe the quantum dynamics of the radiation field in a leaky cavity using Fox-Li modes. The main result is a Hamiltonian, derived from first principles, involving ''creation'' and ''annihilation'' operators for cavity and external (a concept introduced here) Fox-Li modes that, together with the commutation rules for these operators, provides such a framework. We also develop a unifying formalism where Fox-Li modes are shown to follow from the same Sturm-Liouville treatment [7] that is used for normal modes.

In quantum optics there are a plethora of alternative definitions of what a quantized mode of a leaky cavity, often called a quasimode, should be [8,9]. Most of them seem to completely ignore the ideas that were already developed in classical resonator theory. We argue in Sec. V that the proper generalization of the concept of mode for leaky cavities is the notion of natural modes of oscillation. Natural modes were introduced in standard resonator theory by Vaĭnshteĭn [10], but they can be traced back to Thomson's investigation of a simple model of an electromagnetic oscillator [11]. These are the modes in which the leaky cavity will oscillate naturally after an initial excitation is withdrawn, just as a glass of wine vibrates in its own natural frequencies after being hit with a spoon. Applying the same Sturm-Liouville treatment used for normal modes, we show that the introduction of an inner product for natural mode expansions leads to a traveling-wave representation of such a mode. This representation is a Fox-Li mode [12]. So the extension of the concept of mode to a leaky cavity brings us naturally to Fox-Li modes.

An exact quantum description of the field in a leaky cavity has been developed using the normal modes of the closed system formed by the cavity and the rest of the ''universe'' [8]. However, these modes of the ''universe'' often conceal the essential physics because they do not single out the cavity from its environment, describing everything in terms of global universe photons. An approach involving normal cavity modes [13], where the damping is modeled by coupling these normal modes to the normal modes of a reservoir, has been adopted since the early days of the laser. However, this Senitzky-Gardiner-Collett Hamiltonian is a good approximation only when the cavity quality factor (Q) is high [8,14,15]. For arbitrary Q, the usual quantum optics treatment involves either the modes of the universe or abandoning the idea of a mode expansion altogether [16]. The quantum formulation in terms of Fox-Li modes that we present here has three main advantages over this usual treatment. First, as a generalization to arbitrary Q of the Senitzky-Gardiner-Collett Hamiltonian, it is much more intuitively appealing than a modes-of-the-universe formulation. Second, it connects in a straightforward way to the semiclassical theories widely adopted in laser physics, allowing laser physicists and quantum opticians to finally ''speak the same language.'' Third, Fox-Li modes acquired a new significance in the late eighties when they were used by Siegman [3,17] as the basis of a unifying semiclassical theory of excess noise. Excess noise is a curious effect, which was first predicted by Petermann [18], where the usual Schawlow-Townes linewidth of a laser is enhanced by a multiplicative factor. This factor can be quite large, with measured values of the order of 60, a few hundreds, and even as high as 10^3 for some lasers [19]. In Siegman's unifying theory, excess noise is a consequence of mode nonorthogonality. An approximate description in terms of normal cavity modes, such as the Senitzky-Gardiner-Collett Hamiltonian [13], cannot describe excess noise because these modes are always orthogonal [20].

Recently, Lamprecht and Ritsch proposed a quantum theory of excess noise involving quantized Fox-Li modes [21]. In this theory, the cavity damping (a necessary ingredient to have excess noise) is described only at a master equation level by the *ad hoc* introduction of a Lindblad term. This phenomenological approach is very similar to the one that is usually adopted for high-Q cavities [13]. A master equation treatment is valid, however, only when the correlation time of the reservoir is much shorter than the damping time, which is usually not the case for cavities showing excess noise. For example, longitudinal excess noise [22] is only non-negligible if the cavity-damping time is of the order of the roundtrip time, which is roughly the correlation time of the reservoir. The approach presented here avoids this problem because it is developed at the level of a fundamental Hamiltonian description.

Another recent theory involving modes of a leaky cavity is the very interesting toy-model proposed by Grangier and Poizat [23]. They assume that the modes of the universe can be divided into two parts: cavity modes and loss modes. Excess noise appears because different cavity modes couple to the *same* loss mode. However, keeping with the spirit of a toy model, they do not specify how the modes of the universe can be split into these two parts nor how their cavity modes relate to Fox-Li modes.

There are also quantum theories of leaky cavities involving Thomson's natural modes in their plain standing-wave form rather than their Fox-Li representation. Ujihara [8,24] constructs a theory based entirely on a modes-of-the-universe description, but he uses the notion of natural modes to identify the cavity resonances. Leung and collaborators [25], on the other hand, do construct a quantum formalism entirely based on Thomson modes. To expand the field into these modes, they adopt a bilinear form (not an inner product) based on the norm of a decaying state introduced by Zel'dovich [26]. This bilinear form, though, is completely different from the inner product widely adopted in the semiclassical theory by Siegman and others. The connection between their work and the semiclassical theory is still unknown. Our approach has the advantage of being a direct quantum-optics implementation of those semiclassical concepts using a familiar inner product.

In the next section, we describe a simple model of a leaky cavity and set the stage for the introduction of modes. Sections III and IV briefly review the exact normal modes approach to a leaky cavity (modes of the universe) and the usual intuitive idea of Fox-Li modes, respectively. In Sec. V, we introduce the natural modes and develop a unifying formalism for Fox-Li and normal modes. In Sec. VI, we derive the Hamiltonian describing the cavity and outside in terms of these Fox-Li modes. Section VII makes the connection with Siegman's formulation explicit and also discusses the relation between natural modes and an alternative formulation of cavity modes in terms of scattering [27]. Section VIII summarizes our main results and discusses the path that they suggest for further research. In the appendix, we derive a physical picture for the modes of a lossy cavity, and demonstrate that the set of all cavity and external modes together with their adjoints is complete.

II. A SIMPLE MODEL OF A LEAKY CAVITY

A typical laser cavity is leaky mainly because of mirror transmissivity and diffraction losses. Diffraction requires a three-dimensional treatment. However, the key feature of a leaky cavity for the purposes of a quantum description is that it is an open system. This feature can be captured already in a simple one-dimensional model, where leakage is entirely due to a nonvanishing transmissivity. In quantum optics, in fact, the vast majority of treatments of leaky cavities is one-dimensional [8,9,13,14]. Our model is a modified version of a simple model of a one-dimensional cavity [28] introduced by Ujihara [8] who analyzed it using a modes-of-the-universe approach. In our model, the cavity is formed by a perfect mirror at $x=-L$ and a nonabsorptive and nondispersive dielectric extending all the way from $x=0$ to infinity and described by the permittivity

$$\epsilon(x)=\Theta(-x)+\Theta(x)n_d^2, \tag{1}$$

where n_d is the refractive index of the dielectric and $\Theta(x)$ is Heavyside's step function, which is unity for positive x and vanishes for negative x. The reason we have chosen to have the dielectric filling the external region rather than the cavity is simply a pedagogic one: this way, we can recover the case of a perfect cavity by making the permittivity of the dielectric very large so that it becomes a perfect mirror at $x=0$. The more realistic case of a dielectric filling the cavity can easily be accounted for with only a few minor changes that do not affect the main results presented here.

In our one-dimensional model, we consider only linearly polarized electromagnetic waves propagating in the x direction. The polarization of the electric field defines the y axis and that of the magnetic field, the z axis. For simplicity, we rescale the fields multiplying them by the square root of the transverse area in the yz plane as in the paper by Lang, Scully, and Lamb and the one by Baseia and Nussenzveig [8]. Then, Maxwell's equations take the following simple form:

$$\frac{\partial}{\partial x}E(x,t)=-\frac{\partial}{\partial t}B(x,t), \tag{2a}$$

$$\frac{\partial}{\partial x}B(x,t)=-\frac{\epsilon(x)}{c^2}\frac{\partial}{\partial t}E(x,t). \tag{2b}$$

From Eqs. (2), we obtain the following wave equation:

$$\frac{\partial^2}{\partial x^2}E(x,t)-\frac{\epsilon(x)}{c^2}\frac{\partial^2}{\partial t^2}E(x,t)=0. \tag{3}$$

Then the standard method of separation of variables [7] shows that any solution of this equation can be written as a linear combination of the solutions $\chi(\omega,x)$ of the associated Helmholtz equation:

$$\frac{\partial^2}{\partial x^2}\chi(\omega,x)+\frac{\omega^2}{c^2}\epsilon(x)\chi(\omega,x)=0, \tag{4}$$

with time-dependent coefficients $q(\omega,t)$ that obey a simple harmonic oscillator equation

$$\ddot{q}+\omega^2 q=0. \tag{5}$$

Usually, χ is defined in the whole space and has to satisfy the physical boundary conditions at $x=-L$ and at infinity. The functions χ then form a basis set that can represent any physical field configuration: the modes of the universe reviewed in the next section. In Sec. V, we show that a different class of solutions χ of Eq. (4) can also be used as a complete basis set. These are the solutions that satisfy either the boundary conditions at the interface and at $x=-L$ only or at the interface and at infinity only. They correspond to the Fox-Li modes of the cavity and outside, respectively.

