Quantum computation using decoherence-free states of the physical operator algebra

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The states of the physical algebra, namely the algebra generated by the operators involved in encoding and processing qubits, are considered instead of those of the whole-system algebra. If the physical algebra commutes with the interaction Hamiltonian, and the system Hamiltonian is the sum of arbitrary terms either commuting with or belonging to the physical algebra, then its states are decoherence free. One of the examples considered shows that, for a uniform collective coupling to the environment, the smallest number of physical qubits encoding a decoherence-free logical qubit is reduced from four to three.

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I. INTRODUCTION

Environment-induced decoherence [1–3] is the main obstruction to the physical viability of quantum computing [4]. To overcome this obstacle, quantum error correcting codes have been devised [5,4]. Besides these *active* methods, where decoherence is controlled by repeated application of error correction procedures, a more recent *passive* approach has emerged, where logical qubits are encoded in decoherence-free (DF) subspaces [6–11]. In them coherence is protected by the peculiar structure of the coupling Hamiltonian.

So far the notion of a DF state has been considered within the total Hilbert space of the considered system, namely with reference to the whole operator algebra of the system, whereas a more physical approach consists of confining the consideration to the space of the states on the physical algebra, that is the operator algebra involved in encoding and manipulating qubits. The characterization of such state spaces corresponds to the construction of the irreducible representations of the aforementioned algebra. Quantum computing without active error correcting codes requires the use of physical algebras admitting DF irreducible representations, which therefore will be called DF algebras. The construction of such representations is performed here by showing that suitable factorizations of the total Hilbert space exist, where entanglement with the environment (or equivalently decoherence, once this is traced out) is confined to only one factor, the other factor carrying an irreducible representation of the DF algebra.

This more physical approach leads to a fruitful generalization of the notion of a DF state. It is shown, for instance, that for a generic uniform coupling of an array of physical qubits to an arbitrary environment, while the conventional notion of DF space requires at least four physical qubits to encode a logical one [12], three are enough in this new setting.

As to the plan of the paper, since it is addressed to a wide range of theoreticians and experimentalists, the general notion of a quantum state as a functional on a given C^* algebra, instead of a density matrix on a preassigned Hilbert space, is briefly introduced in Sec. II. This is done with explicit reference to the ensuing relativity of the notion of state purity, which is illustrated by the simplest possible example.

In Sec. III the concept of a decoherence-free algebra is presented with reference to a generic system, its Hamiltonian, and its coupling to the environment. In particular, the mentioned example and arrays of qubits uniformly coupled to the environment are considered.

Then in Sec. IV specific examples of three and four qubit arrays are analyzed, giving explicit realizations of the DF algebras in terms of the original physical qubit operators. In particular, it is shown how the present generalized pure states allow for the aforementioned DF logical qubit with only three physical ones, while four physical qubits are shown to encode, in addition to the known DF logical qubit [12], a DF logical qutrit.

Finally some concluding remarks follow in Sec. V.

II. C^* ALGEBRAS AND THEIR PURE STATES

A quantum physical system is characterized by a C^* algebra, namely a normed complex associative algebra $\mathcal{A}$ with conjugation * and unity **1**, whose Hermitian elements are its observables, corresponding in the usual operator setting to Hermitian bounded operators [13]. Conjugation is an antilinear involution

$$*: A \in \mathcal{A} \mapsto A^* \in \mathcal{A}, (A^*)^* = A,$$

$$(cA+B)^* = \bar{c}A^* + B^* \forall c \in C, \tag{1}$$

such that

$$(AB)^* = B^*A^*, \tag{2}$$

and the norm, endowing $\mathcal{A}$ with the structure of a Banach space, is such that

$$\|AB\| \leqslant \|A\|\|B\|, \quad \|A^*\| = \|A\|, \quad \|AA^*\| = \|A\|^2, \quad \|\mathbf{1}\| = 1. \tag{3}$$

Boundedness is not a severe restriction, since every measurement apparatus can detect only a finite range of values of

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an unbounded observable, by which these observables play only a formal role as generators of groups of unitary operators and can be eliminated altogether as primary physical objects.

