Optimal entanglement purification via entanglement swapping

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It is known that entanglement swapping can be used to realize entanglement purification. In this way, two particles belonging to different nonmaximally entangled pairs can be projected probabilistically to a maximally entangled state or to a less entangled state. In this paper, we show, when the less entangled state is obtained, then a maximally entangled state can be obtained probabilistically from this less entangled state if a unitary transformation is introduced locally. The probability of success of our scheme is equal to the entanglement of a single pair purification (if two original pairs are in the same nonmaximally entangled states) or to the smaller entanglement of a single pair purification of these two pairs (if two original pairs are not in the same nonmaximally entangled states). The advantage of our scheme is that no continuous indefinite iterative procedure is needed to achieve optimal purification.

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Entanglement is at the source of a number of pure quantum phenomena, such as the correlations violating Bell's inequalities [1], quantum key distribution [2], quantum teleportation [3], Greanberger-Horne-Zeilinger correlations [4], and various other nonclassical interference phenomena [5]. Polarization entangled photons have been used to demonstrate both dense coding [6] and teleportation [7] in the laboratory. Teleportation has also been realized using path-entangled photons [8] and entangled electromagnetic field modes [9]. In order to realize these schemes, maximum entanglement between distant particles should be set up. One possible way is entanglement swapping [10], which has been demonstrated experimentally [11]. Recently, Bose *et al.* [12] showed that entanglement swapping can be used to realize entanglement purification. In their scheme, if an ensemble of two photon pairs is given, and all pairs are in the same nonmaximally entangled states, then two photons belonging to different photon pairs can be projected probabilistically into a maximally entangled Bell state or into a less entangled state. If one continues this process indefinitely, in the limit of an infinite sequence, the final ensemble generated will be comprised of a certain fraction of Bell pairs and a certain fraction of completely disentangled pairs. The fraction of Bell pairs is equal to twice the modulus square of the Schmidt coefficient, which is the smaller one in the original pair, i.e., to the entanglement of a single pair purification. In this paper, we show that if a unitary transformation follows when a less entangled state is obtained by entanglement swapping, then, a maximally entangled Bell state can be obtained probabilistically from this less entangled state. The maximum probability with which a Bell state can be obtained by our scheme is equal to the entanglement of a single pair purification. This means our scheme is optimal. One advantage of our scheme is that no continuous indenfinite iterative procedure is needed. Furthermore, if two original particle pairs are not the same type of entangled states, two particles belonging to different pairs can also be projected in the same way to a maximally entangled state with a certain probability. This probability is equal to the smaller entanglement of a single pair purification of these two pairs; this also means our scheme is optimal.

Let pairs of particles (1,2) and (3,4) be in the following entangled states, respectively:

$$|\Phi\rangle_{12}=\alpha|00\rangle_{12}+\beta|11\rangle_{12}, \tag{1}$$

$$|\Phi\rangle_{34}=\alpha|00\rangle_{34}+\beta|11\rangle_{134}, \tag{2}$$

where, $|\alpha|>|\beta|$, and $|\alpha|^2+|\beta|^2=1$. Suppose that the particle pair (1,2) and the particle 3 belong to Alice and the particle 4 belongs to Bob. If a Bell state measurement on particles 2 and 3 is performed by Alice, then particles 1 and 4 will be projected into one of the following states:

$$\langle\Phi^{\pm}|_{23}\Phi\rangle_{12}\otimes|\Phi\rangle_{34}=\frac{\alpha^2}{\sqrt{2}}|00\rangle_{14}\pm\frac{\beta^2}{\sqrt{2}}|11\rangle_{14}, \tag{3}$$

$$\langle\Psi^{\pm}|_{23}\Phi\rangle_{12}\otimes|\Phi\rangle_{34}=\alpha\beta\left[\frac{1}{\sqrt{2}}(|01\rangle_{14}\pm|10\rangle_{14})\right], \tag{4}$$

