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Trapped ions in laser fields: A benchmark for deformed quantum oscillators

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Some properties of the nonlinear coherent states (NCS), recognized by Vogel and de Matos Filho as dark states of a trapped ion, are extended to NCS on a circle, for which the Wigner functions are presented. These states are obtained by applying a suitable displacement operator $D_h(\alpha)$ to the vacuum state. The unity resolutions in terms of the projectors $|\alpha,h\rangle\langle\alpha,h^{-1}|,|\alpha,h^{-1}\rangle\langle\alpha,h|$ are presented together with a measure allowing a resolution in terms of $|\alpha,h\rangle\langle\alpha,h|$. $D_h(\alpha)$ is also used for introducing the probability distribution funtion $\rho_{A,h}(z)$ while the existence of a measure is exploited for extending the P representation to these states. The weight of the nth Fock state of the NCS relative to a trapped ion with Lamb-Dicke parameter η, oscillates so wildly as n grows up to infinity that the normalized NCS fill the open circle η^{-1} in the complex α plane. In addition, this prevents the existence of a measure including normalizable states only. This difficulty is overcome by introducing a family of deformations that are rational functions of n, each of them admitting a measure. By increasing the degree of these rational approximations, the deformation of a trapped ion can be approximated with any degree of accuracy and the formalism of the P representation can be applied.

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I. INTRODUCTION

The theory of certain one-parameter (q) deformations of Lie algebras, the so-called quantum groups, has been of great interest in the last decade in several areas of physics. In 1989 Biedenharn [1] and McFarlane [2] independently defined the q-analogue coherent state of a deformed q oscillator, for which Gray and Nelson [3] were able to obtain the resolution of unity. Since then the properties of a class of deformations of the harmonic oscillator were considered by several authors (see e.g., [4]). Using classical version of the q-deformed oscillator it was found [5] that the oscillator can be considered as conventional nonlinear oscillator with amplitude-dependent frequency. Deformed quantum oscillators are represented by dynamical variables A, $A^\dagger$, and N_A satisfying the commutation relations $[A,N_A]=A$, $[A^\dagger,N_A]=-A^\dagger$, and $[A,A^\dagger]=f(N_A)$, with $f(N_A)$ an arbitrary real function of N_A. All such variables are constructed in terms of single-mode field operators[1] a, $a^\dagger$, and $a^\dagger a$. Deformed oscillators were necessary [6] to study the Wigner problem of commutation-relations ambiguity in quantum mechanics.

In 1993 Črnugelj *et al.* [7] observed that the multiphoton interaction of a single-mode laser field with a two-level atom is described by deformed-oscillator creation and annihilation operators that in combination with the pseudospin atomic operators σ_+ and σ_- form the potential

$$W_{JC}=A^\dagger\sigma_-+A\sigma_+$$

used in the Jaynes-Cummings model (JCM) [8].

In that period, the JCM was at center of the attention for the study of laser cooling of ions placed in parabolic traps, with the quantized center-of-mass motion of the ion playing the role of the boson mode, coupled via the laser to the internal degrees of freedom. In the case of cooling, the operator A is represented by a combination of some power of the annihilation operator a times a function of n. When some bosons of the oscillator mode are destroyed, the ion is excited to the upper level from where it decays radiatively. Cooling was investigated in Lamb-Dicke [9] and strong-sideband [10] limits, that is, for ion excursions small compared with the radiation wavelength. This study led to the discovery of many intriguing effects connected with the nonclassical properties of the field, such as a long-time sensitivity to the statistical properties of the radiation field [11]. For example, the mean excitation number of the quantized oscillations of a ion driven by a squeezed field exhibited periodic collapses and revivals [12].

The interest for the vibrational motion of trapped ions was also motivated by the connection between the state of motion and the properties of the fluorescence spectra [13,14]. This link led some experimentalists to look for new nonclassical radiation states generated by trapped ions forced into some unusual vibrational states. In analogy to the preparation of nonclassical states of light in quantum optics, several authors examined the preparation of the center-of-mass motion in a quantum state having no classical counterpart. Worthy examples were those of Cirac *et al.* [15] who considered the possibility of generating squeezed states of the vibrational motion by irradiating the trapped ion with two standing-wave light fields of different frequencies and locating the center of the trap potential at a common node of both waves. In all these cases, the nonlinear dependence of A on a, $a^\dagger$, and $\hat{n}$ stemmed from the ion motion in the trap potential.

de Matos Filho and Vogel [16] observed in 1994 that the

[1]Wherever possible operators will be indicated by simple letters, except for the addition of a caret when confusion could arise with c-number quantities.

center-of-mass state of a trapped ion driven by a two-mode laser field decays toward a dark state coincident with a nonlinear coherent state (NCS) of a deformed oscillator. This result brought new fuel to the study of deformed oscillators describing different classes of states arising in the trapped ion motion under the action of two or three fields detuned by multiples of the vibrational frequency (see, e.g., [17] for nonlinear cat states that generalize the even and odd cat states of [18]). In the wake of this interest attention was paid to theoretical models of deformed oscillators, like those connected with excited coherent states and binomial states [19]. All these nonlinear oscillators differ for the deformation function $h(\hat{n})$ connecting the annihilation operator a to the deformed one $A=ah(\hat{n})$. The ancestor of these realizations were the q oscillators characterized by a deformation $h_q(\hat{n}) =\sqrt{\sinh(\lambda\hat{n})/\hat{n}\sinh\lambda}$ increasing with n. Contrarily, the trapped ion deformation is a very irregular function of n, taking positive and negative values. What is worse, for some combinations of the Lamb-Dicke parameter η^2 and n it can vanish or become infinite. As a consequence, it is hard to capitalize on the work done for the q oscillator for studying the NCS of a trapped ion. In particular, while for the q case it has been found a measure resolving the unity, the same is not exactly true for the ion case. As a consequence, the formalism of the Bargmann spaces [20], that has been extended from the linear oscillators to the q ones, cannot be applied exactly to the ion case. In fact, it will be shown in the following that this can be done by considering a class of rational deformations that approximate to any degree of accuracy the ion deformation. In most experimental cases, the statistical state of a trapped ion is limited to a finite number of Fock states so that these rational deformations may adequately approximate the ion deformation. Only in this ''weak'' sense is it possible to construct an ion analogue of a Bargmann space, on which the deformed creation and annihilation operators are represented as multiplication by z and differentiation with respect to z, respectively.

This paper is dedicated to an extension of the theory of the usual coherent states to NCS using as examples the deformation relative to the dark states of trapped ions. We start with a single-mode excitation field A (Sec. II), by discussing some properties of NCS, and introducing a deformed version $D_h(\alpha)$ of the displacement operator (Sec. III). In Sec. IV, we discuss some aspects of the resolution of unity for these NCS. The operator $D_h(\alpha)$ is used in Sec. V for associating the density matrix operator $\hat{\rho}$ to a linear functional $\rho_{A,h}(z)$ mapping the test function $\exp(\alpha z^*-\alpha^* z)$ into the expectation value $\langle D_h(\alpha)\rangle$, by extending the construction of the antinormal probability distribution function [21]. The connection with the P representation is also briefly examined. Section VI is dedicated to NCS on a circle, for which the Wigner functions are presented. Finally, the last section is dedicated to the dark states, arising when a trapped ion is driven by a bichromatic laser field. An asymptotic expression of the deformation and the relative factorial is obtained and its implication on the convergence of the NCS series is discussed. It comes out that it converges only for α in a circle of radius equal to the inverse of η. On the other hand, the weight of each Fock state can take values so large to prevent the resolution of unity in terms of normalized NCS. Some approximate expressions of the deformation are discussed together with the possibility of using these NCS for representing the ion statistical state.

II. MOTION OF A TRAPPED AND LASER-DRIVEN ION

We consider an ideal two-level ion of mass M constrained to move in a three-dimensional harmonic potential. Taking the principal trap (x axis) axis to coincide with the direction of propagation of the driving field, one quantum number suffices to label the vibrational states of the trap. The other two are traced out by summing over the corresponding degrees of freedom.

The ion's internal and external degrees of freedom are coupled together by a light field $\mathcal{E}e^{i\omega_L t+i\varphi(t)}$ periodically modulated at the frequency ν of the ion trap

$$E(x,t)=\mathcal{E}e^{i\omega_L t+i\varphi(t)}g(t)f(x)+\text{H.c.},$$

where $g(t)=g(t+2\pi/\nu)$ is a generally complex periodic function of frequency ν and H.c. represents for the Hermitian conjugate. The function $f(x)$ stands for $e^{-ik_L x}$ or $\sin(k_L x+\phi)$, respectively, for a progressive or standing wave, with the phase ϕ determining the position of the trap potential with respect to the standing wave.