III. MODES-OF-THE-UNIVERSE DESCRIPTION

To represent an arbitrary spatial configuration of the field, i.e., a given solution of Eq. (4), by a mode expansion, it is convenient to introduce an inner product. The most convenient inner product is one for which the modes are orthogonal or, if this is not possible, at least biorthogonal [7]. We can arrive at such an inner product by first deriving a so-called orthogonality relation for the mode functions. Let us call the modes of the universe $\mathcal{U}(\omega,x)$, to distinguish them from a general χ. For our model, the boundary conditions demand that $\mathcal{U}$ vanish at the perfect mirror and at infinity, and that both $\mathcal{U}$ and $\partial\mathcal{U}/\partial x$ be continuous at $x=0$. With these boundary conditions, the eigenvalue ω^2 associated with each of these modes is always real [7], its positive root can be interpreted as the frequency of a mode. So $\mathcal{U}^*$ also obeys Eq. (4). Then the standard Sturm-Liouville procedure [7] yields

$$\begin{aligned}\frac{\omega^2-\omega'^2}{c^2}&\int_{-L}^{\infty}dx\,\mathcal{U}^*(\omega',x)\mathcal{U}(\omega,x)\epsilon(x)\\ &=\left\{\mathcal{U}^*(\omega',x)\frac{\partial}{\partial x}\mathcal{U}(\omega,x)\right.\\ &\qquad\left.-\mathcal{U}(\omega,x)\frac{\partial}{\partial x}\mathcal{U}^*(\omega',x)\right\}_{x=-L}^{x\to\infty}.\end{aligned} \tag{6}$$

As both $\mathcal{U}$ and $\mathcal{U}^*$ vanish at $x=-L$ and at infinity, Eq. (6) leads to the following orthogonality relation for the modes of the universe:

$$\frac{\omega^2-\omega'^2}{c^2}\int_{-L}^{\infty}dx\,\mathcal{U}^*(\omega',x)\mathcal{U}(\omega,x)\epsilon(x)=0. \tag{7}$$

This relation tells us that the modes of the universe are orthogonal to each other under the inner product:

$$(\psi,\phi)=\int_{-L}^{\infty}dx\,\psi^*(x)\phi(x)\epsilon(x), \tag{8}$$

where ψ and ϕ are two members of the abstract space formed by the solutions of Eq. (4) that satisfy these boundary conditions. It is convenient to normalize the $\mathcal{U}$ as follows:

$$\int_{-L}^{\infty}dx\,\mathcal{U}^*(\omega',x)\mathcal{U}(\omega,x)\epsilon(x)=\delta(\omega-\omega'). \tag{9}$$

From Eq. (9), we can derive the closure relation for these functions [7,29]:

$$\int_0^{\infty}d\omega'\,\mathcal{U}^*(\omega',x)\mathcal{U}(\omega',x')=\frac{\delta(x-x')-\delta(x+x'+2L)}{\epsilon(x)}, \tag{10}$$

where the second delta function appears because of the perfect mirror at $x=-L$ [30]. The delta functions on the right-hand side of (10) are defined in the space of functions that are continuous at the origin (see Ujihara [8]). As any physical electric field will vanish at the perfect mirror and will be continuous across the dielectric interface, Eq. (10) shows that the modes of the ''universe'' form a complete set. Completeness guarantees that any arbitrary physical field can indeed be represented by a modes-of-the-universe expansion.

The modes of the universe are given by

$$\mathcal{U}(\omega,x)=-i\sqrt{\frac{2}{\pi c}}e^{i(\omega/c)L}\mathcal{L}\left(\frac{\omega}{c}\right)\times\begin{cases}\mathcal{J}_{\text{cav}}(\omega,x), & -L\leqslant x<0\\ \mathcal{J}_{\text{out}}(\omega,x), & x>0,\end{cases} \tag{11}$$

where

$$\mathcal{J}_{\text{cav}}(\omega,x)=\sin\left([x+L]\frac{\omega}{c}\right), \tag{12}$$

$$\mathcal{J}_{\text{out}}(\omega,x)=\frac{1-n_d}{2n_d}\sin\left([n_d x-L]\frac{\omega}{c}\right)+\frac{1+n_d}{2n_d}\sin\left([n_d x+L]\frac{\omega}{c}\right), \tag{13}$$

$$\mathcal{L}(k)=(1+r)\sum_{l=0}^{\infty}(-re^{i2kL})^l, \tag{14}$$

and $r=(1-n_d)/(n_d+1)$ is the reflectivity of the left side of the interface. Introducing now the continuous annihilation and creation operators $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ associated with each mode $\mathcal{U}(x,\omega)$, the quantized field operators are given by [8,29]

$$\hat{E}(x)=\int_0^{\infty}d\omega\sqrt{\frac{\hbar\omega}{\epsilon_0}}\mathcal{U}(\omega,x)\hat{a}(\omega)+\text{H.c.}, \tag{15a}$$

$$\hat{B}(x)=-i\int_0^{\infty}d\omega\sqrt{\frac{\hbar\omega}{\epsilon_0}}\frac{1}{\omega}\frac{\partial}{\partial x}\mathcal{U}(\omega,x)\hat{a}(\omega)+\text{H.c.} \tag{15b}$$

From Eqs. (10) and (15), we recover the ordinary commutation relation between the fields in the presence of a perfect mirror at $x=-L$ [30],

$$[\hat{D}(x),\hat{B}(x')]=i\hbar\frac{\partial}{\partial x'}\{\delta(x-x')-\delta(x+x'+2L)\}, \tag{16}$$

where $\hat{D}(x)=\epsilon_0\epsilon(x)\hat{E}(x)$ is the electric displacement operator. The Hamiltonian, derived by substituting Eq. (15) in the expression for the total energy,

$$\hat{H}=\frac{\epsilon_0}{2}\int_{-L}^{\infty}dx\{\epsilon(x)\hat{E}^2(x)+c^2\hat{B}^2(x)\}, \tag{17}$$

is given by

$$\hat{H}=\frac{\hbar}{2}\int_0^{\infty}d\omega\{\hat{a}^{\dagger}(\omega)\hat{a}(\omega)+\hat{a}(\omega)\hat{a}^{\dagger}(\omega)\}\omega, \tag{18}$$

which is analogous to a continuum of uncoupled quantum harmonic oscillators, one oscillator associated with each mode.

IV. FOX-LI MODES AS SELF-REPEATING TRAVELING WAVES

The problem of what is a mode of a leaky cavity gained prominence when the Fabry-Perot interferometer was suggested as a cavity for the first laser [31]. Because the Fabry-Perot is not enclosed by reflecting side walls, it should have a continuum of modes. In fact, as its name says, before the advent of the laser, it had been traditionally used as an interferometer rather than a resonator. The essential point is that Fabry-Perot rings exist for any frequency, whereas a resonator is expected to have a large response only for a discrete spectrum of frequencies [32]. Fox and Li [4] addressed this problem and showed that the diffraction losses, due to the finite surface area of the end mirrors, effectively turn the continuum into a discrete set of modes of unexpectedly high Q (i.e., with low diffraction losses). They considered a propagating wave that was reflected back and forth by the two end mirrors of the Fabry-Perot. These mirrors were assumed to be perfect reflectors but of finite area. The propagated wave was calculated using the scalar formulation of Huygens's principle. Then they looked for field distributions whose profile was self-repeating (apart from a decay factor) in a complete round trip of the leaky cavity, i.e., eigenfunctions of the Huygens's propagation integral. This is their most natural and intuitive definition of mode.

Fox and Li considered only diffraction losses [4]. In the jargon of laser physics, their ''leaky'' modes were transverse, not longitudinal. But their intuitive concept of mode as a self-repeating field distribution, which is the essence of their approach, can be generalized to semitransparent mirrors and even to closed (but leaky) cavities. In fact, Hamel and Woerdman [33] have used this intuitive concept to define longitudinal Fox-Li modes in one-dimensional leaky cavities relating, semiclassically, the excess noise in these cavities [22] to the nonorthogonality of these longitudinal modes. The same definition of longitudinal Fox-Li mode is also implicit in an earlier paper by Lugiato and Narducci [34].

In the Hamel-Woerdman one-dimensional treatment, the solution to the propagation problem, which is given by the Huygens's integral in three dimensions, reduces to one simple exponential $E_+\exp(ikx)$ representing a forward propagating wave and another exponential $E_-\exp(-ikx)$ corresponding to a backward propagating wave. The constants $E_\pm$ are not independent. Their mutual relation is determined by the boundary condition at the perfect mirror, $E_+\exp(-ikL)=-E_-\exp(ikL)$. The self-repeating condition is given then by the boundary condition at the leaky interface, $\exp(-i2kL)=-r$, which yields the allowed values of k:

$$k=k_n-i\gamma\equiv\kappa_n, \tag{19}$$

where $k_n=(\pi/L)n$, with $n=0,\pm1,\pm2,\ldots,$ are the cavity resonances, and $\gamma=-\ln|r|/(2L)$ is the width of the resonances. To distinguish between forward and backward propagating components, Hamel and Woerdman adopt a spinor notation instead of the transmission medium analog (Siegman's lens-guide picture [3]) introduced by Fox and Li [4]. Despite this minor formal difference, the Hamel-Woerdman modes are indeed the appropriate version of Fox-Li modes for the case of a one-dimensional leaky cavity. In their spinor notation, the inner product between two fields $\mathbf{E}_1$ and $\mathbf{E}_2$ is given by

$$(\mathbf{E}_1,\mathbf{E}_2)=\int_{-L}^{0}dx\,\mathbf{E}_1^{\dagger}(x)\cdot\mathbf{E}_2(x). \tag{20}$$

The modes are given by

$$\mathbf{u}_n(x)=\frac{1}{\sqrt{2L}}\begin{bmatrix}e^{i\kappa_n x}\\ re^{-i\kappa_n x}\end{bmatrix}, \tag{21}$$

where the factor of $1/\sqrt{2L}$ is introduced so that the inner product between a mode and its adjoint [33]

$$\mathbf{w}_n(x)=\frac{1}{\sqrt{2L}}\begin{bmatrix}e^{i\kappa_n^* x}\\ \frac{1}{r}e^{-i\kappa_n^* x}\end{bmatrix} \tag{22}$$

be given by $(\mathbf{w}_m,\mathbf{u}_n)=\delta_{n,m}$.

