Systematic method to study the general structure of Bose-Einstein condensates with arbitrary spin

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We develop a systematical method by which the investigation of the general structure of Bose-Einstein condensates with arbitrary spin has been reduced to solving a one-order linear partial-differential equation of a very simple form. Its general solutions for arbitrary spin are studied and a procedure is also developed to seek systematically the building blocks for constructing the general structure of Bose-Einstein condensates with arbitrary spin.

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Over the last few years there has been much theoretical effort focused on Bose-Einstein condensates with multiple internal spin degrees due to their wide and important applications [1–5]. The mean-field theory for a vectorial Bose-Einstein condensate has been developed by several groups to predict various spin textures and topological excitations [1]. Law *et al.* [2] have utilized an algebraic method found in quantum optics [6,7] to study many-body states of spin-1 Bose-Einstein condensate in the absence of external fields and found rather complicated dynamical behaviors that the mean-field theory fails to capture. Subsequently, Koashi and Ueda [3] have discussed the magnetical response of spin-1 and spin-2 Bose-Einstein condensates by explicitly constructing exact eigenspectra and eigenstates. Recently, Ho and Yin have developed an elegant generating function method to deal with the general structure of Bose-Einstein condensates with arbitrary spin and its building blocks [4]. However, there appears not to be any systematical method yet to express the states of the angular momentum (in particular the building blocks) of Bose gases in terms of the Fock states describing hyperfine spin states of their particles, which will be the subject of the present paper. By establishing a relation of the raising operator of Bose gases' angular momentum with the creation and annihilation operators describing their hyperfine spin states, the problem of investigating the general structure of Bose-Einstein condensates with arbitrary spin is reduced to investigating the solutions of a one-order partial-differential equation. We also present a procedure to solve this equation and make some conclusions concerning the building blocks and the general structure of Bose gases with *arbitrary* spin.

We consider a Bose-Einstein condensate (BEC) of N spin-f particles all in BEC's lowest external mode (the zero momentum mode in the case of an untrapped BEC or lowest self-consistent mode that the system condenses into in the trapped case), denoted by the annihilation operators a_j with j labeling the $2f+1$ spin components [4]. The angular momentum operator then becomes $\mathbf{F}_{\mu\nu}=a_\mu^\dagger \mathbf{f}_{\mu\nu} a_\nu$, where $\mathbf{f}_{\mu\nu}$ is the spin matrix for a spin-f particle [4]. To construct the angular momentum eigenstates, it is sufficient to focus on the states $|F,F_z=F\rangle$ with maximum spin projections since other states $|F,F_z=m\rangle$ with $m<F$ can be obtained by the formula

$$|F,m\rangle=\frac{1}{\sqrt{(2F)!}}\sqrt{\frac{(F+m)!}{(F-m)!}}F_-^{F-m}|F,F\rangle, \tag{1}$$

where $F_-=F_x-iF_y$ is the spin-lowering operator.