We will dwell on monochromatic [at frequency $\omega_L-(N+1)\nu$]

$$g(t)=g^{(1)}_{N+1}(t)=e^{-i(N+1)\nu t}$$

and bichromatic driving fields at frequencies $\omega_L-(N+1)\nu$ and ω_L, respectively,

$$g(t)=g^{(2)}_{N+1}(t)=e^{-i(N+1)\nu t}-\alpha_{N+1},$$

with the parameter N taking non-negative integer values, and α_{N+1} a complex coefficient depending on the amplitudes of the two waves.

Now, introducing the Lamb-Dicke parameter $\eta=\hbar k_L/\sqrt{2M\hbar\nu}$ we put as usual $e^{-ikx}=e^{-i\eta(a_v^\dagger+a_v)}$. In the classical limit, η is large and the absorption or emission of a photon will always cause some change in the vibrational state of the atom. In the nonclassical Lamb-Dicke limit of small η, many photons may need to be absorbed or emitted before the atom changes vibrational state. For example, in the sideband cooling experiment carried out by Diedrich *et al.* [22] the parameter η was equal to 0.06.

The Hamiltonian for a trapped ion interacting with a bichromatic field can be split in two parts

$$H=H_0+H_{int},$$

where ($\hbar=1$)

$$H_0=\omega_{12}\sigma_3+\nu\hat{n}, \tag{1}$$

and, in the electric dipole approximation,

$$H_{int}=\wp[\sigma_-E^*(x,t)+\sigma_+E(x,t)].$$

When the Rabi frequency Ω, relative to the laser induced transition between the ion ground and excited levels, is much smaller than the trapping potential frequency ν, a perturbation expansion can be carried out in Ω/ν, as discussed in Ref. [11]. This expansion allows a division into quickly and slowly varying density operator matrix elements, the former of which can be adiabatically eliminated.

Arresting the calculation to the zeroth order in Ω/ν amounts to applying the rotating wave approximation. This approach can be easily pursued by switching to the interaction picture defined by the unitary operator $U_{rw} = \exp[-i(\omega_L\sigma_3+\nu\hat{n})t]$ and retaining in the transformed Hamiltonian H' the time-independent terms together with the slowly varying phase $\varphi(t)$ of the laser field,

$$H' = [\Delta - \dot{\varphi}(t)]\sigma_3 + \Omega(\sigma_- A + \sigma_+ A^\dagger) \tag{2}$$

with $\Delta = \omega_{12} - \omega_L$ the detuning parameter, $\Omega = e^{-\eta^2/2}\wp\mathcal{E}$ the vibronic Rabi frequency and

$$A = e^{-\eta^2/2}\overline{g(t) f[\eta(e^{-i\nu t}a^\dagger + e^{i\nu t}a)]}, \tag{3}$$

the overbar indicating the time average.

Expanding the factor $e^{-i\eta(e^{-i\nu t}a^\dagger + e^{i\nu t}a)}$ in power series in a and $a^\dagger$, introducing the operator

$$f_k(\hat{n},\eta^2) = \sum_{m=0}^{\infty} \frac{(\hat{n}-m+1)_m}{(k+1)_m m!}(-\eta^2)^m = k!\frac{L_{\hat{n}}^k(\eta^2)}{(\hat{n}+1)_k}, \tag{4}$$

with $(\hat{n}-m+1)_m = (a^\dagger)^m a^m = \hat{n}(\hat{n}-1)\cdots(\hat{n}-m+1)$ and $L_{\hat{n}}^k(\eta^2)$ reducing in the Fock basis to the generalized Laguerre polynomials, we obtain, respectively, for progressive

$$e^{-i\eta(e^{-i\nu t}a^\dagger + e^{i\nu t}a)} = e^{-\eta^2/2}\sum_{k=0}^{\infty} \epsilon_k \frac{(-i\eta)^k}{k!} \times [f_k(\hat{n})a^k e^{ik\nu t} + (a^\dagger)^k f_k(\hat{n}) e^{-ik\nu t}] \tag{5}$$

and standing waves

$$\sin[\eta(a^\dagger + a) + \phi] = e^{-\eta^2/2}\sum_{k=0}^{\infty} \epsilon_k \frac{(-\eta)^k}{k!}\sin\left(\phi + k\frac{\pi}{2}\right) \times [a^k f_k(\hat{n}+k)e^{ik\nu t} + f_k(\hat{n}+k)(a^\dagger)^k e^{-ik\nu t}], \tag{6}$$

with k a positive integer and $\epsilon_k = \frac{1}{2}$ for $k=0$ and $\epsilon_k = 1$ otherwise.

For progressive (p) and stationary (s) monochromatic waves with $g(t) = e^{-i(N+1)\nu t}$ the operator A [see Eq. (3)] is given by

$$A_p^{(1)} = \frac{(-i\eta)^{N+1}}{(N+1)!} f_{N+1}(\hat{n},\eta^2) a^{N+1},$$

$$A_s^{(1)} = (-i)^{N+1}\sin\left(\phi + (N+1)\frac{\pi}{2}\right) A_p^{(1)}, \tag{7}$$

while for two modes bichromatic driving fields

$$A_p^{(2)} = f_{N+1}(\hat{n})a^{N+1} - \alpha_{N+1} f_0(\hat{n}),$$

$$A_s^{(2)} = (-i)^{N+1}\sin\left(\phi + (N+1)\frac{\pi}{2}\right) f_{N+1}(\hat{n}) a^{N+1} + \sin(\phi)\alpha_{N+1} f_0(\hat{n}). \tag{8}$$

III. NONLINEAR COHERENT STATES

Coherent states were originally introduced as eigenstates of the annihilation operator for the harmonic oscillator [23]. They have been generalized (see [3,4,16,19]) by labeling as nonlinear coherent states $|\alpha,h\rangle$ the right-hand eigenstates

$$A|\alpha,h\rangle = \alpha|\alpha,h\rangle \tag{9}$$

of operators[2] A of the form

$$A = a h(\hat{n}) \tag{10}$$

where $h(\hat{n})$ is an operator-valued real function of the number operator. It is immediate to show that

$$|\alpha,h\rangle = N_{h,\alpha}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}[h(n)]!}|n\rangle \tag{11}$$

with $[h(n)]! = h(0)h(1)\cdots h(n)$ and normalizing factor $N_{h,\alpha}$

$$N_{\alpha,h} = \frac{1}{\sqrt{E_h(|\alpha|^2)}}$$

expressed in terms of the entire function

$$E_h(v) = \sum_{n=0}^{\infty} \frac{v^n}{n!([h(n)]!)^2}$$

referred to in the following as h exponential in analogy with the q exponential used in Ref. [3].

The deformation functions $h(n,\eta^2)$ associated with the dark states of the trapped ions are represented by the ratio of two Laguerre polynomials of argument equal to the Lamb-Dicke parameter η^2 so that they vanish or become infinite for some isolated combinations of η^2 and n. We are obliged to explicitly assume that this situation does not occur for the values of η^2 considered.

In Sec. VII we will obtain an asymptotic expression of the weights of the Fock states occurring in the series expansion of the NCS relative to trapped ions. They take very large and

[2]For the sake of notational simplicity we will use the same symbol A for indicating fields of the form (7) and (8).

very small values for increasing n, so that these NCS can be normalized only for α inside the circle $1/\eta$. For convenience of discussion we shall ignore this problem by restricting our treatment here to normalized NCS states.

It may be worth noting at this point that many of the foregoing formulas may be abbreviated by adopting a normalization different from the conventional one for the coherent state. Accordingly, we introduce the symbol $\|\alpha;h\rangle$ for the states normalized in the new way,

$$\|\alpha;h\rangle = N_{\alpha,h}^{-1}|\alpha,h\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n![h(n)]!}}|n\rangle$$

and

$$\langle\alpha;h\|\beta;h\rangle = E_h(\alpha^*\beta).$$

Since the commutator $[A^\dagger,A] = \hat{n}h^2(\hat{n}) - (\hat{n}+1)h^2(\hat{n}+1)$ is not a c number it is worth introducing the operator [19]

$$A_h^\dagger = \frac{1}{h(\hat{n})}a^\dagger. \tag{12}$$

With these alterations, we have

$$A\|\alpha;h\rangle = \alpha\|\alpha;h\rangle,$$

$$\hat{n}\|\alpha;h\rangle = \alpha\frac{\partial}{\partial\alpha}\|\alpha;h\rangle,$$

$$A_h^\dagger\|\alpha;h\rangle = \frac{\partial}{\partial\alpha}\|\alpha;h\rangle.$$

In addition,

$$A^\dagger\|\alpha,h\rangle = h^2(\hat{n})A_h^\dagger\|\alpha,h\rangle = h^2\left(\alpha\frac{\partial}{\partial\alpha}\right)\frac{\partial}{\partial\alpha}\|\alpha,h\rangle. \tag{13}$$

In all the above right-hand side, the operators α,∂_α and their combination are intended to act on the coefficients of the Fock states series.