To construct a quantum theory based on the one-dimensional Fox-Li modes introduced by Hamel and Woerdman, we must first make their intuitive approach more rigorous and answer some lingering questions. The inner product given by Eq. (20), for instance, is completely different from that adopted in normal mode expansions, e.g., Eq. (8), where there are additional cross terms mixing forward and backward propagating components in the integrand. How can this inner product be introduced from a standard Sturm-Liouville treatment as that of Sec. III for the modes of the universe? Can these modes be used to construct any possible realization of the field in the cavity? Why do we have to keep track of forward and backward propagating components separately? How do we calculate what are the forward and backward propagating components of the total field? Moreover as Leung and collaborators [25] point out, because the wave equation is a second-order differential equation, the dynamics cannot be specified by knowing the initial electric field alone but requires its time derivative, or the magnetic field, as well. Where is the magnetic field in the Hamel and Woerdman formulation? We answer these questions now with a rigorous formulation in terms of the general theory presented in Sec. II.

V. NATURAL MODES DESCRIPTION

The obvious requirement that any definition of cavity mode has to meet is that such a mode must correspond to a field configuration determined by the cavity alone, regardless of what might lie outside. This requirement automatically leads us to Thomson's definition of cavity modes as the field configurations that will oscillate naturally in the cavity, after an initial excitation is withdrawn [11]. Mathematically, these modes are solutions of Helmholtz equations that satisfy the boundary conditions at the cavity and contain only outgoing waves outside. They are analogous to decaying states in the Gamow-Condon-Gurney theory of α decay [35,36].

The Thomson modes for our one-dimensional cavity are solutions of Eq. (4) that satisfy the boundary conditions at $x=-L$ and $x=0$, but that contain only outgoing waves for $x>0$ (so they cannot satisfy the boundary condition at infinity). They are given by

$$g(c\kappa_n,x)=\begin{cases} e^{i\kappa_n x}+re^{-i\kappa_n x} & \text{for } -L\leqslant x<0 \\ (1+r)e^{i\kappa_n n_d x} & \text{for } x>0 \end{cases} \tag{23}$$

where, unlike the modes of the universe, the ''frequency'' $c\kappa_n$ is complex now and can no longer be interpreted as a physical frequency. Because these modes are purely outgoing, they have to decay in time. Thus they are associated with the time dependence $\exp(-ic\kappa_n t)$ rather than with $\exp(-ick_n t)$ as an ordinary plane would.

As these natural cavity modes diverge at infinity, we want to use them to represent the radiation field only in the cavity [37]. Then to cover the whole of space, we have to use external natural modes for the region outside the cavity. These modes are solutions of Eq. (4) that satisfy the boundary condition at infinity and $x=0$, but that contain only outgoing waves for $x<0$ (so they cannot satisfy the boundary condition on the perfect mirror at $x=-L$, as that implies a reflected incoming wave). These solutions of Eq. (4) are given by

$$G(ck,x)=\begin{cases} (1-r)e^{-ikx} & \text{for } x<0 \\ e^{-ikn_d x}-re^{ikn_d x} & \text{for } x>0, \end{cases} \tag{24}$$

where k is real.

Neither g nor G form an orthogonal set with respect to the inner product (8) that we have introduced in Sec. III. But this should not be a surprise, because that inner product is based on the orthogonality relation (7) that assumes that the mode functions satisfy *all* the physical boundary conditions in the entire space, i.e., the conditions at $x=-L$, $x=0$, and at infinity. As g and G only satisfy two of these boundary conditions each, neither of these new sets of modes obeys the orthogonality relation (7).

Unifying formalism for Fox-Li modes

To be able to use natural modes as a basis for the radiation field, we must have an inner product and show that such a basis is complete. We can proceed as in Sec. III, except for one important difference. The outgoing requirement turns what was previously a Hermitian eigenproblem giving rise to modes of the ''universe'' into a non-Hermitian one. It is well-known that non-Hermitian eigenproblems do not yield orthogonal eigenmodes [7]. Instead, these eigenmodes obey a biorthogonality relation, which is an orthogonality relation between modes and their adjoints. So now the most convenient inner product is one for which the mode functions and their adjoints are *biorthogonal*. It is this inner product that will lead us naturally to the Fox-Li traveling-wave representation of Thomson's natural modes.

Analogously to Sec. III, we obtain the appropriate inner product by first deriving a biorthogonality relation. To find this biorthogonal relation, consider the following function that vanishes at the borders of our cavity (i.e., both at $x=-L$ and $x=0$):

$$\zeta_{n,m}(x)=g(c\kappa_n,x)\frac{1}{\kappa_m}\frac{\partial}{\partial x}\tilde{g}^*(c\kappa_m^*,x) - \tilde{g}^*(c\kappa_m^*,x)\frac{1}{\kappa_n}\frac{\partial}{\partial x}g(c\kappa_n,x), \tag{25}$$

where

$$\tilde{g}(c\kappa_m^*,x)=\begin{cases} e^{i\kappa_m^* x}+\dfrac{e^{-i\kappa_m^* x}}{r} & \text{for } -L\leqslant x<0 \\ \dfrac{1+r}{r}e^{-i\kappa_m^* n_d x} & \text{for } x>0 \end{cases} \tag{26}$$

is the adjoint of $g(c\kappa_m,x)$, i.e., the solution of Eq. (4) of frequency $c\kappa_m^*$ that satisfies the boundary conditions at the cavity and contains only incoming waves at infinity [7,38]. Differentiating Eq. (25) with respect to x, using Eq. (4), and then integrating the result over the cavity, we find

$$(\kappa_n-\kappa_m)\int_{-L}^{0}dx\left\{n^2(x)\tilde{g}^*(c\kappa_m^*,x)g(c\kappa_n,x) + \frac{1}{\kappa_m}\frac{\partial}{\partial x}\tilde{g}^*(c\kappa_m^*,x)\frac{1}{\kappa_n}\frac{\partial}{\partial x}g(c\kappa_n,x)\right\}=0, \tag{27}$$

where $n(x)=\sqrt{\epsilon(x)}$ is the refractive index. Equation (27) is analogous to the orthogonality relation (7) but there are two important differences. First, it is restricted to the space inside the cavity, while Eq. (7) is an orthogonality relation for functions defined in the whole of space. Second, unlike Eq. (7), Eq. (27) involves the derivatives of the mode functions. The restriction to the cavity is not a problem because, from the start, we only intended to use $g(c\kappa_m,x)$ and $\tilde{g}(c\kappa_m^*,x)$ to describe the cavity field anyway. The second difference, however, is apparently an obstacle to the introduction of an inner product based on Eq. (27).

To understand how Eq. (27) can be used to introduce an inner product appropriate to expansions in these modes, consider first what the spatial derivatives in Eq. (27) mean. The general mode functions $\chi(x,\omega)$, of which the present ones are a special case, were originated from a separation of variables of the wave equation for the electric field (3) that gave rise to Helmholtz equation (4). So $g(c\kappa_m,x)$ and $\tilde{g}(c\kappa_m^*,x)$ are related to the electric field. To find out what $(1/\kappa_n)\partial g(c\kappa_n,x)/\partial x$ and $(1/\kappa_m^*)\partial\tilde{g}(c\kappa_m^*,x)/\partial x$ are related to, consider Maxwell's equations (2) after such separation of variables. Let $\mathcal{E}(c\kappa_n,x)$ and $\mathcal{B}(c\kappa_n,x)$ be the spatially-dependent parts of the electric and magnetic fields, respectively, then Eqs. (2) take the following form:

$$\frac{-i}{\kappa_n}\frac{\partial}{\partial x}\mathcal{E}(c\kappa_n,x)=c\mathcal{B}(c\kappa_n,x), \tag{28a}$$

$$\frac{-i}{\kappa_n}\frac{\partial}{\partial x}\mathcal{B}(c\kappa_n,x)=\frac{n^2(x)}{c}\mathcal{E}(c\kappa_n,x). \tag{28b}$$

So according to Eq. (28a), $(-i/\kappa_n)\partial g(c\kappa_n,x)/\partial x$ and $(i/\kappa_m^*)\partial\tilde{g}(c\kappa_m^*,x)/\partial x$ are related to the product of the magnetic field by the speed of light. Now it also follows from Maxwell's equations (2) that $E+cB$ is a purely forward propagating wave in the cavity, while $E-cB$ is a purely backward propagating wave. This suggests that Eq. (27) should be rewritten in the following form that is completely analogous to Eq. (7):

$$(\kappa_n-\kappa_m)\int_{-L}^{0}dx\,\tilde{\mathcal{G}}^\dagger(c\kappa_m^*,x)\mathcal{G}(c\kappa_n,x)n^2(x)=0, \tag{29}$$

where

$$\mathcal{G}(c\kappa_n,x)=\frac{1}{\sqrt{8L}}\begin{bmatrix} g(c\kappa_n,x)-\dfrac{i}{n(x)\kappa_n}\dfrac{\partial}{\partial x}g(c\kappa_n,x)\\[2ex] g(c\kappa_n,x)+\dfrac{i}{n(x)\kappa_n}\dfrac{\partial}{\partial x}g(c\kappa_n,x)\end{bmatrix} \tag{30}$$

and

$$\tilde{\mathcal{G}}(c\kappa_m^*,x)=\frac{1}{\sqrt{8L}}\begin{bmatrix} \tilde{g}(c\kappa_m^*,x)-\dfrac{i}{n(x)\kappa_m^*}\dfrac{\partial}{\partial x}\tilde{g}(c\kappa_m^*,x)\\[2ex] \tilde{g}(c\kappa_m^*,x)+\dfrac{i}{n(x)\kappa_m^*}\dfrac{\partial}{\partial x}\tilde{g}(c\kappa_m^*,x)\end{bmatrix} \tag{31}$$

are the new mode and adjoint functions, respectively. Their upper components are forward propagating waves and the lower, backward propagating waves. The factor of $1/\sqrt{8L}$ was introduced so that

$$\int_{-L}^{0}dx\,\tilde{\mathcal{G}}^\dagger(c\kappa_m^*,x)\mathcal{G}(c\kappa_n,x)n^2(x)=\delta_{n,m}. \tag{32}$$

Thus the proper inner product under which the modes $\mathcal{G}$ are orthogonal to their adjoints $\tilde{\mathcal{G}}$ is

$$(\mathbf{F}_1,\mathbf{F}_2)=\int_{-L}^{0}dx\,\mathbf{F}_1^\dagger(x)\cdot\mathbf{F}_2(x)n^2(x), \tag{33}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are two ''spinor fields.'' Associating E to g and cB to $-(i/\kappa_n)\partial g/\partial x$ in Eq. (30), we construct a spinor field $\mathbf{F}$ from given electric and magnetic fields as follows:

$$\mathbf{F}(x)=\frac{1}{2}\begin{bmatrix} E(x)+\dfrac{c}{n(x)}B(x)\\[2ex] E(x)-\dfrac{c}{n(x)}B(x)\end{bmatrix}, \tag{34}$$

where the factor 1/2 is introduced just for later convenience.

