Methodology for quantum logic gate construction

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We present a general method to construct fault-tolerant quantum logic gates with a simple primitive, which is an analog of quantum teleportation. The technique extends previous results based on traditional quantum teleportation [Gottesman and Chuang, Nature (London) **402**, 390 (1999)] and leads to straightforward and systematic construction of many fault-tolerant encoded operations, including the $\pi/8$ and Toffoli gates. The technique can also be applied to the construction of remote quantum operations that cannot be directly performed.

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I. INTRODUCTION

Practical realization of quantum information processing requires specific types of quantum operations that may be difficult to construct. In particular, to perform quantum computation robustly in the presence of noise, one needs fault-tolerant implementation of quantum gates acting on states that are block-encoded using quantum error-correcting codes [1–4]. Fault-tolerant quantum gates must prevent propagation of single qubit errors to multiple qubits within any code block so that small correctable errors will not grow to exceed the correction capability of the code. This requirement greatly restricts the types of unitary operations that can be performed on the encoded qubits. Certain fault-tolerant operations can be implemented easily by performing direct *transversal* operations on the encoded qubits, in which each qubit in a block interacts only with one corresponding qubit, either in another block or in a specialized ancilla. Unfortunately, for a given code, only a few useful operations can be done transversely, and these are not universal in that they cannot be composed to approximate an arbitrary quantum circuit. To obtain a universal set of gates, additional gates have to be constructed using ancilla states and fault-tolerant measurement. Although these additional gates have been constructed successfully [1,3,5,6], their *ad hoc* construction is complicated and is not easily generalized.

Another kind of application in which we are challenged to construct useful quantum operations from a limited set of primitives is in distributed quantum information processing. In this problem, certain kinds of communication between different parties are constrained or prohibited, but prior distribution of standard states may be allowed. For example, quantum teleportation [7] demonstrates how an unknown quantum state can be sent between two parties without sending any quantum information, using only classical communication and prior entanglement. Protocols for distributed state preparation and computation are also known [8], but again, they have been largely constructed by hand and offer neither an explanation of why a particular ancilla state is required nor a systematic path for generalization.

A general framework for addressing such problems has been presented in [9]; it uses quantum teleportation as a basic primitive to enable construction of quantum operations that cannot be directly performed through unitary operations. This framework provides systematic and generalizable construction for an infinite family of fault-tolerant gates, including the $\pi/8$ and Toffoli gates. It does not, however, lead to circuits equivalent to (or as simple as) prior *ad hoc* construction for the same gates.

In this paper, we provide an extension to the teleportation method of gate construction with a similar but simpler primitive, which we call ''one-bit teleportation'' because it uses one qubit instead of two as ancilla. This method simplifies the construction of [9] and, furthermore, provides strikingly unified construction of the $\pi/8$, controlled-phase, and Toffoli gates. An infinite hierarchy of gates, including the controlled rotations $\mathrm{diag}(1,1,1,e^{i2\pi/2^k})$ used in the quantum factoring algorithm [10], can be constructed with the present scheme.

The structure of the paper is as follows. First, in Sec. II, we define one-bit teleportation, and describe its properties and various guises. Its application to fault-tolerant gate construction is presented in Sec. III, which is followed in Sec. IV with specific circuits for the $\pi/8$, controlled-phase, and Toffoli gates. In Sec. V, we describe the use of one-bit teleportation to derive the two-bit quantum teleportation protocol and to construct a remote quantum gate. We summarize our results in Sec. VI.

II. ONE-BIT TELEPORTATION

In standard quantum teleportation, Alice performs a joint measurement of the unknown qubit and some ancilla, and sends the classical measurement outcome to Bob, who subsequently reconstructs the unknown state. No quantum operation is performed jointly by Alice and Bob, but they need a certain *two*-qubit entangled *ancilla* state. (We refer to any state other than the original unknown qubit state as an ancilla state.) The same objective, communicating a qubit, can be accomplished in a simpler manner if Alice and Bob are al-

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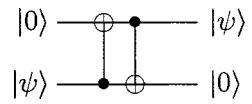
lowed to perform a quantum gate (such as a controlled-NOT gate, a CNOT) between their respective qubits. In this case, only a *single* qubit ancilla in Bob's possession is required. We call such a quantum circuit one-bit teleportation, which can be derived using the following facts.

Fact 1. An unknown qubit state $|\psi\rangle$ can be swapped with the state $|0\rangle$ using only two CNOT gates, as shown in the following circuit:



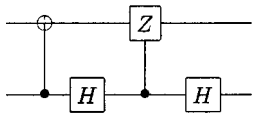
(1)

Note that in all circuits we show, time proceeds from left to right as is usual, and conventions are as in [11]. Throughout this section, the first and second qubits refer to the registers with respective initial states $|0\rangle$ and $|\psi\rangle$.

Fact 2. $X=HZH$, where X and Z are Pauli operators, and H is the Hadamard gate defined as

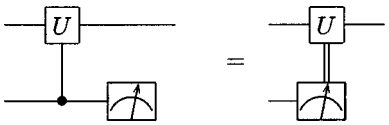
$$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}. \tag{2}$$

Then Eq. (1) is equivalent to the following circuit:



(3)

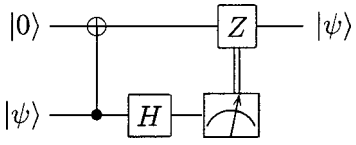
Fact 3. A quantum-controlled gate can be replaced by a classically controlled operation when the control qubit is measured.



(4)

The meter represents the measurement of Z, which projects the measured state onto $|0\rangle$ or $|1\rangle$. The double line coming out of the meter carries the *classical* measurement result, and U is performed if the measurement result is $|1\rangle$.

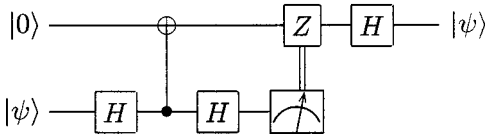
In Eq. (3), the two qubits are disentangled before the second Hadamard gate. Therefore, the second qubit can be measured before the second Hadamard gate without affecting the unknown state in the first qubit. Applying fact 3 to Eq. (3) results in the following circuit:



(5)

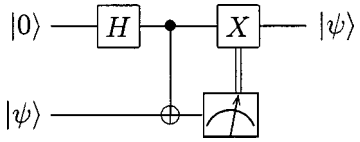
The circuit in Eq. (5) uses a controlled-NOT (CNOT) gate and only one qubit for the ancilla. Therefore it is a one-bit teleportation circuit, which we refer to as ''Z-teleportation'' because a classically controlled-Z is applied after the measurement.

Using Z-teleportation we can derive other one-bit teleportation circuits. For instance, the following circuit first teleports the state $H|\psi\rangle$ using Z-teleportation, and then applies $H^{\dagger}=H$ to the teleported state $H|\psi\rangle$ to obtain the original state $|\psi\rangle$:



(6)

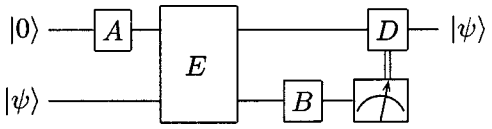
This circuit can be simplified to



(7)

which we refer to as ''X-teleportation.'' Similarly, we can derive other one-bit teleportation circuits as discussed in Appendix A. We will focus on X- and Z-teleportation circuits because they are sufficient for our construction in this paper.

X- and Z-teleportation circuits can both be represented using the same general structure:



(8)

where the first qubit (the ancilla qubit) is initially in the $|0\rangle$ state. For Z-teleportation, $A=I$ (I is the 2×2 identity operator), $B=H$, $D=Z$, and E is a CNOT gate with the first qubit as its target. For X-teleportation, $A=H$, $B=I$, $D=X$, and E is a CNOT gate with the first qubit as its control.

III. FAULT-TOLERANT GATE CONSTRUCTION USING ONE-BIT TELEPORTATION

In this section, we develop a general method for fault-tolerant gate construction using one-bit teleportation as a basic primitive. We will confine our attention to the Calderbank-Shor-Steane (CSS) codes that are doubly even and self-dual [12,13,1], although the results can be extended to any other stabilizer codes [9].