It should be noticed that A and $A_h^\dagger$ provide a (new) different realization of Heisenberg-Weyl algebra, with $A_h^\dagger$ playing the role of the adjoint with respect to a new Hermitian product.

A. Displacement and deformation operators

It is well known that coherent state $|\alpha\rangle$ can be also introduced by displacing the Fock vacuum state $|0\rangle$ by means of the operator

$$\mathcal{D}(\alpha) = \exp(-\alpha^* a + \alpha a^\dagger) \tag{14}$$

due to its property of displacing the annihilation operator a by the generally complex quantity α,

$$\mathcal{D}(\alpha)a\mathcal{D}(-\alpha) = a - \alpha.$$

Unfortunately, $\mathcal{D}(\alpha)$ is unable to displace the deformed operator A. Alternately, $\mathcal{D}(\alpha)$ could be replaced by the unitary operator obtained by replacing in Eq. (14) a and $a^\dagger$ by A and $A^\dagger$, respectively, but also this operator does not displace A by the complex quantity α. The difficulties in dealing with exponentials of linear combinations of A and $A^\dagger$ originate from the circumstance that their commutator is not a c number. These problems can be overcome by using $A_h^\dagger$ [see Eq. (12)] in place of $A^\dagger$ and defining the ''deformed'' version of the displacement operator as

$$\mathcal{D}_h(\alpha) = \exp(-\alpha^* A + \alpha A_h^\dagger) = e^{|\alpha|^2/2}e^{-\alpha^* A}e^{\alpha A_h^\dagger} = e^{-|\alpha|^2/2}e^{\alpha A_h^\dagger}e^{-\alpha^* A}. \tag{15}$$

$\mathcal{D}_h(\alpha)$ shares many properties of the standard operator $\mathcal{D}(\alpha)$ as

$$\mathcal{D}_h^{-1}(\alpha) = \mathcal{D}_h(-\alpha)$$

and

$$\mathcal{D}_h(\beta)\mathcal{D}_h(\alpha) = \exp\left[\frac{1}{2}(\beta\alpha^* - \beta^*\alpha)\right]\mathcal{D}_h(\beta+\alpha).$$

However, $\mathcal{D}_h(\alpha)$ is not a unitary operator,

$$\mathcal{D}_h^\dagger(\alpha) = \mathcal{D}_{1/h}^{-1}(\alpha) = \mathcal{D}_{1/h}(-\alpha)$$

so that it does not preserve the norm of a state.

$\mathcal{D}_h(\alpha)$ and $\mathcal{D}_{1/h}(\alpha)$ displace A and $A^\dagger$, respectively, by α and α^*,

$$\mathcal{D}_h(\alpha)A\mathcal{D}_h(-\alpha) = A - \alpha,$$

$$\mathcal{D}_{1/h}(\alpha)A^\dagger\mathcal{D}_{1/h}(-\alpha) = A^\dagger - \alpha^*, \tag{16}$$

and $A_h^\dagger$ by α^*

$$\mathcal{D}_h(\alpha)A_h^\dagger\mathcal{D}_h(-\alpha) = A_h^\dagger - \alpha^*. \tag{17}$$

Accordingly, the NCS $|\alpha;h\rangle$ can be obtained by applying $\mathcal{D}_h(\alpha)$ to the vacuum state,

$$\|\alpha;h\rangle = e^{\alpha A_h^\dagger}|0\rangle = e^{|\alpha|^2/2}\mathcal{D}_h(\alpha)|0\rangle.$$

In conclusion, the NCS $\|\alpha;h\rangle$ can be obtained by deforming the usual coherent state $\|\alpha\rangle$ by means of the deformation operator

$$d_h = \mathcal{D}_h(\alpha)\mathcal{D}(-\alpha), \tag{18}$$

namely,

$$\|\alpha;h\rangle = d_h\|\alpha\rangle.$$

Although expressed as a product of operators depending on the complex parameter α, d_h is independent of α. In a Fock basis, it is diagonal with components equal to $[h(n)]!^{-1}$. Since $h(n)$ does not vanish, as already assumed, d_h is not singular.

Finally, we note that

$$\langle m|\mathcal{D}_h(\alpha)|n\rangle = \frac{[h(m)]!}{[h(n)]!}\langle m|\mathcal{D}(\alpha)|n\rangle,$$

so that the matrix representations of $\mathcal{D}$ and $\mathcal{D}_h$ have the same diagonal part.

A further remark is that the set of operators $\mathcal{D}_h(\alpha)$ constitutes a Weyl system that does not lead to the canonical quantization for not being unitary.

B. Nonlinear displaced Fock states

In Sec. V, we will use the Fock states displaced by $\mathcal{D}_h(\alpha)$ [see Eq. (27)]

$$|\varphi_m,\alpha,h\rangle = \mathcal{D}_h(\alpha)|m\rangle, \tag{19}$$

which can be shown with the help of Eqs. (16) and (17) to be the right eigenstates of the operator $(A_h^\dagger-\alpha^*)(A-\alpha) = \mathcal{D}_h(\alpha)\hat{n}\mathcal{D}_h(-\alpha)$,

$$(A_h^\dagger-\alpha^*)(A-\alpha)|\varphi_m,\alpha,h\rangle = m|\varphi_m,\alpha,h\rangle.$$

Analogously, we can introduce the left eigenstates defined by

$$\langle\psi_m,\alpha,h|(A_h^\dagger-\alpha^*)(A-\alpha) = m\langle\psi_m,\alpha,h|,$$

which are obtained by displacing $\langle m|$ by $\mathcal{D}_h(-\alpha)$, i.e., $\langle\psi_m,\alpha,h| = \langle m|\mathcal{D}_h(-\alpha)$.

It is noteworthy that the left and right displaced Fock states are mutually orthogonal,

$$\langle\psi_m,\alpha,h|\varphi_n,\alpha,h\rangle = 0$$

for $m\neq n$.

On the other hand, these states can be also expressed in the form

$$|\varphi_m,\alpha,h\rangle = \frac{[h(m)]!(A_h^\dagger-\alpha^*)^m}{\sqrt{m!}}|\alpha,h\rangle = \frac{[h(m)]!}{\sqrt{m!}}\sum_n \binom{m}{n}(-\alpha^*)^{m-n}|\alpha,h,m\rangle,$$

$$\langle\psi_m,\alpha,h| = \langle m|D_h(-\alpha) = \langle\alpha,h|\frac{(A-\alpha)^m}{[h(m)]!\sqrt{m!}} = \frac{1}{[h(m)]!\sqrt{m!}}\sum_n \binom{m}{n}(-\alpha)^{m-n}\langle\alpha,h,m|,$$

where $|\alpha,h,m\rangle = A_h^{\dagger m}|\alpha,h\rangle$ and $\langle\alpha,h,m| = \langle\alpha,h|A^m$ stand for the deformed versions of the excited coherent states [23] (see also [24]).

IV. RESOLUTION OF UNITY

From the completeness relation of coherent states

$$1 = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha|d^2\alpha,$$

it follows that

$$1 = \frac{1}{\pi}\int d_h|\alpha\rangle\langle\alpha|d_h^{-1}d^2\alpha = \frac{1}{\pi}\int d_h^{-1}|\alpha\rangle\langle\alpha|d_h d^2\alpha.$$

Next, using the relation

$$d_h^{-1} = \mathcal{D}(\alpha)\mathcal{D}_{1/h}^\dagger(\alpha) = (\mathcal{D}_{1/h}(\alpha)\mathcal{D}(-\alpha))^\dagger = d_{1/h}^\dagger,$$

the above resolution of unity can be expressed in terms of deformed coherent states

$$1 = \frac{1}{\pi}\int \frac{e^{-\alpha\alpha^*}}{N_{\alpha,h}N_{\alpha,1/h}}|\alpha,h\rangle\langle\alpha,h^{-1}|d^2\alpha = 1/\pi\int \frac{e^{-\alpha\alpha^*}}{N_{\alpha,h}N_{\alpha,1/h}}|\alpha,h^{-1}\rangle\langle\alpha,h|d^2\alpha. \tag{20}$$

It goes without saying, that this resolution holds true only if the NCS relative to the deformations h and $1/h$ are both normalizable in the whole complex α plane.