Substituting Eq. (23) in Eq. (30) and comparing with Eq. (21), we see that these new mode functions are, in fact, the one-dimensional Fox-Li modes introduced by Hamel and Woerdman, which we have now put in the same Sturm-Liouville context as ordinary normal modes. The inner product (20), which seemed to appear out of the blue before, now follows naturally from Eq. (29). The Hamel and Woerdman inner product is Eq. (33) with $n\mathbf{F}_1\to\mathbf{E}_1$ and $n\mathbf{F}_2\to\mathbf{E}_2$.

To expand the field outside the cavity, we introduce external Fox-Li modes $\mathbf{G}(ck,x)$ and their adjoints $\tilde{\mathbf{G}}(ck,x)$ in an analogous way. They are defined just as Eqs. (30) and (31) with the discrete normalization factor $1/\sqrt{8L}$ replaced by the continuum one $1/\sqrt{8\pi n_d}$, $g(c\kappa_n,x)$ replaced by $G(ck,x)$, $\tilde{g}(c\kappa_n^*,x)$ by $\tilde{G}(ck,x)$, and κ_n by k. The external Thomson adjoint modes $\tilde{G}(ck,x)$ are incoming solutions of the Helmholtz equation given by

$$\tilde{G}(ck,x)=\begin{cases}\dfrac{r-1}{r}e^{ikx} & \text{for } x<0\\[2ex] e^{-ikn_dx}-\dfrac{1}{r}e^{ikn_dx} & \text{for } x>0.\end{cases} \tag{35}$$

The external biorthogonality relation is given by

$$\int_{0}^{\infty}dx\,\tilde{\mathbf{G}}^\dagger(ck',x)\mathbf{G}(ck,x)n^2(x)=\delta(k-k'). \tag{36}$$

The origin plays a very important role in the dynamics of the fields because it is the place where the leakage occurs. For this reason, it is crucial that our expansions, not only reproduce the fields for $-L\leqslant x<0$ and $x>0$, but also at $x=0$ [39]. We show in the Appendix that, even though none of the functions g, $\tilde{g}$, G, $\tilde{G}$ satisfies all the boundary conditions, this strict completeness can be achieved, if we allow each side of the interface to have contributions from both outgoing modes and incoming adjoints. In other words, if we use all four different expansions that we can construct with these modes and their adjoints,

$$\mathbf{F}_{\mathrm{cav}}(x)=\sum_{n=-\infty}^{\infty}\mathcal{G}(c\kappa_n,x)\int_{-L}^{0}dx'\,\tilde{\mathcal{G}}^{\dagger}(c\kappa_n^*,x')\mathbf{F}(x')n^2(x') \tag{37}$$

the cavity Fox-Li mode expansion of Eq. (34),

$$\tilde{\mathbf{F}}_{\mathrm{cav}}(x)=\sum_{n=-\infty}^{\infty}\tilde{\mathcal{G}}(c\kappa_n^*,x)\int_{-L}^{0}dx'\,\mathcal{G}^{\dagger}(c\kappa_n,x')\mathbf{F}(x')n^2(x') \tag{38}$$

the cavity Fox-Li adjoint-mode expansion

$$\mathbf{F}_{\mathrm{out}}(x)=\int_{-\infty}^{\infty}dk\,\mathbf{G}(ck,x)\int_{0}^{\infty}dx'\,\tilde{\mathbf{G}}^{\dagger}(ck,x')\mathbf{F}(x')n^2(x') \tag{39}$$

the external Fox-Li mode expansion,

$$\tilde{\mathbf{F}}_{\mathrm{out}}(x)=\int_{-\infty}^{\infty}dk\,\tilde{\mathbf{G}}(ck,x)\int_{0}^{\infty}dx'\,\mathbf{G}^{\dagger}(ck,x')\mathbf{F}(x')n^2(x') \tag{40}$$

the external Fox-Li adjoint-mode expansion, and represent the field by

$$\mathbf{F}_{\mathrm{exp}}(x)=\frac{1}{2}\lim_{\varepsilon\to 0^+}\{[\mathbf{F}_{\mathrm{cav}}(x)+\tilde{\mathbf{F}}_{\mathrm{cav}}(x)]\Theta(\varepsilon-x)+[\mathbf{F}_{\mathrm{out}}(x)+\tilde{\mathbf{F}}_{\mathrm{out}}(x)]\Theta(\varepsilon+x)\}, \tag{41}$$

then $\mathbf{F}_{\mathrm{exp}}(x)=\mathbf{F}(x)$ at every point $x\geqslant -L$.

VI. QUANTUM THEORY

To construct a quantum theory, we rewrite the expansions (37)–(40) in the following form:

$$\hat{\mathbf{F}}_{\mathrm{cav}}(x)=\sum_{n=-\infty}^{\infty}\sqrt{\frac{\hbar c\kappa_n}{2\epsilon_0}}\,\hat{a}_n\mathcal{G}(c\kappa_n,x), \tag{42}$$

$$\hat{\tilde{\mathbf{F}}}_{\mathrm{cav}}(x)=\sum_{n=-\infty}^{\infty}\sqrt{\frac{\hbar c\kappa_n^*}{2\epsilon_0}}\,\hat{b}_n\tilde{\mathcal{G}}(c\kappa_n^*,x), \tag{43}$$

$$\hat{\mathbf{F}}_{\mathrm{out}}(x)=\int_{-\infty}^{\infty}dk\sqrt{\frac{\hbar ck}{2\epsilon_0}}\,\hat{a}_{\mathrm{out}}(k)\mathbf{G}(ck,x), \tag{44}$$

$$\hat{\tilde{\mathbf{F}}}_{\mathrm{out}}(x)=\int_{-\infty}^{\infty}dk\sqrt{\frac{\hbar ck}{2\epsilon_0}}\,\hat{b}_{\mathrm{out}}(k)\tilde{\mathbf{G}}(ck,x), \tag{45}$$

where the non-Hermitean operators introduced above are given by

$$\begin{aligned}\hat{a}_n&=\sqrt{\frac{2\epsilon_0}{\hbar c\kappa_n}}\int_{-L}^{0}dx'\,\tilde{\mathcal{G}}^{\dagger}(c\kappa_n^*,x')\cdot\mathbf{F}(x')n^2(x')\\&=\sqrt{\frac{\epsilon_0}{4L\hbar c\kappa_n}}\int_{-L}^{0}dx'\left\{\frac{\hat{D}(x')}{\epsilon_0}\tilde{g}^*(c\kappa_n^*,x')+\frac{i}{\kappa_n}\frac{\partial}{\partial x'}\tilde{g}^*(c\kappa_n^*,x')c\hat{B}(x')\right\},\end{aligned} \tag{46}$$

$$\begin{aligned}\hat{b}_n&=\sqrt{\frac{2\epsilon_0}{\hbar c\kappa_n^*}}\int_{-L}^{0}dx'\,\mathcal{G}^{\dagger}(c\kappa_n,x')\cdot\mathbf{F}(x')n^2(x')\\&=\sqrt{\frac{\epsilon_0}{4L\hbar c\kappa_n^*}}\int_{-L}^{0}dx'\left\{\frac{\hat{D}(x')}{\epsilon_0}g^*(c\kappa_n,x')+\frac{i}{\kappa_n^*}\frac{\partial}{\partial x'}g^*(c\kappa_n,x')c\hat{B}(x')\right\},\end{aligned} \tag{47}$$

$$\begin{aligned}\hat{a}_{\mathrm{out}}(k)&=\sqrt{\frac{2\epsilon_0}{\hbar ck}}\int_{0}^{\infty}dx'\,\tilde{\mathbf{G}}^{\dagger}(ck,x')\cdot\mathbf{F}(x')n^2(x')\\&=\sqrt{\frac{\epsilon_0}{4\pi n_d\hbar ck}}\int_{0}^{\infty}dx'\left\{\frac{\hat{D}(x')}{\epsilon_0}\tilde{G}^*(ck,x')+\frac{i}{k}\frac{\partial}{\partial x'}\tilde{G}^*(ck,x')c\hat{B}(x')\right\},\end{aligned} \tag{48}$$

$$\begin{aligned}\hat{b}_{\mathrm{out}}(k)&=\sqrt{\frac{2\epsilon_0}{\hbar ck}}\int_{0}^{\infty}dx'\,\mathbf{G}^{\dagger}(ck,x')\cdot\mathbf{F}(x')n^2(x')\\&=\sqrt{\frac{\epsilon_0}{4\pi n_d\hbar ck}}\int_{0}^{\infty}dx'\left\{\frac{\hat{D}(x')}{\epsilon_0}G^*(ck,x')+\frac{i}{k}\frac{\partial}{\partial x'}G^*(ck,x')c\hat{B}(x')\right\},\end{aligned} \tag{49}$$

and $\hat{\mathbf{F}}(x)$ is given by Eq. (34) with *quantized* radiation fields rather than classical. The Hamiltonian follows by substituting $\hat{\mathbf{F}}(x)=\hat{\mathbf{F}}_{\mathrm{exp}}(x)$, that is given by Eq. (41) with all fields quantized, in the expression for the total energy (17) that can be rewritten as

$$\hat{H}=\epsilon_0\int_{-L}^{\infty}dx\,\hat{\mathbf{F}}^{\dagger}(x)\cdot\hat{\mathbf{F}}(x)n^2(x). \tag{50}$$

