Bužek-Hillery cloning revisited using the bures metric and trace norm

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Bužek and Hillery have recently analyzed the possibility of cloning imperfectly arbitrary states using a universal quantum cloning machine. They have analyzed their result using the Hilbert-Schmidt norm, although a better measure of fidelity is the Bures metric. In this paper, we repeat the Bužek and Hillery result using the Bures metric and show that we can still obtain the same result with an improved measure. However, unlike the Hilbert-Schmidt norm, this computation may be extended to the case of continuous variable, infinite-dimensional spaces, or even Hilbert spaces of larger dimensions. For completeness, a similar analysis using the trace norm is also performed.

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I. INTRODUCTION

Quantum information theory differs in many aspects from classical information theory. An important difference is the impossibility of reproducing arbitrary unknown pure quantum states faithfully [1]. For mixed states, such an impossibility condition continues to plague the quantum version and the proof has been provided by Barnum, Caves, Fuchs, Jozsa, and Schumacher [2]. Specifically, they have shown that if a mixed state is fed into a cloning device, it is still not possible for the reduced density matrices of the output to be identical to the input density matrix.

Despite the impossiblity of constructing perfect cloning machines, the no-cloning theorem does not forbid imperfect cloning. The idea of imperfect cloning is to duplicate a quantum state such that the fidelity between the reduced density matrices of the output is closed to the input density matrix. There are several ways of performing imperfect cloning and one such machine is the universal quantum cloning machine [3]. This machine has also been shown to be optimal [4,5] in the sense that the errors in the output can be minimized. Recently, a series of state-dependent cloning machines have also been proposed [6–8].

In Bužek and Hillery (BH) cloning, the fidelity between the input and output states were computed using the Hilbert-Schmidt norms. But the reduced density matrices for the output are generally mixed states. It is well-known that the Hilbert-Schmidt norm is generally not the preferred measure of distinguishing mixed states although it serves as a good measure of quantifying the distance between pure states. Computationally however, it is the most convenient one. Indeed, in quantum information theory, a better way of distinguishing mixed states and quantifying distance between different states is the Bures metric. However, as noted by Bužek and Hillery [3], it is generally more complicating to compute distances between different states using this metric. It is therefore the aim of this paper to re-analyze and compare the results in Bužek and Hillery cloning using other well-known measures such as Bures fidelity and trace norm.

In Sec. II, we briefly describe the various notions of distances for quantum states in a projective Hilbert space. We note that in an infinite-dimensional Hilbert space, the Hilbert-Schmidt distance or norm may not be an appropriate measure of distance between quantum states and it is necessary to introduce a more general concept of Bures fidelity or distance. In this section, we also introduce the idea of trace norm, which is sometimes used as a measure of nonclassical distance [9]. In Sec. III, we re-analyze BH cloning using Bures distance. It is reassuring to note that the result for BH cloning using Hilbert-Schmidt norm remains essentially the same even for Bures distance. A similar computation using the trace norm serves to reconfirm the general results. Finally, in Sec. IV, we reiterate our main points and summarize our findings.

II. BURES FIDELITY

A Hilbert space is a complete inner product space and it is therefore a normed space. Quantum states are objects in the Hilbert space and should therefore be distinguishable through a metric defined through a suitably chosen norm. The simplest choice of metric may not be the most suitable one, for instance the discrete metric [10].

For pure states, it is sufficient to employ the Fubini-Study distance defined in the projective Hilbert space of rays between two states Ψ_1 and Ψ_2 by the relation [11–13]

$$d_{\mathrm{FS}}^2(\Psi_1,\Psi_2)=\inf_{\phi_1,\phi_2}||\Psi_1 e^{i\phi_1}-\Psi_2 e^{i\phi_2}||^2, \tag{1}$$

where $\phi_i,(i=1,2)$ are the phases associated with the states Ψ_i. This definition is not appropriate for mixed states. Mixed states are described by density operators or matrices. These operators are trace class operators that also form a normed space. For two mixed states described by the density matrices ρ_1 and ρ_2, the distance can be defined by the trace norm or by the Hilbert-Schmidt norm. The trace norm is defined as

$$d_{\mathrm{trace}}=\mathrm{tr}\sqrt{(\rho_1-\rho_2)^\dagger(\rho_1-\rho_2)}. \tag{2}$$