The states of the system correspondingly are positive and normalized linear functionals,

$$f:\mathcal{A}\rightarrow C, \quad f(cA+B)=cf(A)+f(B)\forall c\in C,$$
$$f(AA^*)\geqslant 0, \quad f(\mathbf{1})=1. \tag{4}$$

States that can be written as linear convex combinations of different states

$$f=\alpha g+(1-\alpha)h, \quad f\neq g\neq h, \quad 0<\alpha<1 \tag{5}$$

are mixed states; otherwise they are pure.

Given a pure state f one can uniquely construct, by the GNS procedure [13], a Hilbert space whose one-dimensional projectors are pure states (including the initial one), giving an irreducible representation of $\mathcal{A}$ in terms of bounded operators. This procedure is the C^* algebra counterpart of the Lie algebraic construction by raising and lowering operators. The mentioned Hilbert space is identified with (the completion of) the space of equivalence classes $\tilde{A}$ of elements A of $\mathcal{A}$, with respect to the equivalence relation

$$\tilde{A}=\tilde{B}\Leftrightarrow f([A^*-B^*][A-B])=0, \tag{6}$$

where $\tilde{A}$ denotes the equivalence class of $A, A\tilde{B}\equiv\widetilde{AB}$, the inner product is given by

$$\langle\tilde{A}|\tilde{B}\rangle\equiv f(A^*B), \tag{7}$$

and transition amplitudes by [16]

$$\langle\tilde{A}|C|\tilde{B}\rangle=f(A^*CB). \tag{8}$$

In general such a Hilbert space may not span the whole set of states, namely inequivalent representations may ensue, starting from different states. When this happens superselection rules are present, i.e., no observable connects states belonging to inequivalent representations.

While superselection rules usually arise only in connection with infinitely many degrees of freedom when $\mathcal{A}$ is defined in the usual way, as corresponding, for instance, to all possible measurements on a given set of particles, this is not so if somehow it is restricted. In such a case the restricted algebra, which is called here the physical algebra, may have several different (in general) reducible, representations inside the Hilbert space corresponding to the unrestricted algebra.

The main idea in the present paper is to exploit the freedom in choosing the physical algebra with reference to the notion of state pureness. A mixed state of the whole algebra may be a pure state when restricted, as a functional, to the physical algebra. In fact the pure states of the physical algebra can be identified with equivalence classes of (in general not pure) density matrices in the mentioned reducible representations. This may lead to the physical equivalence between a nonunitary evolution of the system state in the usual sense (once the environment has been traced out) and a unitary evolution with respect to the physical algebra if this one is properly chosen.

To give the simplest possible illustration of the foregoing idea, consider a two qubit system in the usual sense, namely a system consisting of two atomic two-state systems. The corresponding operator algebra is generated by

$$\sigma_j\otimes\sigma_k, \quad j,k=0,1,2,3, \tag{9}$$

where $\sigma_1,\sigma_2,\sigma_3$ denote Pauli operators of a single atomic system and σ_0 is the corresponding identity operator. The usual product basis of the state space is given by

$$|j,k\rangle\equiv|j\rangle\otimes|k\rangle, \quad j,k=\pm 1 \tag{10}$$

where

$$\sigma_3|j\rangle=j|j\rangle. \tag{11}$$

On the other hand, the operators

$$\pi_1\equiv 1\otimes\sigma_1, \quad \pi_2\equiv\sigma_3\otimes\sigma_2, \quad \pi_3\equiv\sigma_3\otimes\sigma_3,$$
$$\tau_1\equiv\sigma_2\otimes\sigma_1, \quad \tau_2\equiv\sigma_3\otimes 1, \quad \tau_3\equiv\sigma_1\otimes\sigma_1 \tag{12}$$

obey the same commutation relations

$$[\pi_j,\pi_k]_-=2i\varepsilon_{jkl}\pi_l, \quad [\tau_j,\tau_k]_-=2i\varepsilon_{jkl}\tau_l,$$
$$[\tau_j,\pi_k]_-=0, \tag{13}$$

as the single physical qubit operators

$$1\otimes\sigma_1, \quad 1\otimes\sigma_2, \quad 1\otimes\sigma_3,$$
$$\sigma_1\otimes 1, \quad \sigma_2\otimes 1, \quad \sigma_3\otimes 1. \tag{14}$$