Noting that

$$\begin{aligned}&\int_{-L}^{0}dx\,\tilde{\mathcal{G}}^{\dagger}(c\kappa_n^*,x)\tilde{\mathcal{G}}(c\kappa_{n'}^*,x)n^2(x)\\&=\frac{1}{r^2}\int_{-L}^{0}dx\,\mathcal{G}^{\dagger}(c\kappa_{n'},x)\mathcal{G}(c\kappa_n,x)n^2(x)\\&=\frac{i}{2Lr^2}\frac{r^2-1}{\kappa_n-\kappa_{n'}^*},\end{aligned} \tag{51}$$

$$\int_0^\infty dx \tilde{\mathbf{G}}^\dagger(ck,x)\cdot\tilde{\mathbf{G}}(ck',x)n^2(x)=\frac{1}{r^2}\int_0^\infty dx \mathbf{G}^\dagger(ck',x)\cdot\mathbf{G}(ck,x)n^2(x)$$
$$=\frac{1+r^2}{2r^2}\delta(k-k')-\frac{i}{2\pi r^2}(1-r^2)\mathrm{P}\frac{1}{k-k'}, \tag{52}$$

where P stands for Cauchy's principal part [7], we find

$$\hat{H}=\frac{\hbar c}{8}\sum_n\{\kappa_n\hat{b}_n^\dagger\hat{a}_n+\kappa_n^*\hat{a}_n^\dagger\hat{b}_n\}+\frac{i\hbar c}{16L}(r^2-1)\sum_{n,n'}\frac{\sqrt{\kappa_n^*\kappa_{n'}}}{\kappa_{n'}-\kappa_n^*}\hat{a}_n^\dagger\hat{a}_{n'}+\frac{i\hbar c}{16Lr^2}(r^2-1)$$
$$\times\sum_{n,n'}\frac{\sqrt{\kappa_{n'}^*\kappa_n}}{\kappa_n-\kappa_{n'}^*}\hat{b}_n^\dagger\hat{b}_{n'}+\frac{\hbar c}{8}\int_{-\infty}^\infty dk\, k\{\hat{b}_{\mathrm{out}}^\dagger(k)\hat{a}_{\mathrm{out}}(k)+\hat{a}_{\mathrm{out}}^\dagger(k)\hat{b}_{\mathrm{out}}(k)\}+\frac{\hbar c}{16}(1+r^2)$$
$$\times\int_{-\infty}^\infty dk k\hat{a}_{\mathrm{out}}^\dagger(k)\hat{a}_{\mathrm{out}}(k)+i\frac{\hbar c}{16}\frac{1-r^2}{\pi}\int_{-\infty}^\infty dk\int_{-\infty}^\infty dk'\mathrm{P}\frac{\sqrt{kk'}}{k-k'}\hat{a}_{\mathrm{out}}^\dagger(k)\hat{a}_{\mathrm{out}}(k')+\frac{\hbar c}{16r^2}(1+r^2)$$
$$\times\int_{-\infty}^\infty dk k\hat{b}_{\mathrm{out}}^\dagger(k)\hat{b}_{\mathrm{out}}(k)-i\frac{\hbar c}{16r^2}\frac{1-r^2}{\pi}\int_{-\infty}^\infty dk\int_{-\infty}^\infty dk'\mathrm{P}\frac{\sqrt{kk'}}{k-k'}\hat{b}_{\mathrm{out}}^\dagger(k)\hat{b}_{\mathrm{out}}(k'). \tag{53}$$

This Hamiltonian appears to be the sum of two uncoupled Hamiltonians: a ''cavity Hamiltonian,'' given by the first three terms in Eq. (53), and an ''external Hamiltonian,'' given by the remaining five terms in Eq. (53). Were this the case, the cavity and the outside would be two isolated systems and there would be no dissipation. There is dissipation, however, because the cavity resonance ''frequencies'' $c\kappa_n$ are complex. But how can there be a coupling between the cavity and the outside with a Hamiltonian as Eq. (53)? The answer is that the theoretical framework to describe the dynamics of the quantized fields in the leaky cavity is not complete until all the commutation rules are given. In the present case, cavity and external operators do not commute. This very unusual feature accounts for the coupling between the cavity and the outside. Far from being just a theoretical choice, this feature emerges as an unavoidable consequence of describing an open system in terms of modes pertaining to that system alone rather than global modes of the universe [40].

The commutator rules for the new operators can be obtained from their definitions (46)–(49) in terms of the fields and the commutator between D and B [Eq. (16)] that we have derived in Sec. III. Because of the nonorthogonality of these modes, $\hat{a}_n$ and $\hat{a}_{n'}^\dagger$ do not commute even when $n\neq n'$. The same holds for $\hat{b}_n$ and $\hat{b}_{n'}^\dagger$. However, because of the biorthogonality between modes and adjoints, the commutator between $\hat{a}_n$ and $\hat{b}_{n'}^\dagger$ takes the simple and familiar form of the commutator between ordinary annihilation and creation operators

$$[\hat{a}_n,\hat{b}_{n'}^\dagger]=\delta_{n,n'}. \tag{54}$$

Analogously,

$$[\hat{a}_{\mathrm{out}}(k),\hat{b}_{\mathrm{out}}^\dagger E(k')]=\delta(k-k'). \tag{55}$$

The remaining commutation rules are given by

$$[\hat{a}_n,\hat{a}_{n'}^\dagger]=\frac{1}{r^2}[\hat{b}_n,\hat{b}_{n'}^\dagger]^*=\frac{i}{4L\sqrt{\kappa_n\kappa_{n'}^*}}\frac{\kappa_n+\kappa_{n'}^*}{\kappa_n-\kappa_{n'}^*}\frac{r^2-1}{r^2}, \tag{56}$$

$$[\hat{a}_{\mathrm{out}}(k),\hat{a}_{\mathrm{out}}^\dagger(k')]=\frac{1}{r^2}[\hat{b}_{\mathrm{out}}(k),\hat{b}_{\mathrm{out}}^\dagger(k')]^*$$
$$=\frac{1}{\sqrt{kk'}}\left\{\frac{1+r^2}{2r^2}k\delta(k-k')\right.$$
$$\left.-i\frac{1-r^2}{4\pi r^2}(k+k')\mathrm{P}\frac{1}{k-k'}\right\}, \tag{57}$$

$$[\hat{a}_n,\hat{b}_{\mathrm{out}}^\dagger(k)]=r^2[\hat{a}_n,\hat{a}_{\mathrm{out}}(k)]=[\hat{b}_n^\dagger,\hat{a}_{\mathrm{out}}(k)]$$
$$=[\hat{b}_n^\dagger,\hat{b}_{\mathrm{out}}(k)]=\frac{i}{2}\sqrt{\frac{1-r^2}{L\pi\kappa_n k}}. \tag{58}$$

The commutation rules that we have omitted can be obtained simply by using the identities $\hat{a}_n=-i\hat{a}_{-n}^\dagger$, $\hat{b}_n=i\hat{b}_{-n}^\dagger$, $\hat{a}_{\mathrm{out}}(k)=-i\hat{a}_{\mathrm{out}}^\dagger(-k)$, $\hat{b}_{\mathrm{out}}(k)=i\hat{b}_{\mathrm{out}}^\dagger(-k)$ in Eqs. (54)–(58).

Recovering the Senitzky-Gardiner-Collett phenomenological Hamiltonian in the high-Q limit is a delicate problem that we have treated already in a previous work [15]. It is interesting, however, to take the limit of a perfect cavity. This is the limit $r\to-1$. In this limit, the external region cannot support any fields and the cavity modes and their adjoints become ordinary sine functions. Then the distinction between modes and adjoints disappear and the only nonvanishing commutators are $\lim_{r\to -1}[\hat{a}_n,\hat{a}_{n'}^\dagger]=\delta_{n,n'}$. Noting that $(i/2L)\lim_{r\to -1}\sqrt{\kappa_n^*\kappa_{n'}}(r^2-1)/(\kappa_{n'}-\kappa_n^*)=\delta_{n,n'}|k_n|$, using the identities $\hat{a}_{-n}^\dagger=i\hat{a}_n$, $\hat{b}_n=i\hat{b}_{-n}^\dagger$ to eliminate negative values of n, and realizing that $\lim_{r\to -1}\hat{b}_n=\lim_{r\to -1}\hat{a}_n$, we find that the Hamiltonian (53) reduces to the usual Hamiltonian for the field in a perfect cavity.