A. Fault-tolerant gate hierarchy

We first summarize the fault-tolerant gate hierarchy introduced in [9]. Let C_1 denote the *Pauli group*. Then for $k\geqslant 2$, we can recursively define C_k as

$$C_k\equiv\{U|UC_1U^{\dagger}\subseteq C_{k-1}\}. \tag{9}$$

For every k, $C_k\supset C_{k-1}$, and the set difference $C_k\backslash C_{k-1}$ is nonempty. For instance, $\mathrm{diag}(1,e^{i2\pi/2^k})\in C_k\backslash C_{k-1}$.

C_2 is a group called the *Clifford group* [8], which is the set of operators that conjugate Pauli operators into Pauli op-

erators. Besides the Pauli operators, C_2 also contains other important gates, such as the CNOT gate, H, and the phase gate S (defined by $S|x\rangle = i^x|x\rangle$ for $x \in \{0,1\}$). For doubly even and self-dual CSS codes, any encoded C_2 gate has transversal unitary implementation [1,5], which is fault-tolerant.

C_2 gates alone, however, are not sufficient for universal quantum computation [8]. An additional gate outside C_2 is necessary and sufficient to complete universality [14]. In particular, adding any one of the following gates in $C_3 \backslash C_2$ to the Clifford group results in a universal set of unitary operations: the $\pi/8$ gate T ($T|x\rangle = e^{i\pi x/4}|x\rangle$ for $x \in \{0,1\}$), the controlled-phase gate $\Lambda_1(S)$ [$\Lambda_1(S)|xy\rangle = i^{x \cdot y}|xy\rangle$ for $x,y = \{0,1\}$], and the Toffoli gate (controlled-controlled-NOT) [1,15,6].

The construction of an encoded operation in $C_3 \backslash C_2$ is much more complicated than that of an encoded operation in C_2, and requires quantum measurement and a particular ancilla state. But applying a $C_3 \backslash C_2$ gate to certain *known* states can be replaced by direct preparation of the final states, which can be relatively easier as stated in the following.

Theorem 1. Let U be an n-qubit gate in C_3. Then the encoded state $U(|0\rangle^{\otimes n})$ can be prepared fault-tolerantly by applying and measuring C_2 operators.

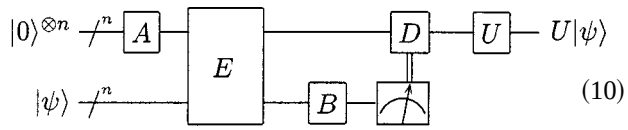
Proof. See Appendix B.

Since C_3 is closed under multiplication by elements in C_2 [16], theorem 1 is also applicable when $|0\rangle^{\otimes n}$ is replaced by $V(|0\rangle^{\otimes n})$ for $V \in C_2$, because $U|\psi\rangle = UV(|0\rangle^{\otimes n})$ with $UV \in C_3$. We will use theorem 1 in our fault-tolerant logic gate construction.

B. C_3 gate construction using one-bit teleportation

We now consider a general method of constructing fault-tolerant gates in C_3 using the one-bit teleportation scheme as a primitive. The basic idea is the following. To apply the encoded operation U to an encoded state $|\psi\rangle$, we can first teleport $|\psi\rangle$ by either X- or Z-teleportation, and apply U to the reconstructed $|\psi\rangle$. The extra teleportation step can be done fault-tolerantly because both X- and Z-teleportation use C_2 gates only. It further reduces the problem of fault-tolerant construction of a quantum logic gate to fault-tolerant preparation of a particular ancilla state. The reason for the reduction is that U is applied to the ancilla, which is originally in the *known* state $|0\rangle$. If U can be commuted backwards until it is applied to a *known* state without introducing more complicated gates, we can prepare the resulting known state, without applying U directly, as the input ancilla. Using such an ancilla, the reconstructed state after the *modified* one-bit teleportation circuit will be $U|\psi\rangle$. That is, the encoded U has been applied to the encoded $|\psi\rangle$ fault-tolerantly.

We now detail the formal construction. Let $U \in C_3$ be an n-qubit gate to be applied to $|\psi\rangle$, an encoded quantum state with n *logical* qubits. We first teleport each logical qubit using either X- or Z-teleportation such that $|\psi\rangle$ is reconstructed in the ancilla, which is initially in the $|0\rangle^{\otimes n}$ state. We then apply U to the reconstructed $|\psi\rangle$ to obtain $U|\psi\rangle$. This is described by the following quantum circuit:

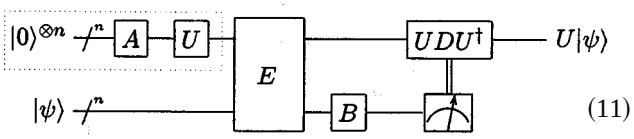


(10)

In Eq. (10), a register (wire) with the symbol "/^{n}" represents a bundle of n logical qubits. A is a bitwise operation, $A = A_1 \otimes \cdots \otimes A_n$, where A_i acts on the ith logical qubit only. B is a bitwise operation similar to A. E is a tensor product such that $E = E_i \otimes \cdots \otimes E_n$, where each E_i is a CNOT gate between the ith logical qubits of $|\psi\rangle$ and the known ancilla. The measurement box measures Z bitwise and the double line represents the n-bit classical outcome. The ith classical bit controls whether an operator D_i is performed on the ith logical state in the first register. This is denoted by D for the sake of simplicity.

According to Sec. II, if Z-teleportation is applied to the ith logical qubit, $A_i = I$, $B_i = H$, $D_i = Z$, and E_i is a CNOT gate with the first qubit as its target; if X teleportation is applied instead, $A_i = H$, $B_i = I$, $D_i = X$, and E_i is a CNOT gate with the first qubit as its control.

We now commute U backwards in time. Commuting U with the classically controlled operation D changes D to $UDU^\dagger$. As $D \in C_1$ and $U \in C_3$, $UDU^\dagger \in C_2$ can still be performed transversely. Likewise, commuting U with E changes E to $UEU^\dagger$ [17]. As the CNOT gate $\notin C_1$, the resulting operation $UEU^\dagger$ may not be in C_2 for an arbitrary $U \in C_3$. To ensure $UEU^\dagger \in C_2$, we only consider U that commutes with E such that $UEU^\dagger = E \in C_2$. Then Eq. (10) becomes



(11)

All the circuit elements outside the dotted box can be performed fault-tolerantly. Therefore, if we can prepare the input ancilla in the state $UA(|0\rangle^{\otimes n})$, we can apply $U \in C_3$ to any encoded state $|\psi\rangle$ fault-tolerantly. As $A \in C_2$, UA is also a C_3 operation. By theorem 1, the ancilla state $UA(|0\rangle^{\otimes n})$ can be created fault-tolerantly. The stabilizers of such an ancilla state, which will be measured in preparing the state, can be easily derived. Recall that when $A_i = I$, $D_i = Z_i$, and when $A_i = H$, $D_i = X_i$. Therefore, $A_i Z_i A_i^\dagger = D_i$ is always true [18], and the stabilizers of $UA(|0\rangle^{\otimes n})$ are $UA_i Z_i A_i^\dagger U^\dagger = UD_i U^\dagger \in C_2$.

Using the above method, we can systematically construct interesting gates in $C_3 \backslash C_2$, including the $\pi/8$, controlled-phase, and Toffoli gate, as will be shown in Sec. IV.

Finally, we remark that the C_3 gates commuting with E are not the only gates that can be performed by the one-bit teleportation scheme. Any C_3 gate of the form $U = G_b V G_a$ for V commuting with E and $G_a, G_b \in C_2$ can be performed using the generalized one-bit teleportation circuits. We discuss this in Appendix A.

C. Recursive construction

In this section, we extend our discussion to the gates in C_k and characterize a class of gates that can be recursively constructed with one-bit teleportation as a basic primitive.

We prove by induction that the diagonal subset of C_k, defined by $F_k=\{U\in C_k$ and U is diagonal$\}$, can be recursively constructed.