This distance has been used to study the idea of nonclassical distance [9]. The Hilbert-Schmidt distance, on the other

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hand, is a more popular measure of distances and is in fact a generalization of Frobenius norm for finite matrices. The Hilbert-Schmidt distance between two mixed states associated with the density matrices ρ_1 and ρ_2 is defined by the relation

$$d^2_{\mathrm{HS}} = \mathrm{tr}\,(\rho_1-\rho_2)^\dagger\,(\rho_1-\rho_2). \tag{3}$$

More generally, if ρ_1 and ρ_2 are the density operators corresponding to two arbitrary mixed states, a good measure of the distance, $d(\rho_1,\rho_2)$, between the two states should satisfy the following properties:

(i) $d(\rho_1,\rho_2)\geqslant 0$ and $d(\rho_1,\rho_2)=0$ if and only if $\rho_1=\rho_2$;

(ii) $d(\rho_1,\rho_2)=d(\rho_2,\rho_1)$;

(iii) $d(\rho_1,\rho_2)+d(\rho_2,\rho_3)\leqslant d(\rho_1,\rho_3)$ (triangle inequality);

(iv) if ρ_1 is a pure state, $\rho_1=|\psi\rangle\langle\psi|$ then $d(\rho_1,\rho_2)=\langle\psi|\rho_2|\psi\rangle$.

Based on this general premise, Uhlmann and Hubner [14–17] showed that it is possible to construct a better formalism suitable for infinite-dimensional Hilbert spaces. This approach possesses elegant geometric properties [18] and its corresponding connection forms have been proposed as a generalization of Berry phase for mixed states. Under this general formalism, the distance between two mixed states, described by the density matrices ρ_1 and ρ_2, is defined by a generalized Fubini-Study distance through the relation

$$d^2_{\mathrm{Bures}}(\rho_1,\rho_2) = 2\,(1-\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}). \tag{4}$$

It is instructive to note that for pure states, the distances for mixed states all reduce to the original Fubini-Study distance.

Finally, we note that the term $\mathrm{tr}[\sqrt{\rho_1}\rho_2\sqrt{\rho_1}]^{1/2}\equiv F_{12}$ in Eq. (4) is also used as a measure of the distinguishability of quantum states and entanglement, being the analogous quantity of the modulus of the inner product for pure states [2,19–21].

III. BUŽEK-HILLERY CLONING

A. Case of Bures fidelity

In Bužek and Hillery (BH) cloning, one starts a set of rules regarding the cloning of basis states. In particular, it was assumed that the following transformation could be achieved:

$$|0\rangle_a|Q\rangle_x \to |0\rangle_a|0\rangle_b|Q_0\rangle_x + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b|0\rangle]|Y_0\rangle_x, \tag{5}$$

$$|1\rangle_a|Q\rangle_x \to |1\rangle_a|1\rangle_b|Q_1\rangle_x + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b|0\rangle]\,|Y_1\rangle_x, \tag{6}$$

where the copying machine states $\{|Q_i\rangle,|Y_i\rangle\}$, $i=0,1$, (Note that the subscrips x have been dropped for convenience) are not necessarily orthonormal and satisfy the conditions

$$\langle Q_i|Y_i\rangle = 0, \quad i=0,1, \tag{7}$$

$$\langle Q_0|Q_1\rangle = 0. \tag{8}$$

It can be shown easily that unitarity of the transformation imposes the condition that

$$\langle Q_i|Q_i\rangle + 2\langle Y_i|Y_i\rangle = 1, \quad i=0,1, \tag{9}$$

$$\langle Y_0|Y_1\rangle = \langle Y_1|Y_0\rangle = 0. \tag{10}$$

To reduce the number of free parameters, Bužek and Hillery further assumed that

$$\langle Y_0|Y_0\rangle = \langle Y_1|Y_1\rangle = \xi, \tag{11}$$

$$\langle Y_0|Q_1\rangle = \langle Q_0|Y_1\rangle = \langle Q_1|Y_0\rangle = \langle Y_1|Q_0\rangle = \frac{\eta}{2}. \tag{12}$$