Furthermore, since $[\tau_j,\tau_k]_+=[\pi_j,\pi_k]_+=2\delta_{jk}$, the Casimir operators $\pi_1^2+\pi_2^2+\pi_3^2$ and $\tau_1^2+\tau_2^2+\tau_3^2$ assume the same value 3 as the Casimir operator $\sigma_1^2+\sigma_2^2+\sigma_3^2$ of the operator algebra corresponding to a single traditional qubit. One can then identify the operator algebra of the two qubit array with the direct product of the two alternative $gl(2,C)$ algebras generated, respectively, by the π and the τ operators.

Similarly the state space of the two qubit array can be realized as the tensor product of two irreducible representations of the two alternative $gl(2,C)$ algebras, which can immediately be built by the GNS construction.

Consider, for instance, the state f of the π algebra uniquely defined by

$$f(\pi_3)=-1, \quad f(\pi_1)=f(\pi_2)=0. \tag{15}$$

Then the equivalence classes of $\mathbf{1}$ and $\pi_+=(\pi_1+i\pi_2)/2$ give the usual basis

$$\tilde{\mathbf{1}}=|-1), \quad \tilde{\pi}_+=|1), \quad \pi_3|k)=k|k), \tag{16}$$

as

$$(1|\pi_3|1)=f(\pi_-\pi_3\pi_+)=f\left(\frac{\mathbf{1}-\pi_3}{2}\right)=1,$$

$$(1|\pi_{1,2}|1)=f(\pi_-\pi_{1,2}\pi_+)=0, \tag{17}$$

and of course

$$\tilde{A}=\tilde{B}\Leftrightarrow A-B=c_1\pi_-+c_2(\mathbf{1}+\pi_3);$$

$$c_1,c_2\in C,\pi_-\equiv\pi_+^*. \tag{18}$$

If the analogous notation is used for the τ algebra, one easily gets the identification

$$|1,1)=\frac{1}{\sqrt{2}}(|1,1\rangle+|-1,-1\rangle),$$

$$|1,-1)=\frac{1}{\sqrt{2}}(|1,1\rangle-|-1,-1\rangle),$$

$$|-1,1)=\frac{1}{\sqrt{2}}(|1,-1\rangle+|-1,1\rangle),$$

$$|-1,-1)=\frac{1}{\sqrt{2}}(|1,-1\rangle-|-1,1\rangle), \tag{19}$$

where

$$|j,k)\equiv|j)\otimes|k),\quad \pi_3|j,k)=j|j,k),\quad \tau_3|j,k)=k|j,k). \tag{20}$$

Assume now that the physical algebra is restricted to the one generated by the π operators. Then, for instance, the state

$$\rho=\frac{|1,1)(1,1|+|1,-1)(1,-1|}{2}$$

$$=|1)(1|\otimes\left(\frac{|1)(1|+|-1)(-1|}{2}\right) \tag{21}$$

is trivially a pure state when restricted to the physical algebra, while its expression in the original basis

$$\rho=\frac{|1,1\rangle\langle 1,1|+|-1,-1\rangle\langle -1,-1|}{2} \tag{22}$$

is entangled and then neither pure with respect to the first physical qubit operator algebra nor to the second one.

III. DECOHERENCE-FREE ALGEBRAS

Consider now the dynamics of a system S coupled to a bath B, the *universe* evolving unitarily under the Hamiltonian $H=H_S\otimes\mathbf{1}_B+\mathbf{1}_S\otimes H_B+H_I$, where H_S and H_B denote, respectively, the system and the bath Hamiltonian, H_I the interaction Hamiltonian, $\mathbf{1}_S$ and $\mathbf{1}_B$ the identity operators on the Hilbert space $\mathcal{H}_S$ of the system and $\mathcal{H}_B$ of the bath, respectively. Let $\mathcal{A}_S\equiv gl(\mathcal{H}_S)$ denote the operator algebra of $\mathcal{H}_S$ (which for simplicity is assumed to be finite dimensional) and $\mathcal{A}_{DF}$ the invariant subalgebra of $\mathcal{A}_S$ consisting of operators commuting with H_I,

$$[\mathcal{A}_{DF},H_I]=0. \tag{23}$$

As a subalgebra of $\mathcal{A}_S$, $\mathcal{A}_{DF}$ has a natural C^* algebra structure, by which, if measurements on the system are confined to those represented by operators in $\mathcal{A}_{DF}$, state spaces can be identified with its irreducible representations.