For some deformations, anyhow it is possible to obtain a resolution of unity in terms of projectors of deformed coherent states, i.e., to find a suitable element of measure $d\mu$ such that

$$1 = \int \|\alpha,h\rangle\langle\alpha,h\| d\mu, \tag{21}$$

$d\mu$ can be considered as an extension of the measure element $d\mu = (1/\pi)e^{-|\alpha|^2}d^2\alpha$ [20] for linear oscillators. Since $\int\langle m\|\alpha,h\rangle\langle\alpha,h\|n\rangle d\mu$ must vanish for $m\neq n$, $d\mu$ can be put in the form

$$d\mu = \frac{1}{\pi}m_h(|\alpha|^2)d^2\alpha,$$

where $m_h(x)$ is a distribution satisfying the set of equations

$$n!([h(n)]!)^2 = \int m_h(x)x^n dx \tag{22}$$

for every integer n.

Treating $n = s-1$ as a continuous variable, the above relation represents a Mellin integral transform,

$$g(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{23}$$

so that $m_h(x)$ is the Mellin antitransform of $g(s) = \Gamma(s)\{[h(s-1)]!\}^2$.

From the relation $\langle\beta,h|A^m|\beta,h\rangle = \beta^m$ it follows that $E_h(\beta^*\alpha)$ is the self-reproducing kernel of the h analogue of the Bargmann space [20], with respect to $d\mu$,

$$\int |E_h(\beta^*\alpha)|^2\alpha^m d\mu=\beta^m,$$

In preparation of the discussion in Sec.VII, it is worth remarking that replacing h by the deformation βh the relative measure $m_{\beta h}(x)$ is given by

$$m_{\beta h}(x)=\beta^{-2}m_h(\beta^{-2}x). \tag{24}$$

This relation can be also used for expressing a thermal density matrix characterized by Boltzmann weight factors $\rho_{nn}\propto\exp(-\beta n)$ in the form

$$\hat{\rho}=e^{2\beta}\int\frac{m_h(e^{2\beta}|\alpha|^2)}{m_h(|\alpha|^2)}\|\alpha,h\rangle\langle\alpha,h\|d\mu.$$

In Ref. [3] it was possible to obtain the resolution of unity for a q oscillator by deforming both the derivative and the integral operators while a resolution for the so-called harmonious states was obtained in [25]. We will see in the following that for the trapped ion deformation the measure is a distributional Laplace antitransform that includes non-normalizable NCS.

For a deformation approximated by a rational function of n, $g(s)$ corresponds to the ratio of products of gamma functions,

$$g(s)=\frac{\Gamma(a_1+s)\cdots\Gamma(a_A+s)}{\Gamma(b_1+s)\cdots\Gamma(b_B+s)}\equiv\Gamma\begin{pmatrix}(a)+s\\(b)+s\end{pmatrix}. \tag{25}$$

For $A\geqslant B$ the relative antitransform is given by a combination of generalized hypergeometric functions

$${}_CF_D\left(\begin{matrix}(c)\\(d)\end{matrix};(-1)^{C+D+1}x\right)=\sum_n\frac{(c_1)_n\cdots(c_C)_n}{(d_1)_n\cdots(d_D)_n}\times\frac{x^n}{n!}(-1)^{n(C+D+1)},$$

namely [26],

$$m_h(x)=\sum_{\mu=1}^{A}\Gamma\begin{pmatrix}(a)'-a_\mu\\(b)-a_\mu\end{pmatrix}\times{}_BF_{A-1}\left(\begin{matrix}1+a_\mu-(b)\\1+a_\mu-(a)'\end{matrix};(-1)^{A+B}x\right)x^{a_\mu}, \tag{26}$$

where $(a)'-a_\mu$ and $(a)'-a_\mu-1$ stand for the sequences $a_1-a_\mu,\ldots,a_A-a_\mu$ and $a_1-a_\mu-1,\ldots,a_A-a_\mu-1$, with the exclusions of the μth term.

V. EXPANSION OF STATISTICAL STATES

The same reasons that led Sudarshan [27] to introduce diagonal coherent states representation to express arbitrary states and operators in terms of coherent states, which in a special case provides the Glauber P representation [28], suggest that we develop expansions in terms of NCS as well.

Following [21] we introduce for a statistical state the deformed quantum linear functional

$$F_h[\alpha]=\mathrm{Tr}\{\hat{\rho}\mathcal{D}_h(\alpha)\}=\sum_m\rho_{mn}\langle n|.\varphi_m,\alpha,h\rangle. \tag{27}$$

In particular, for a diagonal density matrix $F_h[\alpha]$ reduces to the standard $F[\alpha]$.

Using the unity resolution (20) $F_h[\alpha]$ may be rewritten as

$$F_h[\alpha]=e^{\alpha\alpha^*/2}\int\exp(\alpha z^*-\alpha^* z)\rho_{h,A}(z)d^2z, \tag{28}$$

where

$$\rho_{h,A}(z)=\frac{1}{\pi}\frac{e^{-zz^*}}{N_{z,h}N_{z,h^{-1}}}\mathrm{Tr}\{\hat{\rho}|z,h\rangle\langle z,h^{-1}|\}=\frac{1}{\pi}\langle z|\hat{\rho}_h|z\rangle, \tag{29}$$

with

$$\hat{\rho}_h=d_h^{-1}\hat{\rho}d_h$$

the deformed density operator. In other words, $\rho_{h,A}(z)$ stands for the generalized distribution function of the deformed density matrix $d_h^{-1}\hat{\rho}d_h$.

For extending the definition of the characteristic functional $F[\alpha]=\mathrm{Tr}\{\hat{\rho}D(\alpha)\}$ to a deformed oscillator, we pay the penalty of loosing some properties of $\rho_A(z)$. In fact, $\rho_{h,A}(z)$ may take, in general, positive and negative values. It can be regarded as a generalized probability distribution function as long as the association between operators and functions is based on antinormal ordering,

$$\mathrm{Tr}\{\hat{\rho}G_A(A,A_h^\dagger)\}=\int\rho_{h,A}(z)G_A(z,z^*)d^2z, \tag{30}$$

In particular, for a diagonal density matrix $\rho_{h,A}(z)=\rho_A(z)$ while for $\hat{\rho}=|w,h\rangle\langle w,h|$,

$$\rho_{h,A}(z)=\frac{1}{\pi}\frac{E_h(w^*z)}{E_h(w^*w)}\exp[z^*(w-z)].$$

Consequently, transformation (28) applies if there exists the Fourier transform of $\exp[z^*(w-z)]E_h(w^*z)$. Analogously, working with $\hat{\rho}=|w,h^{-1}\rangle\langle w,h^{-1}|$ we arrive at the same conclusion for $\exp[z^*(w-z)]E_{h^{-1}}(w^*z)$. This, in turn, implies that $E_{h^{-1}}(w^*z)$ and $E_h(w^*z)$ cannot grow at infinity as quickly as $\exp(zz^*)$.

We recall that in the coherent state representation of a bounded operator $\hat{O}$, the vanishing of $\langle z|\hat{O}|z\rangle=0$ in a domain of the complex plane of finite area implies the vanishing of $\hat{O}$ itself (see Ref. [21]). Since d_h has been assumed nonsingular, the same theorem holds true for $\langle z|\hat{O}_h|z\rangle$, so that two deformed density matrices having the same function $\rho_{h,A}(z)$ over some area of z , must coincide. In conclusion, Eq. (29) establishes a one-to-one correspondence between the operator $\hat{\rho}$ and the function $\rho_{h,A}(z)$.

When the deformation admits the unity resolution (21), a density matrix can be represented in several cases by a P representation,

$$\hat{\rho}=\int P_h(\alpha)\|\alpha,h\rangle\langle\alpha,h\|d\mu, \tag{31}$$

in which $P_h(\alpha)$ can be regarded as a generalized probability distribution function as long as the association between operators and functions is based on normal ordering,

$$\mathrm{Tr}\{\hat{\rho}G_N(A^\dagger,A)\}=\int P_h(\alpha)G_N(\alpha,\alpha^*)d\mu. \tag{32}$$

When $\hat{\rho}$ is represented in the form of Eq. (31) the master equation of $\hat{\rho}$ can be in many cases transformed in a master equation for P_h. This circumstance becomes particularly valuable in the study of the decay of an excited trapped ion toward the fundamental dark state. In this case, we are faced, for example, with operators of the form

$$A^\dagger\hat{\rho}=\int P(\alpha)m_h(\alpha\alpha^\star)h^2\left(\alpha\frac{\partial}{\partial\alpha}\right)\frac{\partial}{\partial\alpha}\|\alpha\rangle\langle\alpha\|d^2\alpha$$
$$=-\int\|\alpha\rangle\langle\alpha\|\frac{\partial}{\partial\alpha}h^2\left(-1-\alpha\frac{\partial}{\partial\alpha}\right)$$
$$\times\{P(\alpha)m_h(\alpha\alpha^*)\}d^2\alpha,$$

use having been made of Eq. (13) Expanding $P(\alpha)m_h(\alpha\alpha^*)$ in power series of $\alpha^m(\alpha^*)^n$ we see that

$$\frac{\partial}{\partial\alpha}h^2\left(-1-\alpha\frac{\partial}{\partial\alpha}\right)\alpha^m(\alpha^*)^n=mh^2(-1-m)\alpha^{m-1}(\alpha^*)^n.$$

VI. NONLINEAR COHERENT STATES ON A CIRCLE

The above definition of NCS states (we will call them of order 1) can be extended to the eigenstates of the operators A_{N+1} of a more general form $A_{N+1}=a^{N+1}h(\hat{n})$ [29] and so the equation

$$A_{N+1}|\alpha,h,N+1,q\rangle=\alpha|\alpha,h,N+1,q\rangle$$

with $N>0$ is considered.