To relate the states $|F,F_z=F\rangle$ with the states created by applying the creation operators $a_\mu^\dagger$ to the spin vacuum state $|\mathrm{vac}\rangle$ defined by $a_\mu|\mathrm{vac}\rangle=0$ for all $2f+1$ subscripts μ, we express the z-component operator F_z of the angular momentum and corresponding spin-raising (lowering) operator F_+ (F_-) in terms of the spin creation and annihilation operators as follows: $F_z=\Sigma_{j=-f}^{f} j a_j^\dagger a_j \equiv \Sigma_{j=-f}^{f} j n_j$ and $F_+=\Sigma_{j=-f}^{f}\beta_j a_{j+1}^\dagger a_j \equiv F_-^\dagger$ with $\beta_j=\sqrt{(f-j)(f+j+1)}$. Here and hereafter, the notation $\Sigma_{i=-f}^{f}$ denotes the summation over the range $i=-f,-f+1,-f+2,\ldots,f$ so as to cover the cases of both integer and half-integer spins. Note that $\beta_f=0$ and hence $j=f$ in the summation of F_+ is in fact absent. It is easily checked that these introduced operators indeed satisfy the usual commutation relations for angular momentum operators $[F_+,F_-]=2F_z$ and $[F_\pm,F_z]=\mp F_\pm$ as they should be. In addition the squared angular momentum operator $\hat{F}^2$ can easily be expressed in terms of the spin creation and annihilation operators by the relation $\hat{F}^2=F_-F_++F_z^2+F_z$. Obviously, all the states $|F,F_z=F\rangle$ with maximum spin projections satisfy $F_+|F,F_z=F\rangle=0$ and vice versa. In other words, any eigenstate $|\Psi\rangle$ of the operator F_z must be one of the states $|F,F_z=F\rangle$ if it satisfies the relation $F_+|\Psi\rangle=0$ and the reverse is also true. Any common eigenstate of both F_z and the total particle number $N=\Sigma_{j=-f}^{f} n_j$ is of the type $\Gamma|\mathrm{vac}\rangle$ with Γ being a polynomial with each of its terms having the form $\Pi_{j=-f}^{j=+f} a_j^{\dagger n_j}$. Here n_j are $2f+1$ non-negative integers subject to two relations $\Sigma_{j=-f}^{f} n_j=N$ and $\Sigma_{j=-f}^{f} j n_j=F_z$. Utilizing the expression of F_+ in terms of $a_\mu, a_\mu^\dagger$, and noting $F_+|\mathrm{vac}\rangle=0$, $[a_j^\dagger,\Gamma]=-\partial\Gamma/\partial a_j$, and $[a_j,\Gamma]=\partial\Gamma/\partial a_j^\dagger$, it is obvious that constructing the states $|F,F_z=F\rangle$ is equivalent to seeking all the possible nonzero polynomial solutions $\Gamma(a_{-f}^\dagger,a_{-f+1}^\dagger,\ldots,a_{f-1}^\dagger,a_f^\dagger)$ to the one-order partial-differential equation (derived from $F_+\Gamma|\mathrm{vac}\rangle=[F_+,\Gamma]|\mathrm{vac}\rangle=0$) $\Sigma_{j=-f}^{f}\beta_j a_{j+1}^\dagger \partial\Gamma/\partial a_j^\dagger=0$ or

$$\sum_{j=-f}^{f}(1-\delta_{jf})x_{j+1}\frac{\partial\Gamma}{\partial x_j}=0, \tag{2}$$

where $x_f=a_f^\dagger$. Other $2f$ variables x_j are given by $a_j^\dagger=(\beta_j\beta_{j+1}\ldots\beta_{f-1})x_j$ with $\beta_j=\sqrt{(f-j)(f+j+1)}$, and Γ

is linear combinations of the terms $\Pi_{j=-f}^{j=+f}x_j^{n_j}$ with the nonnegative integers n_j subject to the constraints $N=\Sigma_{j=-f}^{f}n_j$ and $F=\Sigma_{j=-f}^{f}jn_j$. Each nonzero polynomial solution Γ results in a nonzero ket vector $\Gamma|\text{vac}\rangle$ which is one of the states $|F,F_z=F\rangle$. We have now completely turned the problem of investigating the general structure of Bose gases with arbitrary spin into finding the solutions to a one-order linear partial-differential equation of a very simple form.