Starting with the input state

$$|\psi\rangle_a = \alpha|0\rangle + \beta|1\rangle, \tag{13}$$

with $\alpha^2+\beta^2=1$, one can easily obtain the output density matrix $\rho^{(\mathrm{out})}_{ab}$ as

$$\begin{aligned}\rho^{(\mathrm{out})}_{ab} = {} & \alpha^2(1-2\xi)\,|00\rangle\langle 00| + \alpha\beta\eta\{|00\rangle\langle 01| + |00\rangle\langle 10| \\ & + |01\rangle\langle 00| + |01\rangle\langle 11| + |10\rangle\langle 00| + |10\rangle\langle 11| \\ & + |11\rangle\langle 01| + |11\rangle\langle 10|\} + \xi\{|01\rangle\langle 01| + |01\rangle\langle 10| \\ & + |10\rangle\langle 01| + |10\rangle\langle 10|\} + \beta^2\,(1-2\xi)|11\rangle\langle 11|.\end{aligned} \tag{14}$$

Tracing out mode b, one gets the reduced density operator for mode a, $\rho^{(\mathrm{out})}_a$ as

$$\begin{aligned}\rho^{(\mathrm{out})}_a = {} & [\alpha^2 + \xi(\beta^2-\alpha^2)]\,|00\rangle\langle 00| + \alpha\beta\eta\,(|01\rangle\langle 10| \\ & + |10\rangle\langle 01|) + [\beta^2 + \xi\,(\alpha^2-\beta^2)]|11\rangle\langle 11|.\end{aligned} \tag{15}$$

This is in general a mixed state. In fact, the entire cloning process is symmetric, namely, if one traces out mode a, the reduced density operator for mode b,

$$\rho^{(\mathrm{out})}_b,$$

is exactly the same as that for mode a. However, the two reduced density matrices are not individually similar to the input density matrix, so that this form of cloning is distorted and imperfect.

To reduce the distortion so that the difference between the input and output density matrices are as close as possible to each other, one needs to introduce an appropriate norm. In BH cloning, the Hilbert-Schmidt norm was used. The Hilbert-Schmidt norm was used because it is convenient and simple to compute. However, as shown in Sec. II, a better measure for mixed states is the Bures distance or fidelity. Indeed it is easy to diagonalize the density matrix $\rho^{(\mathrm{in})}_a$, as

$$\rho^{(\mathrm{in})}_a = U\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}U^{-1}, \tag{16}$$

where

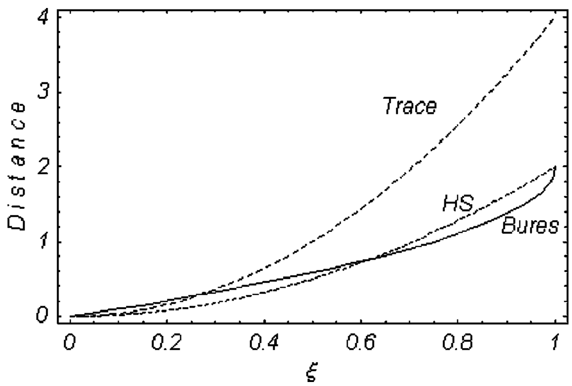

FIG. 1. A comparison of the distance between the input and output density matrices using the various measures: the Bures distance, the Hilbert-Schmidt distance and the trace norm for various values of ξ.

$$U=\begin{pmatrix} -\dfrac{\beta}{\alpha} & \dfrac{\alpha}{\beta} \\ 1 & 1 \end{pmatrix}. \tag{17}$$

A straightforward computation then gives the squared Bures distance D_a^2 between the input and output density operators $\rho_a^{(\mathrm{in})}$ and $\rho_a^{(\mathrm{out})}$ as

$$D_a^2=D_b^2=2-2\sqrt{1-\xi-2\alpha^2(\alpha^2-1)(2\xi+\eta-1)}. \tag{18}$$

As in the Bužek and Hillery paper [3], for an input independent cloning machine, we need to ensure that this distance D_a^2 is independent of the parameter α. To do this, one eliminates one of the parameters from the set $\{\xi,\eta\}$ by using the condition