First, when $U\in F_k$, we choose to apply X teleportation to each logical qubit. In this case, each E_i is a CNOT gate taking the ith logical qubit in the ancilla as the control bit. Therefore, E commutes with U and Eq. (11) holds with $A_i=H$, $B_i=I$, and $D_i=X_i$ for $i=1,\ldots,n$. Second, since for $U\in F_k$ and $P\in C_1$, $UPU^\dagger=\tilde{U}P$ for some $\tilde{U}\in F_{k-1}$ [16], $UD_iU^\dagger=UX_iU^\dagger=U_xX_i$ for some $U_x\in F_{k-1}$. Therefore, if the gates in F_{k-1} can be performed, the classically controlled operation $UD_iU^\dagger$ for $U\in F_k$ can also be performed. Third, the required ancilla $UH^{\otimes n}(|0\rangle^{\otimes n})$ can be prepared fault-tolerantly with recursive construction as shown in Appendix B. Finally, the gates in $F_2\subset C_2$ have transversal implementation. By induction, all the gates in F_k can be performed fault-tolerantly with recursive application of the one-bit teleportation scheme.

The sets F_k contain many interesting gates, such as $V^k=\mathrm{diag}(1,e^{i\pi/2^k})$, which are the single qubit $\pi/2^k$ rotations, and $\Lambda_1(V^{k-1})=\mathrm{diag}(1,1,1,e^{i\pi/2^{k-1}})$, which are the controlled rotations used in the quantum Fourier transform circuit [10,19] essential to Shor's factoring algorithm [10]. F_k also includes the multiple-qubit gates $\Lambda_n(V^l)$ for $n+l\leqslant k$ [16], where $\Lambda_n(V^l)$ applies V^l to the $(n+1)$th qubit if and only if the first n qubits are all in the state $|1\rangle$. By the closure property of F_k [16], all products of $\Lambda_n(V^l)$ for $n+l\leqslant k$ are in F_k. To perform gates in F_k for k small, the recursive construction we have described can be more efficient than approximating these gates to the same accuracy using a universal set of fault-tolerant quantum logic gates.

The gates in F_k are not the only ones that can be constructed using the one-bit teleportation scheme. For instance, if $U\in C_k$ is related to an element in F_k by conjugation with Hadamard gates in the i_1th, $\ldots$, i_lth qubits, E can be made to commute with U by applying Z-teleportation to the i_1th, $\ldots$, i_lth qubits and X-teleportation to the rest. The Toffoli gate is an example. More generally, any gate that is a product of C_2 gates and a single F_k gate can be constructed recursively.

IV. EXAMPLES

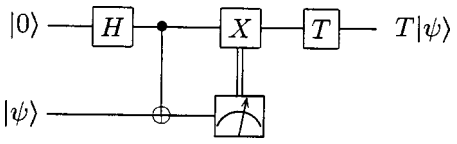
In this section, we systematically construct three important fault-tolerant gates in $C_3\setminus C_2$ using the general method described in Sec. III. Any one of these gates, together with the Clifford group, forms a universal set of gates. For each construction, we will derive the required circuit and the ancilla. The ancilla can always be prepared fault-tolerantly (see Appendix B).

A. The $\pi/8$ gate

The $\pi/8$ gate, T, has the following matrix representation:

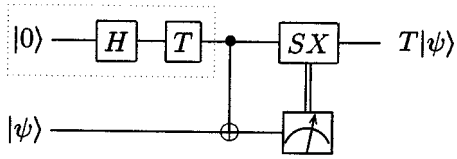
$$T=\begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix}. \tag{12}$$

As T is diagonal, following the recipe in Sec. III, we choose to apply X-teleportation to $|\psi\rangle$ and apply T to the teleported $|\psi\rangle$:



(13)

We commute T backwards using two facts. First, $TXT^\dagger=e^{-i\pi/4}SX$, where the phase gate S is defined in Sec. III A. Second, T commutes with the CNOT gate by construction. Thus, we obtain a circuit to implement the $\pi/8$ gate (where an irrelevant overall phase has been ignored):



(14)

All the circuit elements outside the dotted box can be performed fault-tolerantly. The dotted box, then, can be replaced by an ancilla in the state

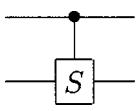
$$|\phi_+\rangle=TH|0\rangle=\frac{|0\rangle+e^{i\pi/4}|1\rangle}{\sqrt{2}}, \tag{15}$$

which can be prepared fault-tolerantly as described in Appendix B. Thus, we have derived a circuit and the corresponding ancilla for performing the fault-tolerant $\pi/8$ gate. We note that this rederives the same circuit and ancilla state used in [6].

B. The controlled-phase gate

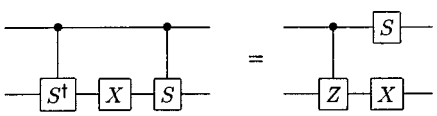
The controlled-phase gate $\Lambda_1(S)$ (defined in Sec. III A) is in C_3, and forms a universal set of gates [15,11] together with H and the CNOT gate.

We use the following circuit symbol for $\Lambda_1(S)$:

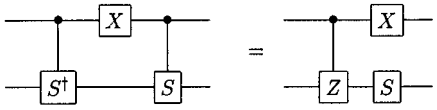


(16)

$\Lambda_1(S)$ commutes with Z_i, and conjugates X_i $(i=1,2)$ as follows:

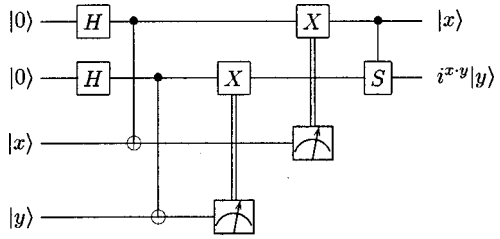


(17)



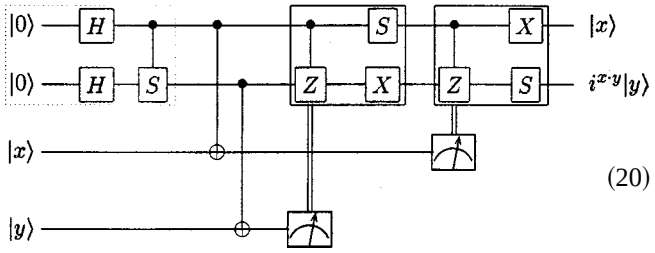
(18)

where the controlled-Z operation, $\Lambda_1(Z)$, acts on basis states as $\Lambda_1(Z)|x\rangle|y\rangle=(-1)^{x\cdot y}|x\rangle|y\rangle$. To construct $\Lambda_1(S)$, we first teleport the two-qubit state $|\psi\rangle$ and apply $\Lambda_1(S)$. This linear transformation preserves phase coherence, and thus, it suffices to consider its action on the basis states $|xy\rangle$. Since $\Lambda_1(S)$ is diagonal, we apply X-teleportation to both qubit states such that the CNOT gates in the circuit commute with $\Lambda_1(S)$.



(19)

Commuting $\Lambda_1(S)$ backwards using the commutation rules in Eqs. (17) and (18), we obtain a circuit for the controlled-phase gate:



(20)

where the double lines control all the operations in the corresponding boxes. All the circuit elements in Eq. (20), except those in the dotted box, can be performed fault-tolerantly. Finally, we can replace the dotted box by an input ancilla in the following state:

$$|\phi_+\rangle=\Lambda_1(S)(H_1\otimes H_2)|00\rangle \tag{21}$$

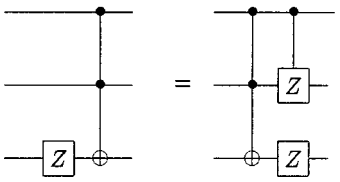
$$=\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+i|11\rangle), \tag{22}$$

which can be prepared fault-tolerantly. This completes the requirement for performing the controlled-phase gate fault-tolerantly.