Our next task is to investigate the general features of the solutions to the partial-differential equation (2). Throughout the rest of this paper, we shall use the notation $\Gamma_F(N)$ to denote one of the polymonial solutions to Eq. (2) with N particles and $F_z=F$. We shall, for the time being, neglect the operator-type characteristic of Eq. (2) by regarding its $2f+1$ operator variables x_k as *c*-number variables. The standard textbook theory of differential equations manifests that the general solution to a *c*-number partial-differential equation of the type in Eq. (2) has the form $\Gamma(\Psi_1,\Psi_2,\ldots,\Psi_{2f})$ with $2f$ functions $\Psi_k\equiv\Psi_k(x_f,x_{f-1},\ldots,x_{-f})$ forming a complete solution set, i.e., all the $2f$ functions Ψ_k are mutually functional independent solutions to Eq. (2) and any solution to Eq. (2) can be expressed as a function of them. They can be obtained by the condition $\Psi_k[x_f(t),x_{f-1}(t),\ldots,x_{-f}(t)]=\text{const}$ along the trajectory $x_k(t)$ determined by a set of equations $dx_f/dt=0,dx_j/dt=x_{j+1}$ for $j\neq f$, or in matrix form $dX(t)/dt=AX(t)$ by introducing $X(t)=(x_f,x_{f-1},\ldots,x_{-f})^T$, and $A_{ij}=\delta_{i+1,j},i,j=-f,-f+1,\ldots,f$. We now explicitly construct a set of polynomial solutions Ψ_k. The solution to the equation $dX(t)/dt=AX(t)$ is $X=\exp(At)X_0$ with $X_0=(x_f^{(0)},x_{f-1}^{(0)},\ldots,x_{-f}^{(0)})^T$ being a *t*-independent column vector. The solution gives immediately $x_f=x_f^{(0)}$ and $x_{f-1}=x_f^{(0)}t+x_{f-1}^{(0)}$. $x_{f-1}^{(0)}$ can be taken as zero by translating t, and hence $X_0=\exp(-At)X$ with $x_{f-1}^{(0)}=0$ and $t=x_{f-1}/x_f^{(0)}$. Using $\exp(-At)=\Sigma_{k=0}^{2f}(-t)^kA^k/k!$ (since $A^m=0$ for $m\geqslant 2f+1$) and $(A^k)_{ij}=\delta_{i+k,j}$, we obtain $x_f=x_f^{(0)}$ and $\Sigma_{k=0}^{f-i}(-1)^k(x_f)^{f-i-1-k}(x_{f-1})^kx_{i+k}/k!=x_i^{(0)}(x_f^{(0)})^{f-i-1}$, $i=-f,-f+1,\ldots,f-2$. It can be shown that the $2f$ polynomial functions in the left-hand side of the above equations are indeed mutually functional independent and hence can be taken as Ψ_k, respectively. What is more, these $2f$ functions, when one considers $2f+1$ variables x_m as operators $a_f^\dagger=x_f,a_j^\dagger=(\beta_j\beta_{j+1}\cdots\beta_{f-1})x_j$, $j\neq f$, are obviously also the independent solutions to the operator-type partial-differential equation (2). Therefore, $2f$ independent operator-type polynomial solutions to Eq. (2) have been obtained by us and they can be written in terms of $a_\mu^\dagger$ as follows:

$$\Gamma_{F=f}(N=1)\equiv a_f^\dagger, \tag{3a}$$

$$\Gamma_{F=j(f-1)}(N=j)=\frac{a_{f-1}^{\dagger j}}{j(j-2)!}+\sum_{k=0}^{j-2}(-1)^{j-1-k}\sqrt{\frac{(2f)^{j-k}(2f-j+k)!}{(k!)^2(2f)!(j-k)!}}i\times(a_f^\dagger)^{j-1-k}a_{f-1}^{\dagger k}a_{f+k-j}^\dagger, \tag{3b}$$