$$\frac{\partial}{\partial\alpha^2}D_a^2=0. \tag{19}$$

From Eq. (18), we find that the above condition yields

$$\eta=1-2\xi, \tag{20}$$

so that the distance D_a^2 takes on the value

$$D_a^2=2-2\sqrt{1-\xi}. \tag{21}$$

It is interesting to note that the condition in Eq. (20) for the distance D_a^2 to be independent of the input parameter α remains the same whether we are using the Hilbert-Schmidt or Bures distance. However, the expression for the Bures distance in Eq. (21), differs from the expression for the distance based on the Hilbert-Schmidt norm. Figure 1 shows the distance graph of the distances computed using the Hilbert-Schmidt norm and the Bures metric. It is indeed interesting to observe that the two distances are never the same except at when $\xi=0,1$ or $(\sqrt{5}-1)/2$, the golden ratio. We can also compare the output density operator $\rho_{ab}^{(\mathrm{out})}$ with a tensor product of the two input density operators $\rho_a^{(\mathrm{in})}$ and $\otimes\rho_b^{(\mathrm{in})}$ using the Bures distance. Since

$$\begin{aligned}\rho_a^{(\mathrm{in})}\otimes\rho_b^{(\mathrm{in})}=&\,\alpha^4|00\rangle\langle 00|+\alpha^3\beta(|00\rangle\langle 01|+|00\rangle\langle 10|\\ &+|01\rangle\langle 00|+|01\rangle\langle 11|+|10\rangle\langle 00|+|10\rangle\langle 11|\\ &+|11\rangle\langle 01|+|11\rangle\langle 10|)\alpha^2\beta^2(|01\rangle\langle 01|\\ &+|01\rangle\langle 10|+|10\rangle\langle 01|+|10\rangle\langle 10|+|00\rangle\langle 11|\\ &+|11\rangle\langle 00|)+\beta^4|11\rangle\langle 11|,\end{aligned} \tag{22}$$

the Bures fidelity between the output density matrix and the tensor product of the input density matrix, namely, $\rho_{ab}^{(\mathrm{out})}$ and $\rho_a^{(\mathrm{in})}\otimes\rho_b^{(\mathrm{in})}$, respectively, can be computed using

$$F_{ab}=\ \mathrm{tr}\sqrt{\sqrt{\rho_a^{(\mathrm{in})}\otimes\rho_b^{(\mathrm{in})}}\rho_{ab}^{(\mathrm{out})}\sqrt{\rho_a^{(\mathrm{in})}\otimes\rho_b^{(\mathrm{in})}}}. \tag{23}$$

Using the Bures fidelity, the square of the Bures distance $D_{ab}^2\equiv 2(1-F_{ab})$ is given by

$$D_{ab}^2=2[1-\sqrt{1+\alpha^2(\alpha^2-1)(3-2\eta-10\xi)-2\xi}]. \tag{24}$$

If one then substitutes the condition in Eq. (21) for the single-mode output device to be independent of the parameters in the input device and demands the additional criterion that

$$\frac{\partial}{\partial\alpha^2}D_{ab}^2=0, \tag{25}$$

we see that $\xi=1/6$, as in the case of the Hilbert-Schmidt norm. However, the distance D_{ab}^2 attains the slightly larger value of $2-2\sqrt{2/3}$ compared with the value of 2/9 for the Hilbert-Schmidt case.

B. Case of trace norm

Aside from the Hilbert-Schmidt norm and Bures fidelity, one sometimes employs the trace norm for measuring distances of nonclassical states [9]. As noted in Ref. [9] and shown in Ref. [22] , the value of the trace norm must always be greater than the Hilbert-Schmidt norm. Since the trace norm involves the computation of the square root of an operator prior to taking the trace, unlike the Hilbert-Schmidt norm in which the process is somewhat reversed, it is generally more complicating to compute distances using the trace norm.

To compute the trace norm $d_{\mathrm{trace}}(\rho_1,\rho_2)$, one simply computes the square of the difference between the density matrices ρ_1 and ρ_2, that is $(\rho_1-\rho_2)^2$, extracts the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$, and then sums the square root of each eigenvalue so that $d_{\mathrm{trace}}=\sqrt{\lambda_1}+\sqrt{\lambda_2}+\cdots+\sqrt{\lambda_n}$.