C. The Toffoli gate

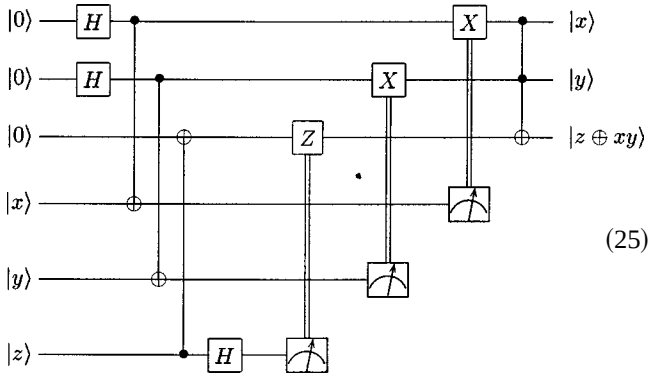
To construct the Toffoli gate (controlled-controlled-NOT), we begin with some useful commutation rules:

(23)



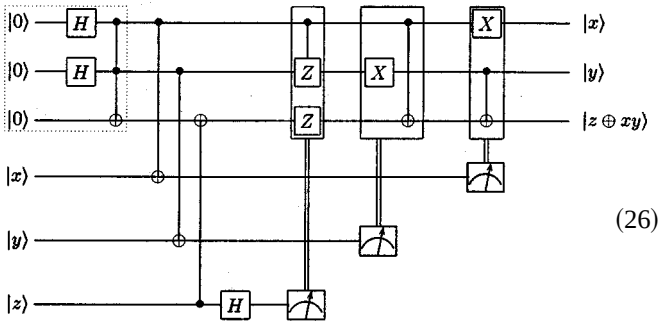
(24)

As in the controlled-phase gate construction, we demonstrate the construction on basis states $|xyz\rangle$ for three qubits. We first teleport $|xyz\rangle$ and then apply a Toffoli gate. Since the Toffoli gate is diagonalized by a Hadamard transform on the target qubit, the choice of X-teleportation for the control qubits and Z-teleportation for the target qubit ensures that the three CNOT gates commute with the Toffoli gate.



(25)

Commuting the Toffoli gate backwards to the far left using Eqs. (23) and (24), Eq. (25) is equivalent to



(26)

All the circuit elements except those in the dotted box can be performed fault-tolerantly. It remains to prepare the state created in the dotted box,

$$|\phi_+\rangle=U(H_1\otimes H_2)|000\rangle \tag{27}$$

$$=\frac{1}{2}(|000\rangle+|010\rangle+|100\rangle+|111\rangle), \tag{28}$$

where U denotes the Toffoli gate. Again this ancilla state can be prepared fault-tolerantly, as described in Appendix B.

The ancilla and the quantum circuit derived here are the same as those in Shor's original construction [1]. The one-bit teleportation scheme elucidates the choice of the ancilla state and the procedure in [1].

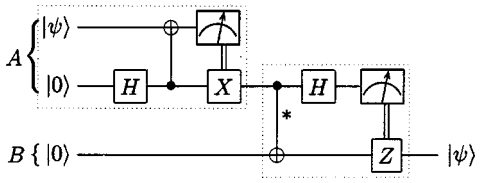
V. REMOTE GATE CONSTRUCTION USING ONE-BIT TELEPORTATION

The one-bit teleportation scheme, in addition to being useful for fault-tolerant gate construction, can also be used to design a variety of remote quantum operations. Constructing remote quantum operations is related to constructing fault-tolerant gates in that both require a particular ancilla state to replace a prohibited operation. In this section, we use one-bit teleportation as a basic primitive to derive the quantum circuits and the required ancilla states for the two-bit quantum teleportation and the remote CNOT gate.

A. Two-bit teleportation

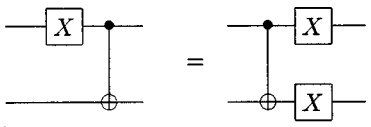
Suppose Alice needs to send a qubit state $|\psi\rangle$ to Bob. Direct quantum communication is not allowed, but Alice and Bob can share some ancilla state. The question is, how can Alice send $|\psi\rangle$ to Bob? A well-known solution to this problem is quantum teleportation [7], which uses an EPR state and classical communication. Using one-bit teleportation, we give an alternative derivation of the required (entangled) ancilla and the required teleportation circuit.

We first construct a circuit to send the unknown state with a prohibited operation. Then we remove the requirement of such a prohibited operation. Let $|\psi\rangle$ be the state to be communicated from Alice to Bob. Alice can send $|\psi\rangle$ to Bob by applying one-bit teleportation twice. Step 1: Alice swaps $|\psi\rangle$ with an ancilla $|0\rangle$ using X-teleportation. Step 2: Alice sends the teleported $|\psi\rangle$ to Bob using Z-teleportation (with a prohibited CNOT gate in this step). The circuit representation for the process is



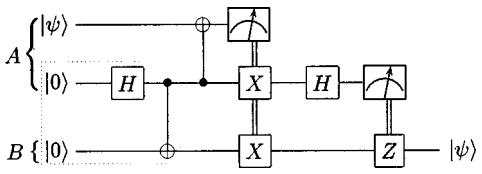
(29)

The prohibited operation (CNOT), which is marked by an asterisk, can be commuted backwards using the commutation relation



(30)

This leads to the usual quantum teleportation circuit



(31)

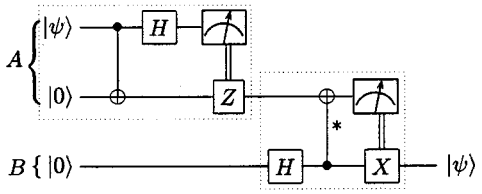
In Eq. (31), the prohibited CNOT gate acts on the *known* state inside the dotted box, which can be replaced by the following state it creates:

$$|\phi\rangle = \Lambda_1(X) H_1 |00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{32}$$

In other words, if Alice shares this entangled state with Bob, the state $|\psi\rangle$ can be sent to Bob without quantum communication.

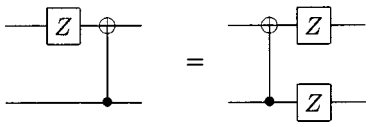
Note that the classically controlled-X on the second register only affects its overall sign, and can be omitted since the second register is subsequently measured.

An alternative circuit, which accomplishes the same task, can be derived when Z- and X-teleportation are used for the two steps instead. We start with the following circuit:



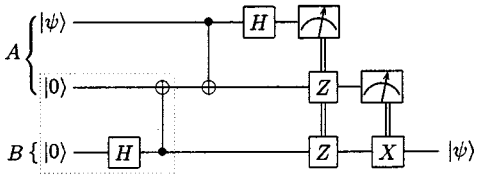
(33)

Using the commutation rule



(34)

we can commute the prohibited CNOT gate backwards to obtain an equivalent quantum teleportation circuit,



(35)

The disallowed element in the dotted box can be replaced by the EPR state of Eq. (32). The irrelevant classically controlled-Z on the second register can be omitted.

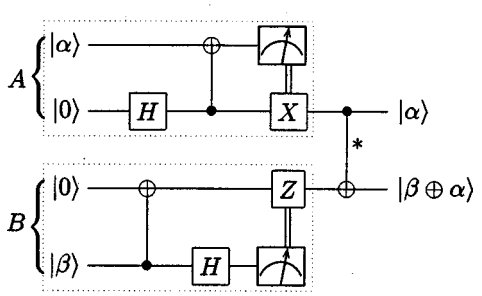
The two-bit teleportation circuits of Eqs. (31) and (35) are equivalent to that in [20], but as mentioned above, they are derived differently.

B. Remote CNOT gate

Suppose Alice and Bob have in their possession quantum states $|\alpha\rangle$ and $|\beta\rangle$, respectively. How can they perform a simple distributed computation, a CNOT gate from $|\alpha\rangle$ to $|\beta\rangle$,

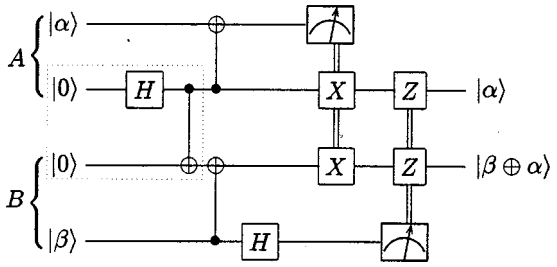
without communicating any quantum information between them, but perhaps with the aid of some initially shared standard quantum state? A solution to this problem is given in [8]. The *ad hoc* method employed, however, does not suggest a systematic technique for deriving the solution, or solutions to generalized versions of this problem. Here, we use one-bit teleportation to present a general technique and derive a different circuit that accomplishes the same task.