The eigenstate belonging to the eigenvalue α is $N+1$-fold degenerate and q is an integer ranging from 0 to N. In terms of Fock states, we have

$$|\alpha,h,N+1,q\rangle=N_{\alpha,h,q}\times\sum_{l=0}^{\infty}\frac{\alpha^{l(N+1)+q}}{\sqrt{(l(N+1)+q)!}[h(l(N+1)+q)]!}\times|l(N+1)+q\rangle, \tag{33}$$

with the normalization factor

$$|N_{\alpha,h,q}|^{-2}=\sum_{l=0}^{\infty}\frac{|\alpha|^{2l(N+1)+2q}}{[l(N+1)+q]!([h\{l(N+1)+q\}]!)^2},$$

where

$$[h\{l(N+1)+q\}]!=h(q)h(N+1+q)\cdots h\{l(N+1)+q\}.$$

Such a state can also be expressed as a sum of NCS [see Eq. (11)]. In fact, by introducing the function $h^{(N+1)}(n)$ defined recursively by

$$h^{(N+1)}\{l(N+1)+q\}=\frac{h(q-1)[h\{l(N+1)+q\}]!}{h(q)[h\{l(N+1)+q-1\}]!}h^{(N+1)}(q),$$

we have also

$$|\alpha,h,N+1,q\rangle=N'_{\alpha,h,q}\sum_{k=0}^{N}(\epsilon^*)^{qk}|\alpha\epsilon^k,h^{(N+1)}\rangle, \tag{34}$$

with $\epsilon=\exp[i2\pi/(N+1)]$ and $N'_{\alpha,h,q}$ a normalization coefficient.

We have obtained that a NCS coherent state of order $N+1$ is decomposed in the sum of $N+1$ first-order NCS of complex amplitudes α, $\alpha\epsilon^q,\ldots,\alpha(\epsilon^q)^N$ distributed uniformly on a circle. These states, referred to as "crystallized cats" in Ref. [30], were introduced for the linear oscillator [31] in the attempt to generalize the optical Schrödinger cats of harmonic oscillators.

Using the deformed displacement operator we have also

$$|\alpha,h,N+1,q\rangle=N'_{\alpha,h,q}\left(\sum_{k=0}^{N}(\epsilon^*)^{qk}\mathcal{D}_{h^{(N+1)}}(\alpha\epsilon^k)\right)|0\rangle.$$

In conclusion, the Hilbert space is the direct sum of $N+1$ spaces $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_N$ $(q=0,1,\ldots N)$, each of them having for basis the Fock states $|\ell(N+1)+q\rangle$, as $l\in(0,\ldots,\infty)$.

For $N=1$, the fundamental states of $\mathcal{H}_0$ and $\mathcal{H}_1$ are, respectively, the even and odd Schrödinger cats. It will be shown in a forthcoming paper that when the radiative damping is negligible, the initial density matrix separate in the product of two matrices evolving, respectively, toward the even and odd Schrödinger cats.

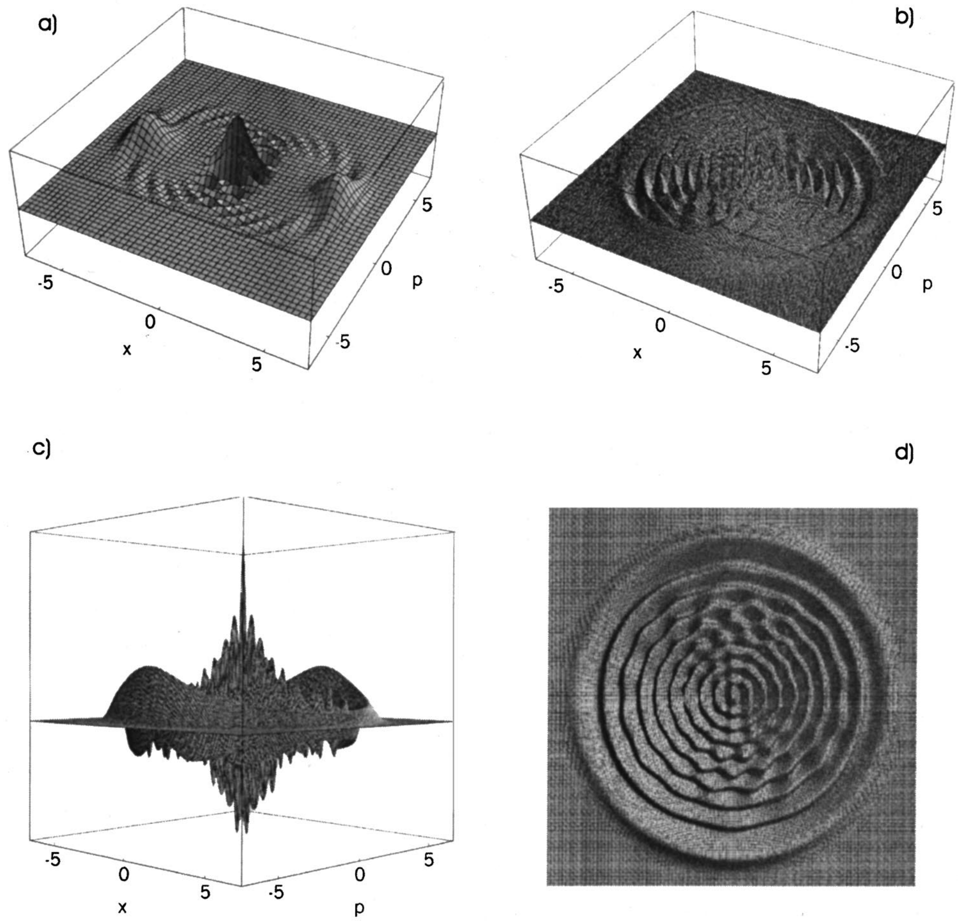

FIG. 1. Top and side views of the Wigner function [Eq. (35)] relative to linear (a) and nonlinear (b), (c), (d) even Schrödinger cat states for α=3.5 (real) and two different values of η. (a) $\eta=0$ (linear case), the two coherent peaks are almost circularly shaped, the interference pattern is axially symmetric along the axis defined by the center of the coherent peaks; [(b) top view, (c) side view, and (d) air view] $\eta=0.5$, the Gaussian peaks are no longer distinguishable from the circularly symmetric interference pattern.

A. Wigner function

The Wigner function [32] relative to these states on a circle can be shown to be given for a generic integer N by

$$W_{N+1}(\tilde{q},\tilde{p})=N^2_{\alpha,h_{N+1},q}e^{-|q+ip|^2}\times\sum_{ll'}[\sqrt{2}(\tilde{q}-i\tilde{p})]^{(l'-l)(N+1)}\times\frac{(-\alpha)^{l(N+1)+q}}{[h_{N+1}\{l(N+1)+q\}]!}\times\frac{(\alpha^*)^{l'(N+1)+q}}{[h_{N+1}\{l'(N+1)+q\}]!}\times\frac{L^{(l'-l)(N+1)}_{l(N+1)+q}(2(\tilde{q}^2+\tilde{p}^2))}{[l'(N+1)+q]!}. \tag{35}$$

Analogously, for the Husimi-Kano [32] Q function

$$Q_{N+1,q}(\tilde{q},\tilde{p})=\left|\left\langle\frac{\tilde{q}+i\tilde{p}}{\sqrt{2}}\right|\alpha,q,h_{N+1}\right\rangle\right|^2,$$

$$Q_{N+1,q}(\tilde{q},\tilde{p})=N^2_{\alpha,h_{N+1},q}e^{-(\tilde{q}^2+\tilde{p}^2)/2}\times\left|\sum_{l=0}^{\infty}\frac{\left(\alpha\frac{\tilde{q}-i\tilde{p}}{\sqrt{2}}\right)^{l(N+1)+q}}{[l(N+1)+q]![h_{N+1}\{l(N+1)+q\}]!}\right|^2, \tag{36}$$

with $|(\tilde{q}+i\tilde{p})/\sqrt{2}\rangle$ a coherent-state vector.