Thus, following the method in Ref. [3], the distances D_a^2 (or D_b^2) between the input and output density matrices $\rho_a^{(\mathrm{in})}$ and $\rho_a^{(\mathrm{out})}$ can be found to be

$$D_a^2=D_b^2=4\{\xi^2+\alpha^2(1-\alpha^2)[(1+\eta)^2-\xi^2]\}, \tag{26}$$

so that for an input-state-independent cloning machine, one gets $\eta=1-2\xi$ as before. It is interesting to note that this distance is exactly twice the magnitude of the analogous distance obtained in Ref. [3] for the Hilbert-Schmidt norm. It is therefore not surprising that the distance between the input and output density matrices, D_a^2, computed using the trace norm is $4\xi^2$. Incidentally, the graphs for the Bures distance and the trace norm intersect precisely at the least positive root of $4\xi^3-4\xi+1=0$, that is $\xi\approx0.2695$. The determination of the optimum value of ξ from the assumption that the distance between the two-mode density operators $\rho_{ab}^{(\mathrm{out})}$ and $\rho_a^{(\mathrm{in})}\otimes\rho_b^{(\mathrm{in})}$ be input-state-independent is more involved, but the result is similar to the Hilbert-Schmidt norm with the optimal value of ξ being 1/6 and the distance D_{ab}^2 being 4/9.

IV. DISCUSSION AND CONCLUSION

The Bures metric or distance has generally been regarded as a better measure of distinguishability of quantum states [17]. Nevertheless, it is also more tedious and difficult to compare density matrices using the Bures metric. Indeed, in the original proposal by Bužek and Hillery, only the Hilbert-Schmidt (HS) metric was employed. For finite-dimensional problems, one does not expect drastically different results through the use of a different metric. However, there is also no reason to assume that the optimal value for ξ should continue to be 1/6 with a different metric.

In this paper, we repeat quantum cloning using the Bures metric (and trace norm) and show that the results are essentially the same, namely, that for optimal cloning, the value of ξ should take the value 1/6. Moreover, although the corresponding values of the distances, namely, D_a^2 and D_{ab}^2, differ in absolute terms, the condition [Eq. (20)], for the norm to be independent of the parameter α, namely, $\eta=1-2\xi$, remains the same for all the different norms. Our results are consistent with the previous results in Ref. [3]. Thus, although the Hilbert-Schmidt norm is not the best norm available for comparing density matrices, it is generally a sufficiently appropriate norm.

In fact, the same analysis using the trace norm shows that the condition for the norm to be independent of the parameter α is also the same as Eq. (20) so that the distance D_a^2 assumes the value of $4\xi^2$, a result consistent with the well-known relation that $d_{\mathrm{trace}}>d_{\mathrm{HS}}$ [22].

The trace norm is not always a useful measure since the computation of the square root prior to the trace is generally harder than the Hilbert-Schmidt norm. Moreover, if the computation of square roots of a large matrices can be done readily, then a better measure would be the Bures fidelity. This is because the Bures fidelity lends itself more naturally to the formalism of positive operator-valued measurements [19]. Using the trace norm, the value of ξ for the minimization of D_{ab}^2 is still 1/6 with $D_{ab}^2=4/9$. Indeed, from the results, one would like to surmise that Eq. (19) seems to lead to the same condition in Eq. (20) so that the condition in Eq. (20) is norm-independent.

It is well known that the Bures fidelity reduces to the Hilbert-Schmidt norm for pure states. Moreover, for the standard optical coherent states $|\alpha\rangle$, defined by

$$|\alpha\rangle\equiv D(\alpha)|0\rangle=\mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{27}$$

the Hilbert-Schmidt norm or equivalently, the Bures fidelity for two pure states $|\alpha\rangle$ and $|\alpha+1\rangle$ is always a constant. This result appears to contradict our intuitive perception of coherent states. Experimentally, we should be able to distinguish physically between the ground (vacuum) state $\alpha=0$, and the low-photon excited state $\alpha=1$, but not for large α in which the two coherent states $|\alpha\rangle$ and $|\alpha+1\rangle$ describe macroscopically indistinguishable states. Recently, there have been some proposals to distinguish states using other means, for instance the mutual information measures [23]. It would be interesting to compare and contrast advantages and disadvantages associated with these new measures for BH cloning.

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