It is also interesting to consider how this formalism reduces to the semiclassical theory developed by Siegman [17], when we apply it to a single mode class-*A* laser and regard the field as classical. Assuming that the lasing mode is resonant with the gain medium and all the nonlasing modes have negligible fields, we only need to modify Eq. (A9) with $F_n=0$ [41] to account for gain amplification and noise. This leads to the following equation:

$$\dot{\alpha}_n=(\eta-ic\kappa_n)\alpha_n+\Gamma_n, \tag{59}$$

where η is the gain coefficient (saturation effects are neglected) and

$$\Gamma_n=\int_{-L}^{0}dx\,\tilde{\mathcal{G}}^\dagger(c\kappa_n,x)\cdot\mathbf{\Gamma}(x)n^2(x) \tag{60}$$

is the ordinary spontaneous emission noise projected onto the lasing mode [33]. From Eq. (59), it follows [17] that the ordinary laser linewidth is enhanced by the logitudinal excess noise factor [22] $K=\int_{-L}^{0}dx\,\tilde{\mathcal{G}}^\dagger(c\kappa_n,x)\tilde{\mathcal{G}}(c\kappa_n,x)n^2(x)$. Incidentally, if instead of a laser with a *gain medium*, we had a single excited atom sitting inside the empty cavity, we can easily verify that its spontaneous emission rate is not enhanced by *K*. Unlike the prediction of Ref. [21], a simple calculation reveals that for an atom in resonance with the *n*th mode sitting at a crest of this mode, the spontaneous emission rate is only enhanced by the cavity quality factor [42]. Further applications of this formalism will be considered elsewhere.

VII. DISCUSSION

Central to our approach is the view that the proper generalization of the concept of mode to a leaky cavity is Thomson's idea of natural modes of oscillation. In Sec. V, we have given a simple argument for this view, and shown that Thomson's idea is in complete accord with the intuitive notion of cavity modes as self-repeating field configurations that is widely adopted in semiclassical laser physics. We should also point out, however, that the same view emerges from a completely independent result. Berry [27] has investigated the possibility of an alternative formulation of what constitutes a mode of a perfect cavity. He argues that every such confined mode would correspond to the continuation to the interior of the cavity of an external superposition of plane waves for which the cavity is effectively transparent, i.e., for which there is no reflected wave. Berry was only concerned about closed cavities made of perfect reflectors in Ref. [27]. Then he recovers the usual normal modes, but the power of his formulation is that it can also be applied to leaky cavities. If we apply it to our one-dimensional leaky cavity, the modes we obtain are the natural modes of oscillation of the cavity, because the adjoint Thomson modes correspond to an incident wave $\exp(-i\kappa_n^*x)$ coming from infinity with no wave being reflected back from the cavity. It is true that this is not an ordinary plane wave (because κ_n^* is complex), but Berry [27] has shown that such waves can also be written as a superposition of plane waves.

Another point worth mentioning about our unifying formalism is that, as the usual derivation of normal modes, it is based on a differential equation: a Helmholtz equation. This equation is derived from the wave equation, which is Hermitian. It is only because of the boundary conditions that a Hermitian equation gives rise to a non-Hermitian eigenproblem [7]. Now Siegman [3,6,17] uses an operator formulation, where the eigenvalue problem can be written in the standard more transparent form

$$\mathbf{M}\mathbf{v}=\mathbf{v}\lambda, \tag{61}$$

with $\mathbf{M}$ being the operator, $\mathbf{v}$ the eigenstate, and λ the eigenvalue. Then it is clear that, when the operator $\mathbf{M}$ is not Hermitian, its eigenvalues λ do not have to be real and its eigenstates $\mathbf{v}$ do not have to be orthogonal. To make the connection with Siegman's work even more explicit, we will rewrite now in this form the eigenproblem that determines the Fox-Li modes of our one-dimensional cavity.

Let us consider the case of a perfect cavity first. Then the cavity modes are solutions χ_p of Helmholtz equation (4) satisfying Dirichlet boundary conditions at the cavity mirrors, i.e., $\chi_p(-L,\omega)=\chi_p(0,\omega)=0$. The connection between this differential formulation and the standard matricial form (61) is given by the Green's function of the one-dimensional Poisson equation [7]. Let $G(x|x')$ be the Green's function that satisfies

$$\frac{\partial^2}{\partial x^2}G(x|x')=-\delta(x-x') \tag{62}$$

with the boundary conditions $G(-L|x')=G(0|x')=0$. Then the matricial formulation (61) of the eigenvalue problem determining the perfect modes is

$$\chi_p(x,\omega)=\left(\frac{\omega}{c}\right)^2\int_{-L}^{0}dx'\,G(x|x')\chi_p(x',\omega), \tag{63}$$

where the Green's function [7]

$$G(x|x')=-\frac{1}{L}\{(x+L)x'\Theta(x'-x)+(x'+L)x\Theta(x-x')\} \tag{64}$$

is equivalent to the inverse of $\mathbf{M}$ and represents a Hermitian matrix.

In the case of our leaky cavity we have shown that, to introduce an inner product appropriate for mode expansions, we must distinguish between forward and backward propagating components of the fields. To do so, we adopted the spinor notation used by Hamel and Woerdman [33]. However, to make the comparison with Siegman's work easier, we switch now (only in this section) to the Fox-Li transmission medium representation. In this representation, the propagation inside the cavity is ''unfolded'' so that a round trip is replaced by forward propagation in a transmission medium: the backward propagation is represented by forward propagation in the extended space from $x=-2L$ to $x=-L$. Within this representation, the Helmholtz equation reduces to the following first-order differential equation

$$\frac{\partial}{\partial x}\Psi(x,\omega)-i\frac{\omega}{c}\Psi(x,\omega)=0, \tag{65}$$

where Ψ is the cavity mode in this transmission-medium representation. The perfect mirror is no longer a boundary now, just an ordinary point in the transmission medium where the wave is continuous. The outcoupling surface at the end of the cavity is now split in two: one at $x=-2L$ and the other at $x=0$. The boundary condition at this surface is given now by $\Psi(-2L,\omega)=-r\Psi(0,\omega)$.

As for the case of the perfect cavity, the matricial formulation (61) of the eigenvalue problem determining the cavity modes is given by

$$\Psi(x,\omega)=-i\frac{\omega}{c}\int_{-2L}^{0}dx' G_{FL}(x|x')\Psi(x',\omega), \tag{66}$$

where the Green's function $G_{FL}(x|x')$ obeys the equation

$$\frac{\partial}{\partial x}G_{FL}(x|x')=-\delta(x-x'), \tag{67}$$

with the boundary condition $G_{FL}(-2L|x')=-rG_{FL}(0|x')$. This Green's function is given by

$$G_{FL}(x|x')=-\frac{\Theta(x-x')-r\Theta(x'-x)}{1+r}, \tag{68}$$

which clearly represents a non-Hermitian matrix.

VIII. CONCLUSIONS

We have unified the concept of normal modes used in quantum optics and that of Fox-Li modes from semiclassical laser physics. The key ingredient in this unification is the view that Thomson's idea of natural modes of oscillation [11] is the proper generalization of the concept of mode for an open system (see Secs. V and VII). We show that because of the constant presence of vacuum fluctuations everywhere, a quantum description of the radiation field in a leaky cavity requires not only cavity Fox-Li modes and their adjoints but also external Fox-Li modes and their adjoints. Then the resulting system-reservoir theory has unavoidably [40] a very unusual feature: cavity and reservoir operators do not commute with each other. The system reservoir Hamiltonian that we have obtained is the arbitrary-Q generalization of the Senitzky-Gardiner-Collett Hamiltonian [13] that is ordinarily adopted for high-Q cavities. Unlike other quantum theories of the radiation field in a leaky cavity, ours is both fundamental and clearly connected to the semiclassical concepts that are widely adopted in laser physics.

Our one-dimensional theory solves the key problem of how to describe the quantized radiation field in a leaky cavity (open system) using Fox-Li modes. It only deals with the simple case of leakage due to transmissivity losses, but it points the way in which a fully three-dimensional theory, including diffraction losses, can be developed. Three-dimensional Fox-Li modes are clearly a paraxial concept. The ingredient that is still missing to construct such a three-dimensional theory is a fundamental way of describing diffraction losses as a coupling of these paraxial Fox-Li modes to nonparaxial reservoir modes. It will be analogous to the present one-dimensional theory, where transmissivity losses are described by the coupling between cavity and external Fox-Li modes that arises from the noncommutability of their respective operators. This description requires space to be split into a paraxial and a non-paraxial part, just as in our one-dimensional theory it was split into a cavity and an external part. Such separation carries some technical difficulties [43] that are not present in our one-dimensional theory, but the principle is the same. We are currently working on this problem and shall report any results in due course.

ACKNOWLEDGMENTS

We thank J. P. Woerdman for fruitful discussions on mode nonorthogonality and excess noise. We are also grateful to P. J. Bardroff and S. Stenholm for sending us a copy of one of their papers prior to publication. This work is part of the research program of the ''Stichting voor Fundamenteel Onderzoek der Materie (FOM),'' and the TMR network ERB4061 PL951021 (Microlasers and Cavity QED).