Alice and Bob first swap their states with their respective ancilla state $|0\rangle$ by one-bit teleportation, and then apply a prohibited CNOT gate. The quantum circuit is chosen so that Alice uses X teleportation and Bob uses Z teleportation:



(36)

The prohibited CNOT gate can be commuted backwards to obtain a remote CNOT circuit:



(37)

The prohibited operation in the dotted box is applied to a known state, and can be replaced by the EPR state of Eq. (32). Provided such a shared entangled state is initially available to Alice and Bob, they can perform a remote CNOT operation using two bits of classical communication.

Note that a remote CNOT gate can also be constructed by using two-bit teleportation twice in an obvious way: Bob first sends his qubit $|\beta\rangle$ to Alice with two-bit teleportation, and then Alice applies the CNOT operation to $|\alpha\rangle|\beta\rangle$ and sends the qubit $|\alpha\oplus\beta\rangle$ to Bob with two-bit teleportation. Such construction, however, requires two pairs of maximally entangled states and four bits of classical communication, which is twice that required for the one-bit teleportation scheme.

Our remote CNOT construction in Eq. (37) is different from that in [8], which can also be derived using the one-bit teleportation scheme, as described in Appendix C.

Finally, we remark that the two examples of constructing remote operations strengthen the concept of teleporting quantum logic gates with one-bit teleportation, as we have shown that if the input ancilla is a special state related to the CNOT gate, the reconstructed state is the one to which a CNOT gate has been applied.

VI. CONCLUSION

We have presented a systematic technique to construct a variety of quantum operations, by using a primitive one-bit teleportation scheme. Such a scheme reduces the problem of constructing a quantum logic gate to preparing an ancilla state created by the same gate applied to a known state. The usefulness of this technique is particularly manifest for two kinds of application: fault-tolerant quantum computation and remote quantum computation, as demonstrated in our construction of the $\pi/8$, controlled-phase, and Toffoli gates, and the remote-CNOT gate. With recursive application of the one-bit teleportation scheme, we can also construct an infinite hierarchy of gates fault-tolerantly.

The idea of teleporting quantum logic gates has been used in [9], with two-bit teleportation as a primitive, to perform universal quantum computation. The two-bit teleportation scheme allows *all* C_3 gates to be teleported fault-tolerantly, and *all* C_k gates to be teleported with recursive application of the scheme. For one-bit teleportation, however, we can only provide *sufficient* conditions for gates in C_3 to be teleportable, namely, any C_3 gate that can be written as a product of C_2 gates and a single C_3 gate that commutes with the CNOT gate. It is not known if this includes all the C_3 gates. The difficulty in describing the exact set of one-bit teleportable C_3 gates arises from the requirement for a $C_2\backslash C_1$ gate in the one-bit teleportation circuit. Such a $C_2\backslash C_1$ gate may be conjugated outside C_2 by a C_3 gate, and therefore cannot be directly performed fault-tolerantly. This places a further constraint on the teleportable $U\in C_k$ for $k>3$. Because of our present lack of understanding of the general structure and nature of C_k gates, the distinction between the ultimate capabilities of the one- and two-bit teleportation schemes remains an interesting and difficult question.

Nevertheless, as we have shown, one-bit teleportation can provide much simpler protocols than two-bit teleportation in constructing quantum logic gates. This is because one-bit teleportation only requires projective measurement of Z and as many ancilla qubits as the state to be transformed; two-bit teleportation, however, requires Bell measurement and twice as many ancilla qubits as the original state.

At a very general level, the logical gate teleportation schemes reduce the difficulty of constructing quantum logic gates by using special ancilla states. This can be useful not only for simplifying hardware requirements, but also for designing and optimizing computation and communication protocols [21,22]. Even more intriguing, perhaps, is that this result gives us a first glimpse at what might someday be a standard architecture for a quantum computer: a simple assembly of one-bit teleportation primitives, capable of universal quantum computation on quantum data, given the assistance of standard quantum states that are obtained as commercial resources. The definition of such a stored-program architecture could be pivotal in the development of this field, much as the von Neumann or Harvard architecture [23] was important in classical computation.

ACKNOWLEDGMENTS

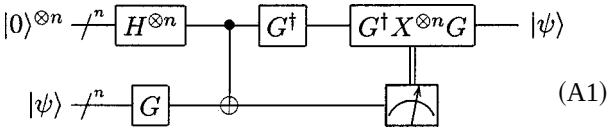
The relation between fault-tolerant quantum logic gate construction and teleportation is first alluded to by Shor [1].

The X and Z teleportation circuits presented in this paper are due to Charles Bennett and Daniel Gottesman (unpublished). We are grateful to Daniel Gottesman for introducing us to the interesting subject of the C_k hierarchy, and for enlightening discussions. We thank Professor James Harris and Yoshihisa Yamamoto for support and encouragement. This work was supported by the DARPA Ultra-scale Program under the NMRQC initiative, Contract No. DAAG55-97-1-0341, administered by the Army Research Office. D.L. acknowledges support from IBM and Nippon Telegraph and Telephone Corporation (NTT).

APPENDIX A: GENERALIZATIONS OF THE ONE-BIT TELEPORTATION CIRCUITS

The one-bit teleportation circuit used in fault-tolerant gate construction has three components: a particular input ancilla, a sequence of C_2 gates, and finally the measurement and classically controlled operation. The teleportability of one-bit teleportation is governed by the sequence of C_2 gates before the measurement. Using the X- and Z-teleportation circuits of Eq. (11), any $U \in C_3$ that commutes with E can be teleported. In this appendix, we derive other one-bit teleportation circuits, which use different C_2 gates, and then discuss their application in constructing fault-tolerant gates.

By teleporting $G|\psi\rangle$ using X-teleportation and applying $G^\dagger$ to the teleported $G|\psi\rangle$, we obtain the following generalized one-bit telelportation circuit:

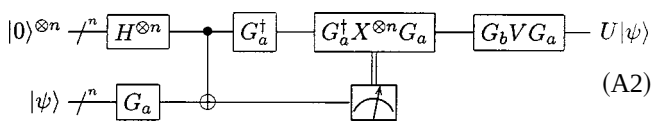


(A1)

When $G = I^{\otimes n}$ and $H^{\otimes n}$, Eq. (A1) reduces to the X and Z-teleportation circuits.

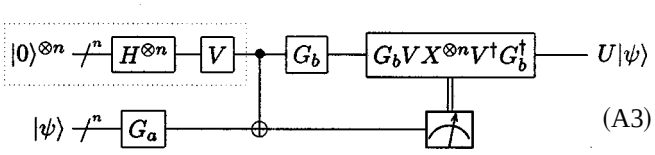
In Sec. III, we showed that all the operations in F_3 can be performed fault-tolerantly using X-teleportation. Here, we generalize the result to show that, if $U \in C_3$ and $U = G_b V G_a$, where $V \in F_3$ and $G_a, G_b \in C_2$, then U can be performed fault-tolerantly using the general one-bit teleportation scheme by the following procedure:

Step 1. Using the circuit of Eq. (A1) with $G = G_a$, we first teleport the state $|\psi\rangle$ to the ancilla initialized in the state $|0\rangle^{\otimes n}$, and then apply U to the ancilla. This can be represented by



(A2)

Step 2. Commuting U backwards, one obtains



(A3)

Note that the new classically controlled operation is $G_b V X^{\otimes n} V^\dagger G_b^\dagger$, which is in C_2 because $V X^{\otimes n} V^\dagger \in C_2$. Therefore, all the circuit elements can be performed fault-tolerantly, except those in the dotted box, which can be replaced by an ancilla in the state $VH^{\otimes n}(|0\rangle^{\otimes n})$.