(While the GNS construction gives a general procedure to construct the representation of $\mathcal{A}_{DF}$ containing a given state of $\mathcal{A}_{DF}$ and, as described below, it is closely connected with what the experimentalist is expected to do in the present context, representations of $\mathcal{A}_{DF}$ in the final examples will be defined explicitly in terms of physical qubit operators.)

A pure state of $\mathcal{A}_{DF}$, namely a state prepared by a complete set of measurements of $\mathcal{A}_{DF}$, such remains under time evolution, if the system Hamiltonian is the sum of an operator belonging to $\mathcal{A}_{DF}$, giving rise to unitary evolution, and an operator that commutes with $\mathcal{A}_{DF}$, which for such a state gives rise to no evolution at all.

As the simplest nontrivial example consider the above two qubit system, assuming that

$$H_S=\sum_{j=1}^{3}\alpha_j\pi_j+\sum_{j=1}^{3}\beta_j\tau_j,\quad \alpha_j,\beta_j\in C,$$

$$H_I=\sum_{j=1}^{3}B_j\tau_j, \tag{24}$$

where π and τ operators are defined in Eq. (12) and B_j denote bath operators; in this case $\mathcal{A}_{DF}$ is the $gl(2,C)$ algebra generated by π operators. Then, for a product state

$$\rho=|\psi)(\psi|\otimes\rho_\tau, \tag{25}$$

the interaction with the environment has no effect on the evolution of the first factor, which then has a unitary evolution even though time evolution of ρ, and specifically of ρ_τ, is not unitary. It should be stressed that while this appears to be rather trivial in terms of π and τ operators, it is quite hidden if the state and the Hamiltonians are expressed in terms of the original physical qubit operators σ.

In order to pass from an *ad hoc* example to a physically more relevant and general setting, consider an array of N qubits. Let $\sigma_0,\sigma_1,\sigma_2,\sigma_3$ be defined as above. If these matrices are intended to be, as usual, representations of pseudospin Hermitian operators in the single qubit state space, the operator algebra for the whole array is generated by

$$M(i_1,i_2,\ldots,i_N)\doteq\bigotimes_{j=1}^{N}\sigma_{i_j},\quad i_j=0,1,2,3. \tag{26}$$

Let

$$S_i=\frac{1}{2}\sum_{j=1}^{N}M(i\delta_{1j},i\delta_{2j},\ldots,i\delta_{Nj}) \tag{27}$$

denote the total pseudospin, where δ_{jk} is the Kronecker symbol, and assume, as frequently done in the literature [8], a uniform collective coupling to the environment

$$H_I=\sum_{i=1}^{3} S_iB_i, \tag{28}$$

where the bath operators B_i commute with $\mathcal{A}_S$ and then with $\mathcal{A}_{DF}$. As to the system Hamiltonian, under the usual hypothesis of equivalent uncoupled qubits [8]

$$H_s=\varepsilon S_3, \tag{29}$$

it commutes with $\mathcal{A}_{DF}$, which, as said above, avoids decoherence of states of $\mathcal{A}_{DF}$, even with the possible addition of terms belonging to $\mathcal{A}_{DF}$, like the scalar couplings

$$\sum_{i=1}^{3} M(i_1=0,i_2=0,\ldots,i_j=i,\ldots,i_k=i,\ldots,i_N=0) \tag{30}$$

due to the exchange interaction present in NMR computing [14]. Let $\mathcal{A}_E$ denote the algebra generated by the *errors* S_i. Of course $\mathcal{A}_{DF}\cap\mathcal{A}_E$ is generated by the identity and by the Casimir operator

$$S^2=\sum_{i=1}^{3} S_i^2, \tag{31}$$

by which, in order to factor the operator algebra as a product of such subalgebras, the state space must be reduced to an S^2 eigenspace. To this end the system Hilbert space $\mathcal{H}_S$, as the tensor product of N fundamental representations of $sl(2,C)$, can be decomposed as the Clebsch-Gordan sum of irreducible representations of the algebra $sl(2,C)$ generated by the operators S_i,

$$\mathcal{H}_S=\bigoplus_j \bigoplus_{k=1}^{n_j} \mathcal{D}_j, \tag{32}$$

where the index j fixes the eigenvalue of the Casimir operator: $S^2\mathcal{D}_j=j(j+1)\mathcal{D}_j$.