For $N=1$, these states reduce to even $(q=0)$ and odd $(q=1)$ Schrödinger cats [17]. In Ref. [33] the squeezing and antibunching effects are examined by using the function $h_1(n)$ introduced in [16] for the NCS. We will see in the following [see Eq. (38)] that the nonlinear cats representing the dark state of a trapped ion are properly described by the deformation $h_2(n;\eta^2)=L^2_{n-2}(\eta^2)/[n(n-1)L_{n-2}(\eta^2)]$.

In Fig. 1, we show the Wigner functions for nonlinear even Schrödinger cats of amplitude $\alpha=3.5$ (real) and different parameters η. In the linear case [$\eta=0$, Fig. 1(a)], the quantum interference is localized around the origin. The two coherent Gaussian peaks are circularly shaped.

For increasing η, the nonlinearity flattens the interference pattern while the central interference fringes, particularly their negative part, become more evident. This is essentially

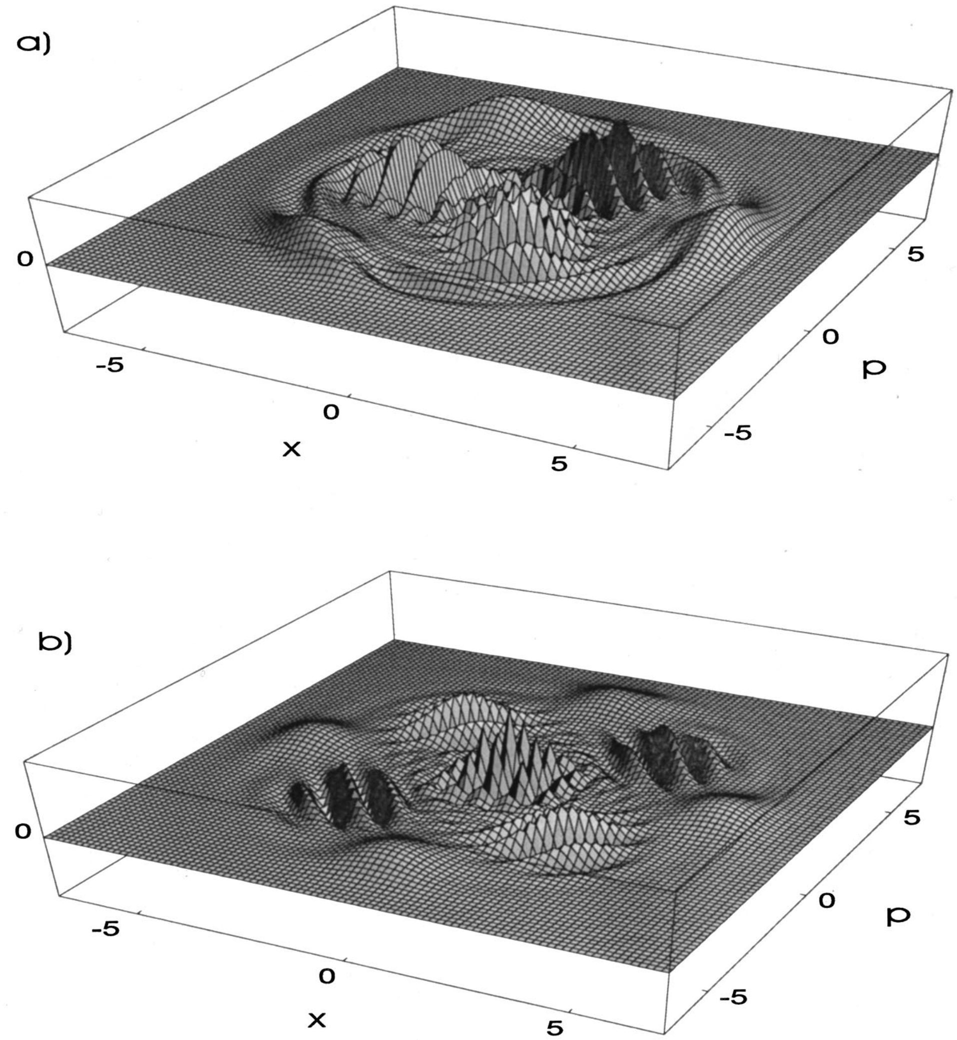

FIG. 2. Top view of the Wigner function [Eq. (35)] for three and four NCS states sitting on the circle. $\alpha=3.5$ (real) and $\eta=0.33$. This value of the nonlinearity η leads to a smooth reshaping of the coherent peaks from a nearly circular shape to an elliptic one.

due to a reshaping of the coherent contribution peak from a Gaussian-like nearly circular shape to an elliptical one with minor axis parallel to the direction connecting the two coherent peaks. These come closer to the origin and the region where the Wigner function is nonzero shrinks notably.

A further increase in η causes the progressive coming closer and closer of the main peaks, while the interference fringes become more localized. For higher η, there are some interference fringes spreading over the two coherent peaks. This phenomenon is dominant for very high η values [Figs. 1(b),1(c),1(d), $\eta=0.5$] where the main peaks come into the interference region and the coherent character of the two states forming the cat is no longer distinguishable. The interference area becomes larger than in the linear case and a circular symmetry of the interference pattern becomes evident.

In Fig. 2, we present two NCS on a circle formed by the superposition of three and four NCS ($\alpha=3.5$ and $\eta=0.33$).

VII. DARK STATES

In 1993 Cirac *et al.*, [15] proposed a scheme for preparing coherent squeezed states of motion in an ion trap based on the multichromatic excitation of a trapped ion. Using two waves with beat frequency equal to twice the trap frequency a ''dark resonance'' appears in the fluorescence emitted by the ion; the ion is placed in a squeezed state. Similar ''dark states'' produced by a bichromatic field with beat frequency equal to the trap frequency were studied by de Matos Filho and Vogel in 1996 [16] dentified as nonlinear coherent states.

We will consider in the following a beat frequency that is a generic multiple of the trap frequency ν, and the ion dark state is described by a generalized coherent state on a circle. We will consider a bichromatic field of the type

$$A_{b,N+1}=f_{N+1}(\hat{n})a^{N+1}-\alpha_{N+1}f_0(\hat{n}), \qquad (37)$$

for which the dark state satisfies the equation

$$\left(a^{N+1}\frac{f_{N+1}(\hat{n}-N-1)}{f_0(\hat{n}-N-1)}-\alpha_{N+1}\right)|\psi_{dark}\rangle=0,$$

that is [see Eq. (33)]

$$|\psi_{dark}\rangle=|\alpha_{N+1},h_{N+1},N+1,q\rangle,$$

with

$$h_{N+1}(\hat{n};\eta^2)=(N+1)!\frac{L_{\hat{n}-N-1}^{N+1}}{(\hat{n}-N)_{N+1}L_{\hat{n}-N-1}} \quad (38)$$

and

$$\alpha_{N+1}=\frac{\Omega_0}{\Omega_{N+1}}\frac{(N+1)!}{(-i\eta)^{N+1}}.$$

In short, the dark state is the superposition of $N+1$ nonlinear coherent states that are equidistantly separated from each other along a circle with modulation factor $\epsilon^k=\exp[2\pi ik/(N+1)]$. These states represent the change of the Schrödinger cat states under the influence of the strong nonlinearity of the trapped ion vibrations (see Fig. 1).

In particular,

$$h_1(\hat{n};\eta^2)=\frac{L_{\hat{n}-1}^1}{\hat{n}L_{\hat{n}-1}}=\frac{L_{\hat{n}-1}-L_{\hat{n}}}{\eta^2L_{\hat{n}-1}},$$

$$h_2(\hat{n};\eta^2)=\frac{2L_{\hat{n}-2}^2}{\hat{n}(\hat{n}-1)L_{\hat{n}-2}}, \quad (39)$$

and

$$\alpha_1=i\frac{\Omega_0}{\Omega_1},\quad \alpha_2=-\frac{\Omega_0}{\Omega_2}\frac{2}{\eta^2}.$$

Expressing the Laguerre polynomials by their asymptotic expression $\sqrt{n\eta^2}h_1(n;\eta^2)$ tends, for $n\rightarrow\infty$, to a function depending on the product $n\eta^2$ only,

$$\sqrt{n\eta^2}h_1(n;\eta^2)=\tan\left(2\sqrt{n\eta^2}-\frac{\pi}{4}\right)+O(n^{-3/4}). \quad (40)$$

Note the oscillating behavior of the eigenvalues $E(n)\sim\tan^2(2\sqrt{n\eta^2}-\pi/4)/\eta^2$. This circumstance implies that each eigenstate is encompassed by an infinite countable set of eigenstates of slightly different energies. For exceptional values of η, some eigenvalues can vanish. In these cases, the series representation of the relative NCS loses its meaning. The behavior of $E(n)$ as $n\rightarrow\infty$ has strong implication on the resolution of unity, as we will see in the following.