There are C_3 gates that cannot be constructed using X- and Z-teleportation *directly*, but can be constructed using other one-bit teleportation circuits. For instance, the controlled-Hadamard gate $\Lambda_1(H_2) \in C_3 \backslash C_2$ does not commute with E in Eq. (10) for all possible combinations of X and Z teleportation circuits, but $\Lambda_1(H_2)$ can be written as $G_b V G_a$ with $G_a = Q_2^\dagger$, $G_b = \Lambda_1(X_2) Q_2$, and $V = T_1 \Lambda_1(S_2^\dagger)$, where $Q = S^\dagger H S \in C_2$. Thus, $\Lambda_1(H_2)$ can still be performed using the general one-bit teleportation scheme.

We remark that a C_3 gate $U = G_b V G_a$ with G_a, $G_b \in C_2$, and $V \in F_3$ can be performed *indirectly* by applying G_a, V, and G_b in sequence, where V is applied by X teleportation. If the operations in the generalized one-bit teleportation circuit [$G_b, \Lambda_1(X)$, and G_a of Eq. (A3)] are also considered, the total requirements to perform U by such indirect implementation and by direct one-bit teleportation are almost the same. But if we are given different one-bit teleportation circuits as primitives, we can use them to directly teleport different sets of C_3 gates. In other words, if we are given the circuit of Eq. (11), using an input ancilla in the state $UA(|0\rangle^{\otimes n})$, we can teleport $U \in C_3$ that commutes with E; if we are given the circuit of Eq. (A3), using an input ancilla in the state $VH^{\otimes n}(|0\rangle^{\otimes n})$, we can teleport U in the form of $G_b V G_a$. In this sense, then, the generalized one-bit teleportation circuits are interesting and allow more gates in C_3 to be teleported directly.

APPENDIX B: FAULT-TOLERANT STATE PREPARATION

In this section, we first prove theorem 1 in Sec. III A by construction. We then show how to create the three ancilla states in Sec. IV fault-tolerantly. Finally, we explain how to prepare a class of encoded quantum states fault-tolerantly by recursive application of the one-bit teleportation scheme.

1. Fault-tolerant preparation of quantum states

A stabilizer of a quantum state is a quantum operator that transforms the state to itself. Let $\mathcal{C}$ be the codeword space corresponding to an $[[m,n]]$ stabilizer code, which encodes n logical qubits using m physical qubits. The stabilizer S of $\mathcal{C}$ is an Abelian subgroup of the Pauli group, or C_1, such that $|\psi\rangle \in \mathcal{C}$ if and only if $\forall\ M \in S, M|\psi\rangle = |\psi\rangle$. By performing error correction for the stabilizer code, we can project an arbitrary state onto an encoded state in $\mathcal{C}$ [8,2].

The stabilizer S has 2^{m-n} elements generated by $m-n$ independent operators in C_1, and defines a quantum code of dimension 2^n. Each encoded state is then determined by n

extra independent stabilizers. (In the following, we will restrict our discussion to the codeword space and exclude the stabilizers of the code from the stabilizers of an encoded state.) For instance, the encoded $|0\rangle^{\otimes n}$ is determined by Z_i for $i=1,\ldots,n$, where Z_i is the encoded Z on each logic qubit. In general, stabilizers need not commute with one another and need not square to the identity. But an independent set of stabilizers can always be chosen to be a mutually commuting set of elements that square to the identity. This is because $|\psi\rangle=U(|0\rangle^{\otimes n})$ for some encoded U, leading to a possible choice $\mathcal{S}(|\psi\rangle)\equiv\{UZ_iU^\dagger, i=1,\ldots,n\}$ with the desired properties. We restate the above as a lemma:

Lemma 1. For any $|\psi\rangle$, $\mathcal{S}(|\psi\rangle)$ can be chosen such that $\forall$ $M,N\in\mathcal{S}(|\psi\rangle)$, (a) $M^2=I$ and (b) $[M,N]=MN-NM=0$. Note that the elements in $\mathcal{S}(|\psi\rangle)$ are all valid encoded operations, and their actions preserve the codeword space.

As a quantum state is the simultaneous $+1$ eigenstate of its stabilizers, the state can be prepared by projecting an arbitrary encoded state onto the simultaneous $+1$ eigenstate of its stabilizers. In the following we will show how to create a class of quantum states fault-tolerantly by measuring their stabilizers.

Given a quantum state $|\psi\rangle$ encoded with an $[[m,n]]$ stabilizer code, the operator $M\in C_2$ with $M^2=I$ can be measured fault-tolerantly on $|\psi\rangle$ as follows. First, we prepare a cat state defined by

$$|\text{cat}\rangle\equiv\frac{1}{\sqrt{2}}(|\overline{0}\rangle+|\overline{1}\rangle), \tag{B1}$$

where $|\overline{i}\rangle$ consists of m physical qubits in the state $|i\rangle$ $(i=0,1)$. (The cat state cannot be created fault-tolerantly, but it can always be verified [1].) For the doubly even and self-dual CSS codes, the encoded $M\in C_2$ can be written as $M=M^1\otimes\cdots\otimes M^m$, where M^j acts only on the jth physical qubit of each block of the encoded state $|\psi\rangle$. For each j, we perform a controlled-M^j gate with the jth qubit of the cat state as the control bit and the j^{th} qubit of $|\psi\rangle$ as the target qubit. Effectively, a cat-state-controlled-M gate is applied to the state $|\text{cat}\rangle|\psi\rangle$ with transversal operations leading to the state

$$\frac{1}{\sqrt{2}}|\overline{0}\rangle|\psi\rangle+\frac{1}{\sqrt{2}}|\overline{1}\rangle M|\psi\rangle \tag{B2}$$

$$=\frac{1}{2}(|\overline{0}\rangle+|\overline{1}\rangle)(I+M)|\psi\rangle+\frac{1}{2}(|\overline{0}\rangle-|\overline{1}\rangle)(I-M)|\psi\rangle. \tag{B3}$$

Note that as $M^2=I$, $(I\pm M)|\psi\rangle$ are ±1 eigenstates of M for any $|\psi\rangle$.

We can measure the cat state fault-tolerantly using the procedure described in [1,2] to distinguish $|\overline{0}\rangle+|\overline{1}\rangle$ from $|\overline{0}\rangle-|\overline{1}\rangle$. (We omit the unimportant normalization factors.) If we obtain $|\overline{0}\rangle+|\overline{1}\rangle$, the encoded state is projected onto $(I+M)|\psi\rangle$, the $+1$ eigenstate of M; otherwise the resulting encoded state is $(I-M)|\psi\rangle$, the -1 eigenstate of M, which may be transformed to a $+1$ eigenstate of M by the following lemma.

Lemma 2. If $M\in C_2, M^2=I$, and there exists $Q\in C_2$ such that $\{M,Q\}=MQ+QM=0$, then we can always transform an arbitrary encoded state $|\psi\rangle$ onto a $+1$ eigenstate of M using fault-tolerant operations. The resulting $+1$ eigenstate is either $(I+M)|\psi\rangle$ or $(I+M)Q|\psi\rangle$, which can be written jointly as $(I+M)Q^a|\psi\rangle$ for $a=0$ or 1.

Proof. We have shown that we can project an arbitrary encoded state $|\psi\rangle$ onto $(I\pm M)|\psi\rangle$, the ±1 eigenstate of M. If the resulting state is $(I+M)|\psi\rangle$, we are done; otherwise, we apply $Q\in C_2$ fault-tolerantly to $(I-M)|\psi\rangle$. Since Q anticommutes with M, it transforms the -1 eigenstate of M to a $+1$ eigenstate of M as follows: $Q(I-M)|\psi\rangle=(I+M)(Q|\psi\rangle)$. Thus, we can always obtain a $+1$ eigenstate of M, which is $(I+M)Q^a|\psi\rangle$ for $a=0$ or 1.

Next we will show that a special class of quantum states can be created fault-tolerantly.