The operator algebra of the generic eigenspace of S^2 can be identified with the product of the representations of the DF and the error algebras on $\oplus_{k=1}^{n_j}\mathcal{D}_j$,

$$_j\mathcal{A}_S\equiv gl\left(\bigoplus_{k=1}^{n_j}\mathcal{D}_j\right)\sim \ _j\mathcal{A}_{DF}\otimes_j\mathcal{A}_E. \tag{33}$$

In fact the S^2 eigenspace in its turn can be identified with the direct product of an n_j-dimensional complex space and just one copy of the irreducible representation

$$\bigoplus_{k=1}^{n_j}\mathcal{D}_j\sim C^{n_j}\otimes\mathcal{D}_j \tag{34}$$

through the one-to-one correspondence $|k,m\rangle\leftrightarrow|k\rangle\otimes|m\rangle$, where $|k,m\rangle$ denotes the eigenvector of S_3 with eigenvalue m in the kth copy of $\mathcal{D}_j$, while $|m\rangle$ denotes the only such eigenvector in $\mathcal{D}_j$ and $|k\rangle$ is the kth element of a basis of C^{n_j}. To be more precise, once the mutually orthogonal vectors $|k,j\rangle$ are fixed, one defines $|k,m\rangle\equiv(S_-)^{j-m}|k,j\rangle$ by means of the lowering operator $S_-=S_1-iS_2$.

Since the generic operator O on C^{n_j} gives through this identification an operator $O\otimes\mathbf{1}_{\mathcal{D}_j}$ on $\oplus_{k=1}^{n_j}\mathcal{D}_j$ commuting with $\mathcal{A}_E$, which is generated by operators of the form $\mathbf{1}_{C^{n_j}}\otimes Q$, and since all operators can be realized in terms of the operators $M(i_1,i_2,\ldots,i_N)$, it follows that operators on C^{n_j} can be identified with (equivalence classes of) elements of $\mathcal{A}_{DF}$. This proves that the generic S^2 eigenspace can be identified with the product of two spaces, carrying irreducible representations of $\mathcal{A}_{DF}$ and $\mathcal{A}_E$, respectively. It should be stressed that coherent superpositions of S^2 eigenstates with different eigenvalues do not exist as states of $\mathcal{A}_{DF}$, as they live in different representations.

As to the operational method to construct the $S^2=j(j+1)$ representation of $\mathcal{A}_{DF}$, it is just the physical translation of the, only seemingly formal, GNS procedure. To be specific, first an arbitrary (mixed or pure) state of the chosen S^2 eigenspace has to be prepared. Then a complete set of measurements corresponding to Hermitian elements of $\mathcal{A}_{DF}$ is performed in order to select a pure state of $\mathcal{A}_{DF}$. Finally the whole representation is spanned by arbitrary unitary evolution generated by Hamiltonian operators belonging to $\mathcal{A}_{DF}$. Of course here unitarity is referred to $\mathcal{A}_{DF}$ only, since the coupling with the environment is simultaneously producing, in general, a nonunitary evolution of the whole-system algebra $\mathcal{A}_S$ and, to be more specific, of the $S^2=j(j+1)$ representation of $\mathcal{A}_E$. Finally, it is worth remarking that possible terms in the system Hamiltonian belonging to $\mathcal{A}_E$ give rise to a further unitary evolution in $\mathcal{A}_S$, which in the present context is physically irrelevant since it does not affect $\mathcal{A}_{DF}$.