As a consequence of Eq. (40), the logarithm of the factorial $[h_1^2(n,\eta^2)]!n!$ times η^{2n} tends asymptotically to

$$\ln\{[h_1^2(n;\eta^2)]!n!\eta^{2n}\}$$

$$\underset{n\rightarrow\infty}{\sim}\frac{\pi}{8\eta^2}\int^{(8/\pi)^2n\eta^2}\ln\left[\tan\left(\frac{\pi}{4}(\sqrt{z}-1)\right)\right]^2dz+\text{const}$$

$$\underset{n\rightarrow\infty}{\sim}\frac{1}{\eta^2}u(\sqrt{n\eta^2})+\text{const},$$

where $u(x)$ is an oscillating entire function

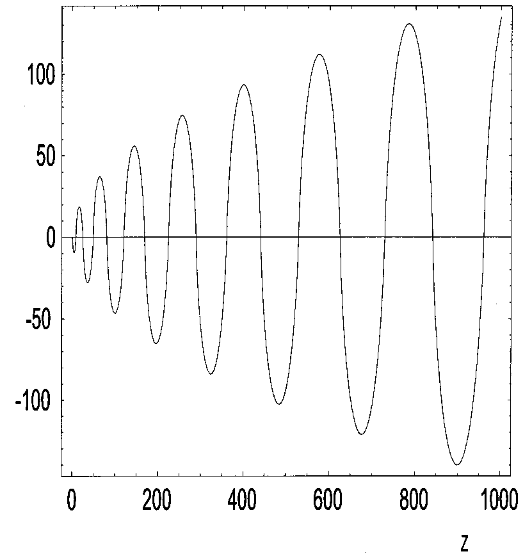

FIG. 3. Asymptotic expression of $u(\sqrt{z})$ versus z [see Eq. (41)].

$$u(x)=\frac{2}{\pi}\{-4x\,\text{Im}[\text{Li}_2(e^{-i\varphi})+\text{Li}_2(-e^{i\varphi})]+\text{Re}[\text{Li}_3(e^{-i\varphi})-\text{Li}_3(-e^{i\varphi})]\}$$
$$=\frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{4x(2k+1)-1}{(2k+1)^3}\cos[(2k+1)4x], \quad (41)$$

with $\text{Li}_n(z)=\Sigma_{k=1}^{\infty}z^k/k^n$ the polylogarithm function. According to Eq. (41), $\ln\{[h_1^2(n,\eta^2)]!n!\eta^{2n}\}$ is an oscillating function of $\sqrt{n\eta^2}$, with the envelope expanding proportionally to $\sqrt{n\eta^2}$, as shown in Fig. 3. This behavior is confirmed by the exact expressions of $\ln\{[h_1^2(n,\eta^2)]!n!\eta^{2n}\}$ plotted in Fig. 4

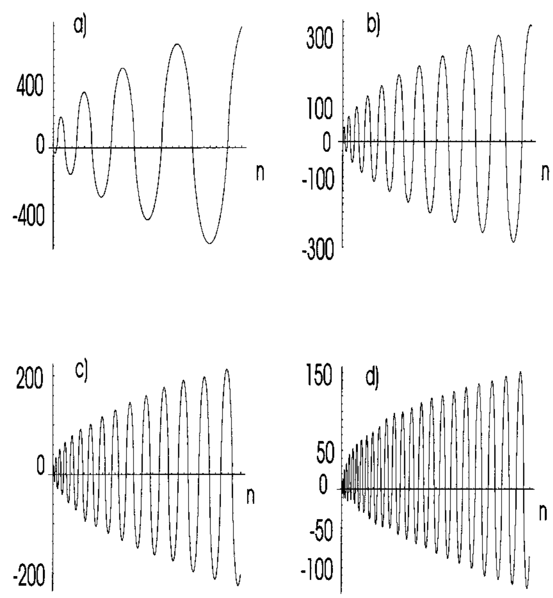

FIG. 4. $\ln\{[h_1^2(n,\eta^2)]!n!\eta^{2n}\}$ versus n for $\eta^2=0.01$ (a), 0.02 (b), 0.1 (c), and 0.2 (d).

versus n for $\eta^2=0.01,0.02,0.1$, and 0.2.

In particular, there exists a countable infinite sequence of values of $n\eta^2$ for which $[h_1^2(n,\eta^2)]!n!\eta^{2n}$ is close to $\exp(-16\sqrt{n}/\pi\eta)$, so that the NCS can be normalized only for $|\alpha\eta|$ less than one. In contrast with the linear coherent states, the NCS relative to a trapped ion fill the open circle $1/\eta$ in the complex plane. As $\eta\to 0$, the domain of existence tends to the whole complex plane, as for the linear coherent states. While these states are normalized for $\alpha\eta$ inside the unit circle, the scalar product of $|\alpha_1,h\rangle$ and $|\alpha_2,h\rangle$ is defined even if one of the numbers α_1 or α_2 has an arbitrary modulus as long as the product of the moduli is less than $1/\eta^2$. A similar situation occurs for the harmonious states [25] described by the deformation function $h(n)=1/\sqrt{n}$.

For generic combinations of vibrational excitation levels and parameter η^2, the ion rovibronic dynamics fully displays its nonlinear character. An example of this feature has been seen above in connection with the discussion of the Wigner functions of some nonlinear Schrödinger cats and states on a circle of order 3 and 4.

A. A resolution of unity

Being that these NCS are restricted to values of α such that $|\alpha\eta|<1$, the Mellin transform (23) reduces to the single-sided Laplace transform

$$g(s)=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} m_h(\eta^{-2}e^{-t})e^{-st}dt.$$

Then, $g(s)$ is the right-sided Laplace transform of the distribution $m_h(\eta^{-2}e^{-t})$ and tends asymptotically to the analytic function $\exp[\eta^{-2}v(\sqrt{s\eta^2})]$ of s in the half-plane Re $s>1$ and is bounded according to

$$g(s)\leqslant K\ \ e^{-T\,\mathrm{Re}\,s},$$

with T a negative infinitesimally small constant and K a constant. Consequently, m_h is a distribution with support bounded on the left at $t=T<0$ (see Ref. [34] corollary 8.4-1a.). This means that it is necessary to include in the unity resolution non-normalizable states of amplitude $|\alpha|>\eta^{-1}$.

B. Approximate deformations

The difficulties in dealing with this deformation can be overcome by using approximate deformations. This is justified by the circumstance that in laser cooling experiments one deals with ions occupying a finite number of vibrational levels.

In particular, when the parameter η^2 is not very large, the deformations $h_{1,2}(n)$ can be approximated by a few terms of the series expansion

$$h_1(\hat{n};\eta^2)=1+\frac{\hat{n}-1}{2}\eta^2+\frac{2\hat{n}^2-3\hat{n}+1}{6}\eta^4+\frac{11\hat{n}^3-22\hat{n}^2+13\hat{n}-2}{48}\eta^6+\cdots,$$

$$h_2(\hat{n};\eta^2)=1+\frac{2}{3}(\hat{n}-2)\eta^2+\frac{11\hat{n}^2-39\hat{n}+34}{24}\eta^4+\frac{19\hat{n}^3-96\hat{n}^2+159\hat{n}-86}{60}\eta^6+\cdots. \quad (42)$$

Approximating the deformation $h_1(n)$ by $1+[(\hat{n}-1)/2]\eta^2$ we have that $m_{(\eta^2/2)h_1}(x)$ coincides with the Mellin antitransform of $\Gamma(s)\Gamma^2(2/\eta^2+s-1)$. Before using Eq. (26) with $A=3$, $B=0$ and $a_1=0$, $a_2=a_3=2/\eta^2-1$, we have to remove the degeneracy $a_2=a_3$ by evaluating Eq. (26) for $a_2-a_3=\varepsilon$ and letting $\varepsilon\to 0$. Since $\lim_{\varepsilon\to 0}\Gamma(\varepsilon)+\Gamma(-\varepsilon)\to -2\sinh(\gamma\varepsilon)/\varepsilon=-2\gamma$ with $\gamma=0.577\,21$ the Euler's constant, then

$$a_2^2 m_{h_1}(x)=\Gamma^2(a_2)\,{}_0F_2\left(;1-a_2,1-a_2;-\frac{x}{a_2^2}\right)-2\gamma\Gamma(-a_2)\Gamma(2a_2)\,{}_0F_2\left(;1,1+a;-\frac{x}{a_2^2}\right)\times\left(\frac{x}{a_2^2}\right)^{a_2}.$$