Lemma 3. If $\mathcal{S}(|\psi\rangle)=\{M_1,\ldots,M_n\}\subset C_2$ and $\forall$ $M_i\in\mathcal{S}(|\psi\rangle)$, there exists $Q_i\in C_2$ such that $\{M_i,Q_i\}=0$, and $[M_i,Q_j]=0$ for $i\neq j$; then $|\psi\rangle$ can be created fault-tolerantly by measuring the elements in $\mathcal{S}(|\psi\rangle)$ fault-tolerantly.

Proof. By Lemma 1, $\forall$ i, $M_i^2=I$. Starting from *any* encoded state $|\phi\rangle$, we measure $M_1,\ldots,M_n$ sequentially, and after each measurement we apply the corresponding operation Q_i if the projected state is the -1 eigenstate of M_i. By lemma 2, the resulting state is

$$|\psi\rangle=(I+M_n)Q_n^{a_n}\cdots(I+M_1)Q_1^{a_1}|\phi\rangle \tag{B4}$$

$$=(I+M_n)\cdots(I+M_1)Q_n^{a_n}\cdots Q_1^{a_1}|\phi\rangle, \tag{B5}$$

where $a_i=0$ or 1, and we have used the fact that $[M_i,Q_j]=0$ for $i\neq j$. As $[M_i,M_j]=0$, it is easily verified that $\forall i$, $M_i|\psi\rangle=|\psi\rangle$, and $|\psi\rangle$ is the desired state that has been created fault-tolerantly.

Theorem 1 in Sec. III A immediately follows.

Theorem 1. $\forall$ $U\in C_3$, U can be applied to the encoded $|0\rangle^{\otimes n}$ state using C_2 operators and fault-tolerant measurement of C_2 operators.

Proof. Applying $U\in C_3$ to the encoded $|0\rangle^{\otimes n}$ state is equivalent to preparing the state $|\psi\rangle=U(|0\rangle^{\otimes n})$, which has stabilizers $M_i=UZ_iU^\dagger$ for $i=1,\ldots,n$. Define $Q_i\equiv UX_iU^\dagger\in C_2$ for each i. Then $\{Z_i,X_i\}=0$ implies $\{M_i,Q_i\}=U\{Z_i,X_i\}U^\dagger=0$, and for $i\neq j$, $[Z_i,X_j]=0$ implies $[M_i,Q_j]=U[Z_i,X_j]U^\dagger=0$. Thus by lemma 3, the state $|\psi\rangle$ can be created fault-tolerantly.

2. Examples

To prepare a specific encoded state from an *unknown* encoded state, we need to measure *all* its independent stabilizers. When the initial state is a *known* encoded state related to the desired state, we may not have to measure all the independent stabilizers. For instance, given two encoded states $|\phi\rangle$ and $|\phi'\rangle$ with $\mathcal{S}(|\phi\rangle)=\{M_1,\ldots,M_k,M_{k+1},\ldots,M_n\}$ and $\mathcal{S}(|\phi'\rangle)=\{M_1,\ldots,M_k,M'_{k+1},\ldots,M'_n\}$, the state

$$(I+M_n)\cdots(I+M_{k+1})Q_n^{a_n}\cdots Q_{k+1}^{a_{k+1}}|\phi'\rangle \tag{B6}$$

is the simultaneous $+1$ eigenstate of M_i for $i=1,\ldots,n$. Thus, starting from $|\phi'\rangle$, we can prepare the encoded state $|\phi\rangle$ by measuring only the $n-k$ different stabilizers. In the following, we will construct an initial state, with which the desired state can be obtained by measuring only a single stabilizer.

Assume we want to prepare the encoded state $|\psi_+\rangle = U(|0\rangle^{\otimes n})$ for $U\in C_3$. Define M_i and Q_i for $i=1,\ldots,n$ as in the proof of theorem 1. Then $Q_i|\psi_+\rangle$ is a -1 eigenstate of M_i such that $\langle\psi_+|Q_i|\psi_+\rangle=0$, and the state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|\psi_+\rangle+Q_i|\psi_+\rangle) \tag{B7}$$

is different from $|\psi_+\rangle$ by only one independent stabilizer: Q_i has replaced M_i. Therefore, the state $|\psi\rangle$ also satisfies the conditions of lemma 3, and can be prepared fault-tolerantly. It follows that to obtain the state $|\psi_+\rangle$, we only need to measure the single stabilizer M_i on $|\psi\rangle$.

To prepare an encoded state $|\psi_+\rangle$ by preparing $|\psi\rangle$ first can be simpler than directly preparing $|\psi_+\rangle$ from an arbitrary encoded state if $|\psi\rangle$ itself can be prepared easily. For instance, when $|\psi\rangle$ is a product state, it can be prepared by measuring only single qubit operators. We will describe how to prepare the required ancilla states for the three gates in Sec. IV. When the required ancilla $|\psi_+\rangle$ is an entangled state with multiple-qubit stabilizers, we will construct it by preparing an easier state $|\psi\rangle$ first.

a. Fault-tolerant preparation of the ancilla required for the T gate

The required ancilla for constructing the $\pi/8$ gate, T, is

$$|\psi_+\rangle=TH|0\rangle=\frac{|0\rangle+e^{i\pi/4}|1\rangle}{\sqrt{2}}, \tag{B8}$$

with stabilizer

$$M=(TH)Z(TH)^\dagger=e^{-i\pi/4}SX, \tag{B9}$$

which anticommutes with $(TH)X(TH)^\dagger=Z$. Then starting from any encoded state, we can measure M, and apply Z if the projected state is the -1 eigenstate, to create the state $|\psi_+\rangle$ fault-tolerantly.

b. Fault-tolerant preparation of the ancilla required for the controlled-phase gate

The required ancilla for constructing the controlled-phase gate is

$$|\psi_+\rangle=\Lambda_1(S)(H_1\otimes H_2)|00\rangle \tag{B10}$$

$$=\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+i|11\rangle), \tag{B11}$$

with stabilizers $M_i=\Lambda_1(S)(H_1\otimes H_2)Z_i(H_1\otimes H_2)\Lambda_1(S^\dagger)$ for $i=1,2$. Using Eqs. (17) and (18),

$$M_1=(X_1\otimes S_2)\Lambda(Z), \tag{B12}$$

$$M_2=(S_1\otimes X_2)\Lambda(Z). \tag{B13}$$

The corresponding operator that anticommutes with M_i is $Q_i=\Lambda_1(S)(H_1\otimes H_2)X_i(H_1\otimes H_2)\Lambda_1(S^\dagger)=Z_i$ for $i=1,2$. $|\psi_+\rangle$ is an entangled state, and both M_1 and M_2 are two-qubit operators. But the state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|\psi_+\rangle+Q_1|\psi_+\rangle) \tag{B14}$$

$$=\frac{1}{\sqrt{2}}|0\rangle(|0\rangle+|1\rangle) \tag{B15}$$

is a product of single qubit states and has stabilizers Z_1 and X_2. Thus, we can first prepare $|\psi\rangle$ fault-tolerantly by measuring Z_1 and X_2, and then measure M_1 alone to get the state $|\psi_+\rangle$. Equivalently, we can also first prepare the state $(|\psi_+\rangle+Q_2|\psi_+\rangle)/\sqrt{2}=(|0\rangle+|1\rangle)|0\rangle/\sqrt{2}$, which has stabilizers X_2 and Z_1, and measure M_2 to obtain the state $|\psi_+\rangle$.