where $|\Phi^{\pm}\rangle_{23}=1/\sqrt{2}(|00\rangle_{23}\pm|11\rangle_{23})$ and $|\Psi^{\pm}\rangle_{23}=1/\sqrt{2}(|01\rangle_{23}\pm|10\rangle_{23})$. Obviously, particles 1 and 4 will be projected into a less entangled state $\alpha^2/\sqrt{2}|00\rangle_{14}\pm\beta^2/\sqrt{2}|11\rangle_{14}$ with probability $(\alpha^4+\beta^4)/2$. In order to get optimal entanglement purification, a unitary transformation performed by Alice follows when a less entangled state is obtained. (This transformation can be made by Alice or by Bob. In this paper, we let Alice perform this transformation.) To carry out this evolution, an auxiliary qubit with the original state $|0\rangle_a$ is introduced by Alice. Under the basis $\{|0\rangle_1|0\rangle_a,|1\rangle_1|0\rangle_a,|0\rangle_1|1\rangle_a,|1\rangle_1|1\rangle_a\}$, this unitary transformation can be written as

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$$\begin{bmatrix} \frac{\beta^2}{\alpha^2} & 0 & \sqrt{1-\frac{\beta^4}{\alpha^4}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \sqrt{1-\frac{\beta^4}{\alpha^4}} & 0 & -\frac{\beta^2}{\alpha^2} & 0 \end{bmatrix}. \tag{5}$$

This transformation will transform Eq. (3) to the following state

$$\beta^2\left[\frac{1}{\sqrt{2}}(|00\rangle_{14}\pm|11\rangle_{14})\right]|0\rangle_a+\frac{\alpha^2}{\sqrt{2}}\sqrt{1-\frac{\beta^4}{\alpha^4}}|1\rangle_1|0\rangle_4|1\rangle_a. \tag{6}$$

Having completed the transformation, Alice makes a measurement on this auxiliary qubit. If the result of the measurement is $|0\rangle_a$, then particles 1 and 4 will be projected into a maximally entangled Bell state. If the result of the measurement is $|1\rangle_a$, particles 1 and 4 are completely disentangled.

The maximally probability with which a Bell state can be obtained by purifying a single entangled pair is $2\beta^2$. In our scheme, the probability of the success is $2\beta^2$, so our scheme is optimal.

Next, we proceed to consider the case in which particle pairs (1,2) and (3,4) are not in the same nonmaximally entangled states. Suppose particles 1 and 2 are in the entangled state $|\Phi\rangle_{12}$ and particles 3 and 4 are in another entangled state $|\Phi\rangle_{34}$, which are the following states, respectively,

$$|\Phi\rangle_{12}=\alpha|00\rangle_{12}+\beta|11\rangle_{12} \tag{7}$$

and

$$|\Phi\rangle_{34}=a|00\rangle_{34}+b|11\rangle_{34}, \tag{8}$$

where $|a|>|b|$, $|a|^2+|b|^2=1$. Suppose that particles 1, 2, and 3 belong to Alice and particle 4 belongs to Bob. If Alice makes a Bell state measurement on particles 2 and 3, then particles 1 and 4 will be projected into one of the following states:

$$\langle\Phi^{\pm}|_{23}\Phi\rangle_{12}\otimes|\Phi\rangle_{34}=\frac{\alpha a}{\sqrt{2}}|00\rangle_{14}\pm\frac{\beta b}{\sqrt{2}}|11\rangle_{14}, \tag{9}$$

$$\langle\Psi^{\pm}|_{23}\Phi\rangle_{12}\otimes|\Phi\rangle_{34}=\frac{\alpha b}{\sqrt{2}}|01\rangle_{14}\pm\frac{\beta a}{\sqrt{2}}|10\rangle_{14}. \tag{10}$$