APPENDIX: PHYSICAL PICTURE AND COMPLETENESS

Completeness can be discussed by deriving a closure relation. Let us consider only the cavity Fox-Li modes for the moment. As we have done for the modes of the universe, a closure relation can be obtained [7,29] from Eq. (32). But, as we have now a biorthogonal basis, there are two closure relations: one for expansions in the modes $\mathcal{G}$,

$$\sum_{n=-\infty}^{\infty}\mathcal{G}(c\kappa_n,x)\tilde{\mathcal{G}}^{\dagger}(c\kappa_n^*,x')=\frac{I(x,x')}{n^2(x')}, \tag{A1}$$

and another for expansions in their adjoints,

$$\sum_{n=-\infty}^{\infty}\tilde{\mathcal{G}}(c\kappa_n^*,x)\mathcal{G}^{\dagger}(c\kappa_n,x')=\frac{\tilde{I}(x,x')}{n^2(x')}, \tag{A2}$$

where $I(x,x')$ is the identity in the space spanned by $\mathcal{G}$ and $\tilde{I}(x,x')$ in the space spanned by $\tilde{\mathcal{G}}$. What are the expressions for I and $\tilde{I}$? In general, the notion that we can separate forward and backward propagating components breaks down at the origin where the refractive index is discontinuous [44]. However, if we assume that the refractive index is still unity at the origin (this assumption will be lifted later on in this Appendix), we can state that the space spanned by $\mathcal{G}$ is the space of outgoing spinors $\mathcal{F}(x)$ defined for $-L\leqslant x\leqslant 0$ by $\mathcal{F}_B(-L)=-\mathcal{F}_F(-L)$ and $\mathcal{F}_B(0)=r\mathcal{F}_F(0)$ with the subscript B standing for the backward propagating lower component and F for the forward propagating top component. Analogously, the space spanned by $\tilde{\mathcal{G}}$ is then the space of incoming spinors $\tilde{\mathcal{F}}(x)$ defined for $-L\leqslant x\leqslant 0$ by $\tilde{\mathcal{F}}_B(-L)=-\tilde{\mathcal{F}}_F(-L)$ and $\tilde{\mathcal{F}}_B(0)=\tilde{\mathcal{F}}_F(0)/r$. This is enough to determine $I(x,x')$ and $\tilde{I}(x,x')$ for $-L\leqslant x,x'\leqslant 0$. They are given by

$$I(x,x')=\begin{bmatrix}\delta(x_-) & \frac{1}{r}\delta(x_+)-\delta(x_++2L)\\ r\,\delta(x_+)-\delta(x_++2L) & \delta(x_-)\end{bmatrix}\tag{A3}$$

and

$$\tilde{I}(x,x')=\begin{bmatrix}\delta(x_-) & r\,\delta(x_+)-\delta(x_++2L)\\ \frac{1}{r}\delta(x_+)-\delta(x_++2L) & \delta(x_-)\end{bmatrix},\tag{A4}$$

with $x_-\equiv x-x'$ and $x_+\equiv x+x'$, as can be verified in a straightforward way by calculating the summations on the left-hand side of (A1) and (A2) using Eqs. (23), (26), (30), and (31) with the refractive index always set to unity. So neither mode nor adjoint expansions are complete, as they fail to reproduce the correct value of an arbitrary $\mathbf{F}(x)$ at the origin.

It is instructive to derivate the equations of motion obeyed by the expansion coefficients α_n of a mode expansion of an arbitrary classical field (34) for $-L\leqslant x\leqslant 0$:

$$\mathbf{F}_{\rm cav}(x)=\sum_{n=-\infty}^{\infty}\alpha_n\mathcal{G}(c\kappa_n,x),\tag{A5}$$

where

$$\alpha_n=\int_{-L}^{0}dx\,\mathcal{G}^{\dagger}(c\kappa_n^*,x)\mathbf{F}(x)n^2(x).\tag{A6}$$

Still taking $n(0)=1$ and interpreting spatial derivatives at $x=0$ as derivatives from the left, we find, from Maxwell equations (2),

$$\frac{\partial}{\partial x}\mathbf{F}(x)=\begin{bmatrix}-\frac{1}{c}\frac{\partial}{\partial t} & 0\\ 0 & \frac{1}{c}\frac{\partial}{\partial t}\end{bmatrix}\mathbf{F}(x)\tag{A7}$$

and from Eq. (30),

$$\frac{\partial}{\partial x}\mathcal{G}(c\kappa_n,x)=\begin{bmatrix}i\kappa_n & 0\\ 0 & -i\kappa_n\end{bmatrix}\mathcal{G}(c\kappa_n,x).\tag{A8}$$

Then using Eqs. (A1) and (A3), we obtain after some straightforward algebra

$$\ddot{\alpha}_n+2\gamma c\dot{\alpha}_n+c^2(k_n^2+\gamma^2)\alpha_n=F_n(t),\tag{A9}$$

where

$$F_n(t)=-\frac{c}{4L}\left\{\frac{r-1}{r}\dot{E}(0,t)+c\frac{r+1}{r}\dot{B}(0,t)+c(\gamma-ik_n)\times\left[\frac{r-1}{r}E(0,t)+c\frac{r+1}{r}B(0,t)\right]\right\}.\tag{A10}$$

So we can think of α_n as the coordinate of a damped harmonic oscillator that is being driven by F_n [41]. The damping rate is just the cavity damping rate γc, and the frequency of the oscillator is the cavity resonance frequency ck_n. If $\mathbf{F}$ is purely outgoing, i.e., $F_B(0,t)=rF_F(0,t)$, the electric and magnetic fields at the origin are related by $(r-1)E(0,t)=-(r+1)cB(0,t)$ making the driving force (A10) vanish. For the coefficients of an adjoint expansion, we find a damped harmonic-oscillator equation with negative damping $-\gamma c$ and a driving force that only vanishes when $\mathbf{F}$ is purely incoming. As the quantum electromagnetic field will always have both incoming and outgoing components, these driving forces will never vanish in a full quantum theory. The driving forces are a consequence of the lack of completeness of these expansions at the origin. The constant presence of these forces in the quantum case tells us that $\mathcal{G}$ and $\tilde{\mathcal{G}}$ alone are not suitable for a quantum description of the field.

To show that Eq. (41) reproduces an arbitrary field (34) everywhere, first we notice that $\mathbf{F}_{\rm cav}(x)=\tilde{\mathbf{F}}_{\rm cav}(x)=\mathbf{F}(x)$ for $-L\leqslant x<0$ and that $\mathbf{F}_{\rm out}(x)=\tilde{\mathbf{F}}_{\rm out}(x)=\mathbf{F}(x)$ for $x>0$. It remains then to show that $\mathbf{F}_{\rm exp}(0)=\mathbf{F}(0)$. We do so calculating each expansion in Eq. (41) separately. The cavity mode expansion (A5) can be written in the following form at the origin:

$$\mathbf{F}_{\rm cav}(0)=\frac{1}{8L}\begin{bmatrix}1+r+\frac{1-r}{n(0)}\\ 1+r-\frac{1-r}{n(0)}\end{bmatrix}\sum_{n=-\infty}^{\infty}\phi_n,\tag{A11}$$

where

$$\phi_n=\int_{-L}^{0}dx\left\{n^2(x)\tilde{g}^*(c\kappa_n^*,x)E(x)+\frac{i}{\kappa_n}\frac{\partial}{\partial x}\tilde{g}^*(c\kappa_n^*,x)cB(x)\right\}.\tag{A12}$$

Now, if we substitute Eq. (A12) as it stands in Eq. (A11) and perform the summation over n before the integration, $\Sigma_n\tilde{g}^*(c\kappa_n,x)$ will yield a combination of delta functions that only differ from zero at the origin where $n^2(x)$ is discontinuous. To avoid the problem of having to integrate the ambiguous combination $n^2(x)\,\delta(x)$, we use Helmholtz equation (4) to eliminate the refractive index in Eq. (A12) followed by an integration by parts to eliminate the discontinuous second derivative of $\tilde{g}^*$. Then,

$$\phi_n = -\frac{i}{\kappa_n}\frac{1-r}{r}E(0) + \frac{1}{r}\int_{-L}^{0} dx\,(e^{i\kappa_n x} - re^{-i\kappa_n x})\times\left\{\frac{i}{\kappa_n}\frac{\partial}{\partial x}E(x) - cB(x)\right\}. \tag{A13}$$

Now, using Eq. (A13) in Eq. (A11) and performing the summation over n and then the integration, we obtain

$$\mathbf{F}_{\rm cav}(0) = \frac{1}{8r}\begin{bmatrix} 1+r+\dfrac{1-r}{n(0)} \\ 1+r-\dfrac{1-r}{n(0)} \end{bmatrix}\times\{(r+1)E(0)-(1-r)cB(0)\}. \tag{A14}$$

Proceeding in the same way for $\tilde{F}_{\rm cav}(0)$, we find

$$\mathbf{F}_{\rm cav}(0) = \frac{1}{8r}\begin{bmatrix} 1+r-\dfrac{1-r}{n(0)} \\ 1+r+\dfrac{1-r}{n(0)} \end{bmatrix}\times\{(r+1)E(0)+(1-r)cB(0)\}. \tag{A15}$$

For $\mathbf{F}_{\rm out}(0)$ and $\tilde{\mathbf{F}}_{\rm out}(0)$, the calculation also follows the same lines but we have to make explicit use of the fact that G and $\tilde{G}$ vanish as $x\rightarrow\infty$ to determine which poles are slightly above or slightly below the real axis. The result is

$$\mathbf{F}_{\rm out}(0) = \frac{1}{8r}\begin{bmatrix} 1-\dfrac{1}{n(0)} \\ 1+\dfrac{1}{n(0)} \end{bmatrix}\times\{-(1-r)^2E(0)-(1+r)^2cB(0)\} \tag{A16}$$

and

$$\tilde{\mathbf{F}}_{\rm out}(0) = \frac{1}{8r}\begin{bmatrix} -1-\dfrac{1}{n(0)} \\ -1+\dfrac{1}{n(0)} \end{bmatrix}\times\{(1-r)^2E(0)-(1+r)^2cB(0)\}. \tag{A17}$$

So, $\mathbf{F}_{\rm exp}(x)=\mathbf{F}(x)$ for any arbitrary $\mathbf{F}$ and regardless of the actual value of the refractive index at the origin.

[1] E. Fermi, Rev. Mod. Phys. **4**, 87 (1932).

[2] M. A. Rippin and G. H. C. New, J. Mod. Opt. **43**, 993 (1996).

[3] A. E. Siegman, *Lasers* (Oxford University Press, Oxford, 1987), Chaps. 14–22.

[4] A. G. Fox and T. Li, Proc. IRE **48**, 1904 (1960); Bell Syst. Tech. J. **40**, 453 (1961).

[5] A. E. Siegman, Appl. Phys. Lett. **36**, 412 (1980).

[6] A. E. Siegman, Opt. Commun. **31**, 369 (1979).