where $j=2,3,\ldots,2f$ (note that there exists no possible j for $f=\frac{1}{2}$). It is emphasized that we can always neglect the operator characteristic of Eq. (2) by considering its x_μ as $2f+1$ *c*-number variables in seeking its *polynomial* solutions as we have done in obtaining results given in Eq. (3), which greatly simplify the solving procedure. However, its operator-type characteristic does make some difference as explained below. In neglecting its operator-type characteristic, the textbook theory of *c*-number differential equations tells us that the $2f$ indepedent solutions $\Gamma_{F_k}(N=N_k)$ given by Eq. (3) completely determine the general structure of solutions to Eq. (2) because any solution to Eq. (2) can always be expressed as their function and any function of them is also its solution. Obviously, the counterpart conclusion for the operator-type equation (2) is that any polynomial solution can always be expressed as a function of the $2f$ independent operator-type solutions $\Gamma_{F_k}(N_k)$ given by Eq. (3) but the reverse may be wrong because a function of them does not need to be a *polynomial* and may, even worse, not be well defined (note that a function of the $2f$ independent operator-type solutions, no matter whether it is involved in any seeming ill-defined operation or not, represents an appropriate solution if the final form of the function can be put into a polynomial of the $2f+1$ variables x_μ, see the paragraph discussing the spin-2 case). For instance, both $[\Gamma_F(N)]^{1/5}|\text{vac}\rangle$ and $[\Gamma_F(N)/\Gamma_{F_1}(N_1)]|\text{vac}\rangle$ may have no meaning at all. Nevertheless, a seemingly ill-defined division operation can in fact be avoided when one considers only polynomial solutions, which is what we are concerned about in this paper. Let $\Gamma=G/H$ be a polynomial solution with G and H being two polynomial functions of the solutions $\Gamma_{F_k}(N_k)$ given by Eq. (3) [note that the solution Γ must be a function of the solutions $\Gamma_{F_k}(N_k)$], then G must have a factorized form with its two factors H and Γ (which can easily be understood by taking all $2f+1$ operators x_μ in H, G, and Γ as complex variables), and therefore the division operation does not occur at all. The fact that each of the above G and H themselves may be certain $\Gamma_{F_k}(N_k)$ also suggests the problem of factorized and nonfactorized solutions which we shall now pay our attention to. Inspecting Eq. (2) leads obviously to the property that the product of two polynomial solutions must also be a polynomial solution, i.e., to be more specific, $\Gamma_{F_1}(N_1)\Gamma_{F_2}(N_2)=\Gamma_{F=F_1+F_2}(N=N_1+N_2)$. This property introduces a concept of the so-called building blocks (those polynomial solutions unable to be expressed as the product of two polynomial solutions) which certainly play an important role in investigating the general structure of Bose-Einstein condensates with arbitrary spin. In addition it also tells us that building blocks tend to have small N, and hence we shall in the next paragraph seek them in a way beginning with $N=1$, then $N=2$, next $N=3$, and so on. We can thus avoid annoying division operations by choosing a solution set composed of $2f$ independent building blocks found in this way. Nevertheless, we still face the problem of the radical operation which tends to make the number of the complete solution set be greater than $2f$. For instance, if a nonfactorized polynomial solution Γ is unable to be ex-

pressed as a polynomial function of $2f$ independent building blocks but its fifth power can, i.e., $\Gamma = A^{1/5}$ with A being a polynomial function of the $2f$ building blocks, then this nonfactorized polynomial solution Γ should be added to the solution set to make the set complete. Nevertheless, such a nonfactorized polynomial Γ can only appear finite times (at most four times for $\Gamma = A^{1/5}$) in constructing the general structure of Bose gases and therefore has no thermodynamic significance. From above discussions, though we are unable to prove it rigorously for arbitrary f at present, it seems reasonable to make the conjecture: *A complete solution set describing the general structure of a Bose-Einstein condensate with arbitrary spin contains at most $2f$ independent building blocks of thermodynamic significance*.

Let us now consider the building blocks for small N. It is worthwhile to point out that one significant advantage of our method, which turns investigating the general structure of Bose gases with arbitrary spin into finding the solutions to a one-order linear partial-differential equation (2), is to allow us to obtain analytically all the polynomial solutions, including the building blocks by a simple MATHEMATICA program whose spirit is outlined in Ref. [8]. The program obtains required results *quickly* so long as N or f is not too big. Here we only list the results for $N=1,2$. The unique building block for $N=1$ is $\Gamma_{F=f}(N=1) \equiv a_f^\dagger$. For $N=2$, all the building blocks are given by $\Gamma_{F=2f-2k}(N=2) = W_k \Sigma_{j=-k}^{k} (-1)^j x_{f-k+j} x_{f-k-j}$ with W_k being constant and the range of integers k being written immediately after Eq. (4). Utilizing the relation between x_j and $a_j^\dagger$ given immediately after Eq. (2), and taking