At the same time, the h exponential reads

$$E_h(v)={}_0F_3\left(;1,a_2+1,a_2+1;\frac{v}{a_2^2}\right).$$

Expanding the generalized factorial $[h_1(\hat{n},\eta^2)]!$ to the second order in η^2, the NCS $|\alpha,h_1\rangle$ can be expressed as a combination of excited coherent states

$$|\alpha,h_1\rangle=\left(1+\eta^2-\frac{\eta^4}{12}\right)|\alpha\rangle+\left(\frac{\eta^2}{2}+\frac{7\eta^4}{72}\right)\alpha|\alpha,1\rangle+\left(-\frac{\eta^2}{2}+\frac{\eta^4}{24}\right)\alpha^2|\alpha,2\rangle-\frac{5\eta^4}{36}\alpha^3|\alpha,3\rangle+O(\eta^6).$$

A better approximation can be obtained by representing the Laguerre polynomials $L_{n-1}(\eta^2)$ by a finite sum of powers of η^2,

$$L_n(\eta^2)\sim\sum_{k=0}^{K}(-1)^k\binom{n}{k}\frac{\eta^{2k}}{k!}=P_K(n),$$

so that $h_1(n;\eta^2)$ [see Eq. (39)] can be replaced by a rational function

$$h_{A,B}(n)=\frac{P_{A+1}(n-1)-P_{A+1}(n)}{\eta^2 P_B(n-1)}=\gamma\frac{\prod_{i=1}^{A}(n+a_i)}{\prod_{j=1}^{B}(n+b_j)}. \quad (43)$$

For using the characteristic function $\rho_{h,A}$, we should choose $A=B$, while for introducing the P representation $A \geqslant B$. In particular, Cirac *et al.*, [10,15] and Blockley *et al.*, [11] have expanded the exponential of Eq. (3) up to second order in η, their case corresponds to $A=B=4$.

If the roots $-a_i$ and $-b_i$ are not integer, we have [cf. Eq. (25)]

$$[h_{AB}(s)]! = \gamma^{s-1}\Gamma^{-1}\binom{(a)}{(b)}\Gamma\binom{(a)+s}{(b)+s}.$$

For this class of deformations, the measure is given by the combination of the generalized hypergeometric functions of Eq. (24), subject to the precaution of removing the degeneracy of the coefficients a_i, b_i.

In particular, for $A=B=1$ the Husimi-Kano Q function (36) relative to the state $|z,h\rangle$ reduces to

$$Q_1(w) \propto e^{-|w-z|^2/2}|\varpi^{\nu}J_{-\nu}(\varpi)|^2 \rightarrow_{|\varpi|\rightarrow\infty} Ce^{-|w-z|^2/2} \times\left[1+\frac{\cot(\nu\pi)}{2}(\zeta w^{*2\nu}+\zeta^* w^{2\nu})\right],$$

with $\nu = 1/\eta^2 + (1/2)$, $\varpi = (i/2)zw^*$ and $\zeta = (iez/4\nu)^{2\nu}$. Consequently, the projector $|z,h\rangle\langle z,h|$ is represented in terms of the undeformed coherent states by a P representation containing derivatives of the Dirac function of the very high order ν. This confirms the advantage of using the representation (31).

For a finite rank density matrix, $m_h(x)$ can be represented by a finite combination of Laguerre polynomials

$$m_h(x) = e^{-x}\sum_{n=0}^{n_{\max}} m_n L_n(x). \tag{44}$$

Imposing the condition (22) for $0 \leqslant n \leqslant n_{\max}$ yields

$$m_n = \sum_m (-1)^m \binom{n}{m}\{[h(m)]!\}^2.$$

In Fig. 5, we have plotted these approximate measure functions for different values of η^2 ($=0.015, 0.0156, 0.0158, 0.016$), representing density operators relative to ions excited up to the level $n=50$. For these values of η^2, the inclusion of a larger number of terms ($n>50$) would lead to measures taking negative values.

VIII. CONCLUSIONS

The vibrational steady states of ions placed in a parabolic trap and driven by bichromatic fields detuned by multiples of the vibrational frequency provide a class of realizations of the nonlinear version of the so-called coherent states on a circle. The most well-known example is that of the nonlinear Schrödinger cat states. As for the linear case, these states can also be decomposed into finite sums of nonlinear coherent states, which can be considered as the building blocks of the vibrational wave functions of systems driven by laser fields detuned by multiples of the vibrational frequency.

This class of states is well described by the Wigner function, which has been computed for states on a circle of degree two (cats), three, and four. The relative patterns show a dramatic dependence on the Lamb-Dicke parameter η, which measures the degree of departure from the linear case. This behavior is due to the irregular dependence of the ion deformation function on the Fock state index n.

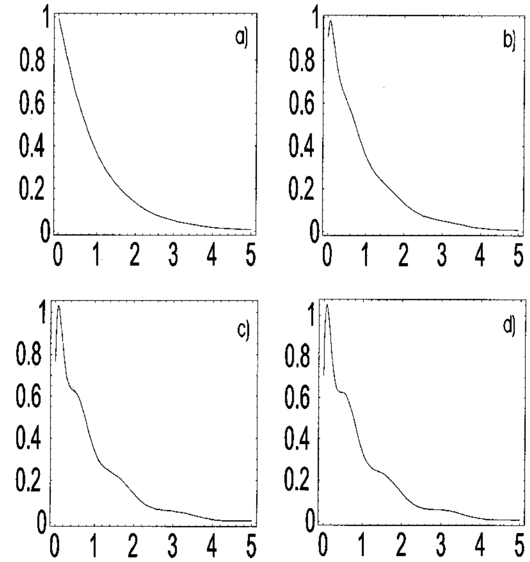

FIG. 5. Measure $m_h(x)$ versus x obtained as a combination of 50 Laguerre polynomials [Eq. (44)] and four different values of $\eta^2 = 0.015, 0.0156, 0.0158, 0.016$.

With the aim of investigating the possibility of extending some mathematical tools of the linear coherent state theory to NCS a deformed displacement operator $\mathcal{D}_h(\alpha)$ has been introduced, which, in analogy with the linear one, generates the NCS $|\alpha,h\rangle$ by displacing the Fock vacuum state $[|\alpha,h\rangle = \mathcal{D}_h(\alpha)|0\rangle]$. The penalty paid for this extension is the loss of the unitarity. However, it allows the construction of a linear functional that can be used for representing density operators by means of a generalized probability distribution function $\rho_{A,h}(z)$.

The peculiarities of NCS connected with trapped ions become evident when the deformation factorial is analyzed for very large n. The weight of the nth Fock state contributing to a NCS exhibits an almost periodic behavior by taking very large and very small values of the order of $e^{\pm C\sqrt{n\eta^2}}$. Consequently, the NCS can be normalized only for α filling the open circle $1/\eta$ in the complex α plane. In addition, this behavior prevents the existence of a regular measure for resolving the unity. These pathologies mark the difference with q oscillators whose deformation function is an increasing function of n. The NCS with radius less than $1/\eta$ are complete but their duals have radius η. This is the same for the harmonious states [25].

For extending to these NCS the P representation formalism, it is necessary to replace the deformation with an approximate one, for which there exists a measure. These approximate NCS can be constructed in different ways. Two examples are provided. In the first case, the deformation is represented by a rational function of the occupation number, obtained by truncating the Laguerre polynomials to some order in η^2. By increasing the degree of these rational functions it is possible to accurately represent the actual deformation for occupation numbers extending up to infinity. The respective measures are the Mellin antitransforms of gamma function products $\Gamma(\binom{(b)}{(a)})$ and are given by combinations of generalized hypergeometric functions. In the second case, the measure is represented by a finite combination of Laguerre polynomials. In this way it is possible to represent exactly the factorials up to a given level n, although it is not possible to obtain, in general, a positive definite measure.

The NCS can provide a basis for studying the trapped ion evolution by representing the statistical expectation values of either antinormal $G_A(A,A_h^\dagger)$ or normal $G_N(A,A_h^\dagger)$ products of A and $A_h^\dagger$ as integrals of the probability distributions $\rho_{h,A}(z)$ or $P_h(z)$ times the classical functions $G_A(z,z^*)$ and $G_N(z,z^*)$. Another possibility consists in transforming the density matrix master equation to an equivalent equation for the P representation. This problem will be addressed in a more systematic way in a forthcoming paper.

Before concluding, it is remarkable that the deformation used for the q oscillators, the ancestors of the NCS, is based on the same transformation used by Heine [35] a century ago for generalizing the Gauss hypergeometric function.

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