[7] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[8] R. Lang, M. O. Scully, and W. E. Lamb, Jr., Phys. Rev. A **7**, 1788 (1973); R. Lang and M. O. Scully, Opt. Commun. **9**, 331 (1973); K. Ujihara, Phys. Rev. A **12**, 148 (1975); B. Baseia and H. M. Nussenzveig, Opt. Acta **31**, 39 (1984); M. Ley and R. Loudon, J. Mod. Opt. **34**, 227 (1987); J. Gea-Banacloche, N. Lu, L. M. Pedrotti, S. Prasad, M. O. Scully, and K. Wódkiewicz, Phys. Rev. A **41**, 369 (1990); S. M. Dutra and P. L. Knight, *ibid.* **53**, 3587 (1996); B. J. Dalton, M. Babiker, and P. L. Knight, Opt. Commun. **152**, 36 (1998).

[9] L. Knöll, W. Vogel, and D. G. Welsch, Phys. Rev. A **43**, 543 (1991); L. Knöll and D.-G. Welsch, Prog. Quantum Electron. **16**, 135 (1992); W. Vogel and D.-G. Welsch, *Lectures on Quantum Optics* (Akademie Verlag, Berlin, 1994), Chap. 8; R. W. F. van der Plank and L. G. Suttorp, Phys. Rev. A **53**, 1791 (1996); R. W. F. van der Plank, Ph.D. thesis, University of Amsterdam, Amsterdam, The Netherlands, 1997; B. M. Garraway and P. L. Knight, Phys. Rev. A **54**, 3592 (1996); B. M. Garraway, *ibid.* **55**, 2290 (1997); **55**, 4636 (1997); B. J. Dalton, S. M. Barnett, and P. L. Knight, J. Mod. Opt. **46**, 1315 (1999); P. J. Bardroff and S. Stenholm, Phys. Rev. A **60**, 2529 (1999); **61**, 023806 (2000).

[10] L. A. Vaĭnshteĭn, J. Tech. Phys. **34**, 1541 (1964); High-Power Electron. **4**, 157 (1965).

[11] J. J. Thomson, Proc. London Math. Soc. **15**, 197 (1884); *Notes on Recent Researches in Electricity and Magnetism: Intended as a Sequel to Professor Clerk-Maxwell's Treatise on Electricity and Magnetism* (Clarendon Press, Oxford, 1893); A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press, New York, 1949), pp. 214–224; this is Sommerfeld's pedagogic account of the original problem of a leaky electromagnetic resonator solved by Thomson.

[12] L. Ronchi, in *Laser Handbook*, edited by F. T. Arecchi and E. O. Schulz-DuBois (North-Holland, Amsterdam, 1972), Vol. 1, Chap. A4, pp. 151–190.

[13] I. R. Senitzky, Phys. Rev. **115**, 227 (1959); I. R. Senitzky, *ibid.* **119**, 670 (1960); M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984); C. W. Gardiner and M. J. Collett, *ibid.* **31**, 3761 (1985); C. W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).

[14] S. M. Barnett and P. M. Radmore, Opt. Commun. **68**, 364 (1988).

[15] S. M. Dutra and G. Nienhuis, Acta Phys. Slov. **50**, 275 (2000); J. Opt. B **2**, 584 (2000).

[16] I. H. Deutsch, Am. J. Phys. **59**, 834 (1991); S. Prasad, Phys. Rev. A **46**, 1540 (1992).

[17] A. E. Siegman, Phys. Rev. A **39**, 1253 (1989); **39**, 1264 (1989).

[18] K. Petermann, IEEE J. Quantum Electron. **15**, 566 (1979).

[19] Y.-J. Cheng, C. G. Fanning, and A. E. Siegman, Phys. Rev. Lett. **77**, 627 (1996); M. A. van Eijkelenborg, A. M. Lindberg, M. S. Thijssen, and J. P. Woerdman, *ibid.* **77**, 4314 (1996); A. M. van der Lee, N. J. van Druten, A. L. Mieremet, M. A. van

Eijkelenborg, A. M. Lindberg, M. P. van Exter, and J. P. Woerdman, *ibid.* **79**, 4357 (1997); G. P. Karman, A. M. Lindberg, and J. P. Woerdman, Opt. Lett. **23**, 1698 (1998) A. M. van der Lee, A. L. Mieremet, M. P. van Exter, N. J. van Druten, and J. P. Woerdman, Phys. Rev. A **61**, 033812 (2000).

[20] There are alternative theories of excess noise that do not involve the concept of mode [see, for instance, P. Goldberg, P. W. Milonni, and B. Sundaram, J. Mod. Opt. **38**, 1421 (1991); Phys. Rev. A **44**, 1969 (1991); **44**, 4556 (1991)] but these theories do not account for all types of excess noise (e.g., transverse, longitudinal, polarization, etc.) in a unifying way as Siegman's theory does under the concept of mode nonorthogonality.

[21] C. Lamprecht and H. Ritsch, Phys. Rev. Lett. **82**, 3787 (1999).

[22] K. Ujihara, IEEE J. Quantum Electron. **20**, 814 (1984).

[23] P. Grangier and J.-P. Poizat, Eur. Phys. J. D **1**, 97 (1998); **7**, 99 (1999).

[24] K. Ujihara, Phys. Rev. A **20**, 1096 (1979); X.-P. Feng and K. Ujihara, IEEE J. Quantum Electron. **25**, 2332 (1989); Phys. Rev. A **41**, 2668 (1990).

[25] P. T. Leung, S. S. Tong, and K. Young, J. Phys. A **30**, 2139 (1997); **30**, 2153 (1997); P. T. Leung, W. M. Suen, C. P. Sun, and K. Young, Phys. Rev. E **57**, 6101 (1998); E. S. C. Ching, P. T. Leung, A. M. V. D. Brink, W. M. Suen, S. S. Tong, and K. Young, Rev. Mod. Phys. **70**, 1545 (1998).

[26] Y. B. Zel'dovich, Zh. Éksp. Teor. Fiz. **39**, 776 (1960) [Sov. Phys. JETP **12**, 542 (1961)].

[27] M. V. Berry, J. Phys. A **27**, L391 (1994).

[28] S. Weigert, Phys. Lett. A **214**, 215 (1996); Dipolmarbeit, Rheinisch-Westfälischen Technischen Hochschule, Aachen, Germany, 1986.

[29] D. J. Santos and R. Loudon, Phys. Rev. A **52**, 1538 (1995).

[30] P. W. Milonni, Phys. Rev. A **25**, 1315 (1982); E. A. Power and T. Thirunamachandran, *ibid.* **25**, 2473 (1982); S. M. Barnett, C. R. Gilson, B. Huttner, and N. Imoto, Phys. Rev. Lett. **77**, 1739 (1996).

[31] R. H. Dicke, U.S. Patent No. 2 851 652 (9 September 1958); A. L. Schawlow and C. H. Townes, Phys. Rev. **29**, 1940 (1958); A. M. Prokhorov, Zh. Éksp. Teor. Fiz. **34**, 1658 (1958) [Sov. Phys. JETP **7**, 1140 (1958)].

[32] S. R. Barone, J. Appl. Phys. **34**, 831 (1963).

[33] W. A. Hamel and J. P. Woerdman, Phys. Rev. A **40**, 2785 (1989); Phys. Rev. Lett. **64**, 1506 (1990); W. A. Hamel, Ph.D. thesis, Leiden University, Leiden, The Netherlands, 1991. Their original definition of mode is, unlike the Fox-Li case, for a cavity with an active medium. But their basic idea is easily applied to an empty cavity, as we do in Sec. IV.

[34] L. A. Lugiato and L. M. Narducci, Z. Phys. B: Condens. Matter **71**, 129 (1988).

[35] R. Peierls, *More Surprises in Theoretical Physics* (Princeton University Press, Princeton, NJ, 1991), pp. 76–82.

[36] G. Beck and H. M. Nussenzveig, Nuovo Cimento **16**, 416 (1960); H. M. Nussenzveig, *ibid.* **20**, 694 (1961); H. M. Nussenzveig, *Causality and Dispersion Relations* (Academic Press, New York, 1972), Chap. 4.

[37] Even though these mode functions diverge at infinity, it is still possible to expand the fields in the *whole of space* in terms of them, provided the initial excitation conditions are explicitly taken into account (see Ref. [36]). However, such an approach is not quite suitable for quantum electrodynamics.

[38] Because $\tilde{g}$ are purely incoming, they have to increase in time rather than decrease.

[39] In high-Q cavities the leakage appears to be delocalized throughout the cavity. Then we can adopt mode functions that do not reproduce the fields faithfully at the place where the outcoupling occurs [15], but this is only an approximation that is violated when Q increases and the true localized nature of the leakage becomes evident.

[40] R. E. Prange, Phys. Rev. **131**, 1083 (1963); J. Fransson, O. Eriksson, B. Johansson, and I. Sandalov, Phys. B **272**, 28 (1999).

[41] Eq. (A9) is equivalent to $\ddot{\alpha}_n + c^2\kappa_n^2\alpha_n = F_n$ which reduces to the form (5) when F_n vanishes.

[42] E. M. Purcell, Phys. Rev. **69**, 681 (1946).

[43] M. Lax, W. H. Louisell, and W. B. McKnight, Phys. Rev. A **11**, 1365 (1975); L. W. Davis, *ibid.* **19**, 1177 (1979); I. H. Deutsch and J. C. Garrison, *ibid.* **43**, 2498 (1991).

[44] The discontinuity of the refractive index at the origin is just a way of modeling the very abrupt variation of the index at an interface. It is this (implicit) fast spatial dependence of the refractive index that is the physical reason behind the breakdown, at the origin, of the notion that we can separate forward and backward propagating components. See F. W. Sluijter, J. Opt. Soc. Am. **60**, 8 (1970).