If Eq. (9) is obtained, in order to get the optimal entanglement purification, a unitary transformation, which is made on the particle 1, and an auxiliary qubit with the original state $|0\rangle_a$, is introduced by Alice. Under the basis $\{|0\rangle_1|0\rangle_a,|1\rangle_1|0\rangle_a,|0\rangle_1|1\rangle_a,|1\rangle_1|1\rangle_a\}$, this unitary transformation is

$$\begin{bmatrix} \frac{\beta b}{\alpha a} & 0 & \sqrt{1-\frac{\beta^2 b^2}{\alpha^2 a^2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \sqrt{1-\frac{\beta^2 b^2}{\alpha^2 a^2}} & 0 & -\frac{\beta b}{\alpha a} & 0 \end{bmatrix}. \tag{11}$$

Under this transformation, Eq. (9) will be transformed into the state

$$\beta b\left[\frac{1}{\sqrt{2}}(|00\rangle_{14}\pm|11\rangle_{14})\right]|0\rangle_a + \frac{\alpha a}{\sqrt{2}}\sqrt{1-\frac{\beta^2 b^2}{\alpha^2 a^2}}|1\rangle_1|0\rangle_4|1\rangle_a. \tag{12}$$

After that, a measurement on the auxiliary particle follows. If the result of the measurement is $|0\rangle_a$, particles 1 and 4 will be projected into a maximally entangled state with probability $\beta^2 b^2$. If the result is $|1\rangle_a$, particles 1 and 4 are completely disentangled.

If Eq. (10) is obtained, two different cases should be considered:

(1) $|\alpha b|>|a\beta|$: In this case, the unitary transformation on the particles 1 and the auxiliary qubit performed by Alice is

$$\begin{bmatrix} \frac{\beta a}{\alpha b} & 0 & \sqrt{1-\frac{\beta^2 a^2}{\alpha^2 b^2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \sqrt{1-\frac{\beta^2 a^2}{\alpha^2 b^2}} & 0 & -\frac{\beta a}{\alpha b} & 0 \end{bmatrix}. \tag{13}$$

By the same procedure, particles 1 and 4 will be projected into a maximally entangled state with probability $a^2\beta^2$.

(2) $|\alpha b|<|a\beta|$: In this case, the probability of obtaining a maximally entangled state is $\alpha^2 b^2$. The unitary transformation is on the particles 1 and the auxiliary qubit performed by Alice is

$$\begin{bmatrix} \frac{\alpha b}{a\beta} & 0 & \sqrt{1-\frac{\alpha^2 b^2}{a^2\beta^2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \sqrt{1-\frac{\alpha^2 b^2}{a^2\beta^2}} & 0 & -\frac{\alpha b}{a\beta} & 0 \end{bmatrix}. \tag{14}$$

The probability of obtaining a maximally entangled state from the original pair is $2\beta^2$ or $2b^2$, respectively, in these two cases. In the first case, the entanglement of a single pair purification of the $|\Phi\rangle_{12}$ is less than that of the state $|\Phi\rangle_{34}$. In the second case, the entanglement of a single pair purification of $|\Phi\rangle_{34}$ is less than that of the state $|\Phi\rangle_{12}$. This probability is equal to the smaller entanglement of a single pair purification of these two pairs $|\Phi\rangle_{12}$ and $|\Phi\rangle_{34}$. Obviously, our scheme is optimal.

In conclusion, in Ref. [12], an indefinite iterative procedure is needed in order to achieve the optimal entanglement purification. In our scheme, when a less entangled state is obtained during the entanglement purification, a maximally entangled state can be obtained with a certain probability if a unitary transformation is introduced locally. The successful probability of our scheme is equal to the entanglement of a single pair purification if two original pairs are in the same nonmaximally entangled states, or to the smaller entanglement of a single pair purification of these two pairs if they are not in the same nonmaximally entangled states; this means our scheme is optimal. No continuous indefinite iterative procedure is needed, which makes our scheme easily implementable in practice.

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