c. Fault-tolerant preparation of the required ancilla for the Toffoli gate

The required ancilla for the Toffoli gate construction is

$$|\psi_+\rangle=U(H_1\otimes H_2)|000\rangle \tag{B16}$$

$$=\frac{1}{2}(|000\rangle+|010\rangle+|100\rangle+|111\rangle), \tag{B17}$$

where U is the Toffoli gate. The stabilizer of this state is $M_i=U(H_1\otimes H_2)Z_i(H_1\otimes H_2)U^\dagger$ for $i=1$, 2, and 3. Using Eqs. (23) and (24),

$$M_1=X_1\otimes CNOT23, \tag{B18}$$

$$M_2=X_2\otimes CNOT13, \tag{B19}$$

$$M_3=Z_3\otimes CZ12, \tag{B20}$$

where CZ represents a controlled-Z, and the *ordered* subscripts for CNOT and CZ specify the control and target bits. The operator that anticommutes with M_i is $Q_i=U(H_1\otimes H_2)X_i(H_1\otimes H_2)U^\dagger$, or Z_1,Z_2 and X_3 for $i=1$, 2, and 3, respectively. Again, each of M_i is a two-qubit operator, but the state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|\psi_+\rangle+Q_1|\psi_+\rangle)=\frac{1}{\sqrt{2}}|0\rangle(|0\rangle+|1\rangle)|0\rangle \tag{B21}$$

can be prepared easily by measuring its stabilizers Z_1, X_2, and Z_3. Then we only need to measure a single two-qubit operator M_1 on $|\psi\rangle$ to obtain $|\psi_+\rangle$. Equivalently, we can also first prepare the state $|\psi\rangle=(1/\sqrt{2})(I+Q_2)|\psi_+\rangle$ with

stabilizers X_1, Z_2, and Z_3 or the state $|\psi\rangle=(1/\sqrt{2})(I+Q_3)|\psi_+\rangle$ with stabilizers X_1, X_2, and X_3, and measure the corresponding single stabilizer to obtain $|\psi_+\rangle$.

3. Recursive preparation

In this subsection, we will prove the following theorem, which is used in Sec. III C.

Theorem 2. The encoded state $|\psi\rangle=UH^{\otimes n}(|0\rangle^{\otimes n})$ for $U\in F_k$ can be prepared fault-tolerantly by recursive application of one-bit teleportation.

First we have the following lemma, which is a generalization of lemma 3.

Lemma 4. $|\psi\rangle$ can be created fault-tolerantly if given $\mathcal{S}(|\psi\rangle)=\{M_1,\ldots,M_n\}$, $\forall\ i,j$: (i) the cat-state-controlled M_i operation can be performed fault-tolerantly; (ii) there exists Q_i such that Q_i can be performed fault-tolerantly, $\{M_i,Q_i\}=0$, and for $i\neq j,[M_i,Q_j]=0$.

Proof. Since $M_i^2=I$, by applying the cat-state-controlled M_i operation and measuring the cat state fault-tolerantly as before, we can project any encoded state onto a ±1 eigenstate of M_i. Then apply Q_i if a -1 eigenstate is obtained. Using the same argument as in the proof of lemma 3, we can fault-tolerantly prepare the state $|\psi\rangle$.

Lemma 5. If operations in F_{k-1} and the cat-state-controlled-V operation for any $V\in F_{k-2}$ can be performed fault-tolerantly using the one-bit teleportation scheme, then $UH^{\otimes n}(|0\rangle^{\otimes n})$ for $U\in F_k$ can be created fault-tolerantly.

Proof. The stabilizers of $|\psi\rangle=UH^{\otimes n}(|0\rangle^{\otimes n})$ are $M_i=UH^{\otimes n}Z_i(UH^{\otimes n})^\dagger=UX_iU^\dagger=U_xX_i$ for some $U_x\in F_{k-1}$. Define $Q_i\equiv UZ_iU^\dagger=Z_i$. Q_i satisfies condition (ii) of lemma 4.

Since $M_i=U_xX_i$, the cat-state-controlled-M_i operation is the product of cat-state-controlled-U_x and cat-state-controlled-X_i operations. The cat-state-controlled-X_i operation is easily performed fault-tolerantly. Thus it remains to show how to perform the cat-state-controlled-U_x operation for $U_x\in F_{k-1}$ fault-tolerantly.

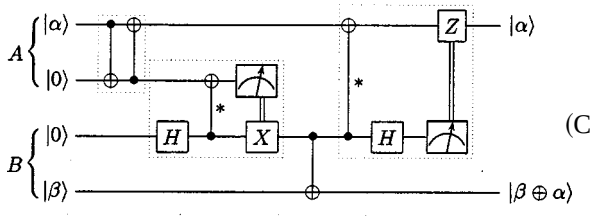
As $U_x\in F_{k-1}$ is constructed with the one-bit teleportation scheme using the circuit of Eq. (11), where $A_i=H$, $B_i=I$, and $D_i=X_i$, to perform the cat-state-controlled U_x operator, we need to perform the cat-state-controlled-E and cat-state-controlled $U_xX_iU_x^\dagger$ operations and to prepare the ancilla $U_xH^{\otimes n}(|0\rangle^{\otimes n})$ fault-tolerantly. As $E\in C_2$ and $U_xX_iU_x^\dagger=U_x'X_i$ with $U_x'\in F_{k-2}$, both the cat-state-controlled E and cat-state-controlled-$U_xD_iU_x^\dagger$ operations can be performed fault-tolerantly. Next, the state $U_xH^{\otimes n}|0\rangle^{\otimes n}$ has stabilizers $M_i'=U_xX_iU_x^\dagger=U_x'X_i$ with $U_x'\in F_{k-2}$, which satisfies both conditions of lemma 4 and can therefore be prepared fault-tolerantly. Thus the cat-state-controlled-U_x operation can be performed fault-tolerantly. This completes the proof of lemma 5.

In fact, what we have shown in the proof of lemma 5 is that if operations in F_{k-1} and the cat-state-controlled V operation for $V\in F_{k-2}$ can be performed fault-tolerantly, then the cat-state-controlled-U operation for $U\in F_{k-1}$ and operations in F_k can be performed fault-tolerantly. This is because according to Sec. III C, fault-tolerant construction of F_k gates only requires fault-tolerant F_{k-1} gates and an ancilla $UH^{\otimes n}(|0\rangle^{\otimes n})$ for $U\in F_k$.

Since both the operations in F_2 and the cat-state-controlled U operation for $U\in F_1$ can be performed fault-tolerantly, by induction, operations in F_k and the cat-state-controlled U operation for $U\in F_{k-1}$ can be constructed fault-tolerantly, with which we can fault-tolerantly prepare the encoded state $UH^{\otimes n}(|0\rangle^{\otimes n})$ for $U\in F_k$.

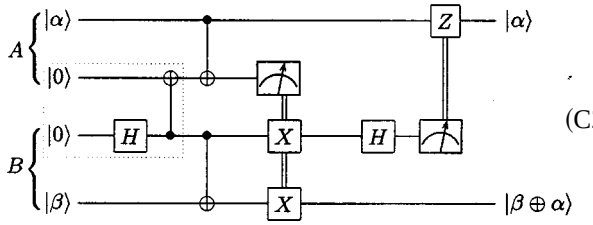
APPENDIX C: ALTERNATIVE REMOTE CNOT CIRCUIT

In this appendix, we rederive the remote CNOT construction, given in [8], using one-bit teleportation. A remote CNOT construction between the states $|\alpha\rangle$ and $|\beta\rangle$ belonging to Alice and Bob, respectively, can be performed by a four-step procedure: (i) Alice swaps her state $|\alpha\rangle$ with an ancilla $|0\rangle$, (ii) Alice sends the teleported $|\alpha\rangle$ to Bob using X teleportation, (iii) Bob applies CNOT construction from $|\alpha\rangle$ to $|\beta\rangle$, and (iv) Bob teleports $|\alpha\rangle$ back to Alice using Z teleportation. Steps (ii) and (iv) involve prohibited operations. Here is a circuit representation:



(C1)

The two prohibited CNOT gates are labeled with asterisks. They can be commuted backwards to obtain the equivalent circuit



(C2)

which again reduces prohibited operations to some specific shared entangled state.

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[17] We write $UEU^\dagger$ instead of $(U\otimes I^{\otimes n})E(U^\dagger\otimes I^{\otimes n})$ for simplicity. Unimportant identity operators are suppressed throughout the paper.

[18] Note that the n operators $UD_iU^\dagger$ to be performed conditioned on the measurement outcomes are precisely the n stabilizers of the ancilla state $UA(|0\rangle^{\otimes n})$. This coincidence is due to the structure of the teleportation circuits. See also the examples in Sec. IV.

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