$$W_k = (\beta_{f-k}\beta_{f-k+1}\cdots\beta_{f-1})^2/2,$$

we obtain

$$\Gamma_{F=2f-2k}(N=2) = \sum_{j=0}^{k} \frac{(-1)^j}{1+\delta_{j0}} \sqrt{\frac{(k!)^2(2f-k-j)!(2f-k+j)!}{(k-j)!(k+j)![(2f-k)!]^2}} \times a_{f-k+j}^\dagger a_{f-k-j}^\dagger, \tag{4}$$

where $k=1,2,\ldots,f$ if $f=1,2,\ldots,$ no possible k exists for $f=\frac{1}{2}$ (i.e., no any two-particle building block exists for $f=\frac{1}{2}$) while $k=1,2,\ldots,f-\frac{1}{2}$ for other half-integer f $(=\frac{3}{2},\frac{5}{2},\frac{7}{2},\ldots)$. We have presented all the building blocks for $N\leqslant 2$ and for arbitrary spin f. In addition we have obtained a three-particle building block for $f\geqslant 1$ given by taking $j=3$ in Eq. (3b). (Note that no possible j exists for $f=\frac{1}{2}$.) Apart from this one, all other possible N-particle building blocks for $N\geqslant 3$ can be obtained by a very simple MATHEMATICA program whose spirit is outlined in Ref. [8].

Now, we apply the above general results and discussions for arbitrary spin f to some concrete values of $f=\frac{1}{2},\frac{3}{2}$ and $f=1,2,3$. It is pointed out that for arbitrary spin f, there exists only one one-particle building block $a_f^\dagger$, and all the two-particle building blocks, if any, are given in Eq. (4). In addition there has been found a three-particle building block (for $f\geqslant 1$) given by taking $j=3$ in Eq. (3b).

Spin-$\frac{1}{2}$ case. Eq. (2) in this case becomes $x_{1/2}\partial\Gamma/\partial x_{-1/2} = 0$ resulting in $\Gamma \equiv \Gamma(x_{1/2})$ independent of the operator $x_{-1/2} \propto a_{-1/2}^\dagger$. This immediately leads to the conclusion that there exists one $(=2f)$ and only one building block $a_{1/2}^\dagger \propto x_{1/2}$, and hence only possible angular momentum states with maximum spin projections are $|F,F_z\rangle = a_{1/2}^{\dagger N}|\text{vac}\rangle$ with $N=2F$.

Spin-$\frac{3}{2}$ case. In this case, we have already found three $(=2f)$ independent building blocks given, respectively, by Eq. (3a) (one-particle), Eq. (4) (two-particle, letting $f=\frac{3}{2}$ and $k=1$) and Eq. (3b) (three-particle, letting $f=\frac{3}{2}$ and $j=3$). They are rewritten here as follows: $\Gamma_{F=3/2}(N=1) = a_{3/2}^\dagger$, and

$$\Gamma_{F=1}(N=2) = \frac{1}{2}[a_{1/2}^{\dagger 2} - \sqrt{3} a_{1/2}^\dagger a_{-1/2}^\dagger], \tag{5a}$$

$$\Gamma_{3/2}(3) = \frac{1}{3} a_{1/2}^{\dagger 3} + \frac{\sqrt{3}}{2}(a_{3/2}^{\dagger 2} a_{-3/2}^\dagger - a_{3/2}^\dagger a_{1/2}^\dagger a_{-1/2}^\dagger). \tag{5b}$$

Ho and Yin [4] have found that there exist four basic building blocks in this case. Though they have not derived their explicit expressions, Ho and Yin have indeed designated their F and N values. Three of them have already been derived by us and are explicitly expressed above while the fourth one in our notation is $\Gamma_{F=0}(N=4)$, which is easily obtained by the MATHEMATICA program and is given as follows:

$$\Gamma_{F=0}(N=4) = a_{-3/2}^{\dagger 2} a_{3/2}^{\dagger 2} + 2 a_{-3/2}^\dagger a_{-1/2}^\dagger a_{3/2}^\dagger a_{1/2}^\dagger + \frac{1}{3} a_{-1/2}^{\dagger 2} a_{1/2}^{\dagger 2} - \frac{4}{3\sqrt{3}}(a_{-3/2}^\dagger a_{1/2}^{\dagger 3} + a_{-3/2}^{\dagger 3} a_{3/2}^\dagger). \tag{6}$$

The fact that the existing four $(=2f+1)$ basic building blocks listed above does not contradict our conjecture of a complete solution set describing the general structure of Bose gases contains at most $2f$ independent building blocks of thermodynamic significance because Ho and Yin have found that $\Gamma_{3/2}(3)$ can appear at most once [4]. However, this indeed represents an annoying circumstance because if we did not know the results of Ho and Yin, we would most likely have ceased seeking four-particle building block(s) because we already had found three $(=2f=3)$ N-particle independent building blocks for $N\leqslant 3$.

Spin-1 case. Equations (3a) and (4) give the two independent building blocks $\Gamma_{F=1}(N=1) \equiv a_1^\dagger$ and $\Gamma_0(2) = (a_0^\dagger)^2/2 - a_1^\dagger a_{-1}^\dagger$, respectively. As has been shown by Wu [7] and subsequently by others [2–4], the possible states $|F,F_z = F\rangle$ must have the form $(a_1^\dagger)^F[\Gamma_0(2)]^{(N-F)/2}|\text{vac}\rangle$ with $N-F$ being non-negative even integers. Therefore these two $(=2f)$ building blocks form a complete set.

Spin-2 case. Four $(=2f)$ independent building blocks have already been explicitly given, respectively, by Eq. (3a) (a one-particle block), Eq. (4) (two two-particle blocks), and Eq. (3b) (a three-particle block). They are explicitly written here as follows: $\Gamma_{F=2}(N=1) \equiv a_2^\dagger$ and

$$\Gamma_{F=0}(N=2)=\tfrac{1}{2}a_0^{\dagger 2}-a_1^{\dagger}a_{-1}^{\dagger}+a_2^{\dagger}a_{-2}^{\dagger}, \tag{7a}$$

$$\Gamma_{F=2}(N=2)=\tfrac{1}{2}a_1^{\dagger 2}-\sqrt{a_0^{\dagger}}, \tag{7b}$$

$$\Gamma_{F=3}(N=3)=\tfrac{1}{3}a_1^{\dagger 3}+\tfrac{2}{3}a_2^{\dagger 2}a_{-1}^{\dagger}-\sqrt{\tfrac{2}{3}}\ a_2^{\dagger}a_1^{\dagger}a_0^{\dagger}. \tag{7c}$$

Again, an annoying situation similar to the spin-$\frac{3}{2}$ case occurs, i.e., $\Gamma_{F=3}(N=3)$ can appear at most once [4] and we need to supply another building block

$$\Gamma_{F=0}(N=3)=2a_2^{\dagger}a_{-2}^{\dagger}a_0^{\dagger}+a_1^{\dagger}a_{-1}^{\dagger}a_0^{\dagger}-\tfrac{1}{3}a_0^{\dagger 3} -\sqrt{\tfrac{3}{2}}a_1^{\dagger 2}a_{-2}^{\dagger}-\sqrt{\tfrac{3}{2}}a_2^{\dagger}a_{-1}^{\dagger 2}, \tag{8}$$

which is easily obtained by the MATHEMATICA program, and also explicitly given by Ho and Yin (they use different notation) [4]. Once again there only exist $2f$ independent building blocks of thermodynamic significance. At this time, let us explain the reason why such an annoying situation will appear here in the spin-$\frac{3}{2}$ case as well. The reason is that a function of $2f$ independent operator-type solutions, no matter whether it involves any seeming ill-defined operation or not, represents an appropriate solution if the final form of the function can be put into a polynomial of the $2f+1$ variables x_μ or $a_\mu^{\dagger}$. Let us take the spin-2 case as an example to illustrate this statement. It can be shown from Eqs. (7) and (8) that $4\sqrt{2}a_2^{\dagger 3}\Gamma_{F=0}(N=3)=-9\sqrt{3}[\Gamma_{F=3}(N=3)]^2+8\sqrt{3}[\Gamma_{F=2}(N=2)]^3-8\Gamma_{F=2}(N=2)\Gamma_{F=0}(N=2)\equiv 4\sqrt{2}G$ (see also the second to last paragraph of p. 2304 of Ref. [4]). From this relation, we realize $\Gamma_{F=0}(N=3)=G/a_2^{\dagger 3}$, whose right-hand side involves a *seemingly* ill-defined division but its final outcome $\Gamma_{F=0}(N=3)$ is a polynomial function of the $2f+1$ variables $a_\mu^{\dagger}$ and hence a well-defined solution. Note that this fact reflects our general conclusion that any polynomial solution to Eq. (2) must be able to be expressed as the function of any $2f$ independent solutions. On the other hand, this relation can also be expressed as $\Gamma_{F=3}(N=3)=\sqrt{A}$ with A being a polynomial function of the $2f=4$ solutions $a_2^{\dagger}$, $\Gamma_{F=2}(N=2)$, $\Gamma_{F=0}(N=2)$, and $\Gamma_{F=0}(N=3)$. The latter fact also explains why $\Gamma_{F=3}(N=3)$ can appear at most once.

Spin-3 case. Four independent building blocks have already been found and are $a_3^{\dagger}$ and three two-particle blocks $\Gamma_{F=0}(N=2)$, $\Gamma_{F=2}(N=2)$, and $\Gamma_{F=4}(N=2)$. All these three two-particle blocks are explicitly given in Eq. (4). Since we can obtain a state $|F,F\rangle$ with $N=3$ and $F=6$ corresponding to $a_3^{\dagger 2}\Gamma_{F=0}(N=2)$, we therefore do not count the three-particle block $\Gamma_{F=6}(N=3)$ explicitly given by taking $j=3$ in Eq. (3b) as a basic building block. Ho and Yin [4] have claimed to have seven $(=2f+1)$ basic building blocks in this case. Besides the four blocks $a_3^{\dagger}$ and $\Gamma_F(N=2)$, $F=0,2,4$ listed above, the other three are (in our notation) $\Gamma_{F=0}(N)$, $N=4,6,8$, though they have not derived the explicit expressions of all the seven blocks. Once again, we do not count $\Gamma_{F=0}(N)$, $N=4,6,8$ as basic blocks because $\Gamma_{F=0}^{k}(N=2)$, $k=2,3,4$ can produce states $|F,F\rangle$ with $N=4$ and $F=4,6,8$. Therefore the four $(<2f=6)$ one and two-building blocks seem to be able to describe completely the general structure of Bose gases with spin-3 particles.

In summary, we have shown that the general structure of Bose-Einstein condensates with arbitrary spin can be investigated by solving a one-order linear partial-differential equation of a very simple form. We have also provided a systematic, and in our view the simplest, procedure to establish explicitly the relations of all the states of the total angular momentum with the Fock states $\Gamma(\{a_\mu^{\dagger}\})|\text{vac}\rangle$. In particular, combining our method with the elegant generating function method developed by Ho and Yin [4] produces a very powerful tool to deal with the general structure of Bose-Einstein condensates with arbitrary spin.

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[8] We can easily assemble a MATHEMATICA program to finish the following two essential ingredients: (1) $\Gamma_F(N)$ in Eq. (2) is a linear combination of at most $p(L)$ terms of the form $\Pi_{j=0}^{j=2f}y_j^{N_j}$ with $y_j\equiv x_{j-f}$ and the non-negative integers $N_j\equiv n_{j-f}(j=0,1,\ldots,2f)$ subject to the constraints $N=\Sigma_{j=0}^{2f}N_j$ and $L=F+fN$ with $L\equiv\Sigma_{j=0}^{2f}jN_j$. $p(L)$ denotes the multiplicity of the partitions of the positive integer L into positive integers. Note that L is always a positive integer no matter whether f is a half integer or an integer. (2) Substituting the form of $\Gamma_F(N)$ containing $p(L)$ coefficients into Eq. (2), we can determine the $p(L)$-dimensional vector(s) made of the $p(L)$ coefficients. Any nonzero vector corresponds to a polynomial solution to Eq. (2) with given N, F, and f.