IV. EXAMPLES

As a first example of a qubit array, collectively and uniformly coupled to the environment, consider a system of three physical qubits. The corresponding DF algebra is generated by

$$b_{23}\doteq 4\vec{S}^2\cdot\vec{S}^3=\sum_{j=1}^{3} 1\otimes\sigma_j\otimes\sigma_j,$$

$$b_{31}\doteq 4\vec{S}^3\cdot\vec{S}^1=\sum_{j=1}^{3} \sigma_j\otimes 1\otimes\sigma_j,$$

$$b_{12}\doteq 4\vec{S}^1\cdot\vec{S}^2=\sum_{j=1}^{3} \sigma_j\otimes\sigma_j\otimes 1, \tag{35}$$

where $\vec{S}^j$ denotes the pseudospin vector of the jth qubit, and the Clebsch-Gordan decomposition in Eq. (32) reads

$$\mathcal{H}_S=\mathcal{D}_{3/2}\oplus\mathcal{D}_{1/2}\oplus\mathcal{D}_{1/2}=\mathcal{H}_{3/2}\oplus\mathcal{H}_{1/2}. \tag{36}$$

Since the factorization of ${}_j\mathcal{A}_S$ in Eq. (33) is trivial for $S^2=15/4(j=3/2)$, as the error algebra generates the whole operator algebra, the analysis is confined to the eigenspace $\mathcal{H}_{1/2}$ with $S^2=3/4$.

One can now apply the general GNS procedure. As a starting point take the pure state of $\mathcal{A}_{DF}$ corresponding to an arbitrary normalized vector of $\mathcal{H}_{1/2}$. The ensuing Hilbert space of equivalence classes of elements of $\mathcal{A}_{DF}$, according to Sec. II, gives the looked for representation of $\mathcal{A}_{DF}$. (Of course even a density matrix on $\mathcal{H}_{1/2}$ that, as a state of $\mathcal{A}_{DF}$, is a pure state, can be taken as an equivalent starting point.) While this procedure can be applied in principle to much more general cases than the present qubit array, the final result given below can easily be checked directly [17].

Using the symbol ${}_{1/2}O$ for the representation of the generic operator O in $\mathcal{H}_{1/2}$, for instance it can be checked that, if one defines the invariant operator

$$E_{123}\doteq\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\sigma_i\otimes\sigma_j\otimes\sigma_k, \tag{37}$$

with ε_{ijk} denoting the usual completely antisymmetric symbol, the $\mathcal{H}_{1/2}$ representations of invariant operators

$$\begin{aligned}{}_{1/2}\tau_1&=({}_{1/2}b_{12}-{}_{1/2}b_{23})/\sqrt{12},\\ {}_{1/2}\tau_2&={}_{1/2}E_{123}/\sqrt{12},\\ {}_{1/2}\tau_3&=({}_{1/2}b_{23}-2\,{}_{1/2}b_{31}+{}_{1/2}b_{12})/6,\end{aligned} \tag{38}$$

are the generators of an $su(2)$ algebra,

$$[{}_{1/2}\tau_i,{}_{1/2}\tau_j]=2i\sum_{k=1}^{3}\varepsilon_{ijk}\,{}_{1/2}\tau_k, \tag{39}$$

with the Casimir given by

$${}_{1/2}\tau^2\equiv\sum_{j=1}^{3}{}_{1/2}\tau_j^2=3\hat{\mathbf{1}}. \tag{40}$$

The corresponding universal enveloping algebra $\mathcal{A}({}_{1/2}\boldsymbol{\tau})$, which coincides with ${}_{1/2}\mathcal{A}_{DF}$, is then the operator algebra of a two-state system and the total operator algebra ${}_{1/2}\mathcal{A}$ is given by the product of this algebra and the universal enveloping algebra $\mathcal{A}({}_{1/2}\mathbf{S})$ of the total pseudospin algebra,

$${}_{1/2}\mathcal{A}=\mathcal{A}({}_{1/2}\boldsymbol{\tau})\otimes\mathcal{A}({}_{1/2}\mathbf{S})={}_{1/2}\mathcal{A}_{DF}\otimes\mathcal{A}({}_{1/2}\mathbf{S}), \tag{41}$$

as a particular instance of Eq. (33). As a consequence the state space $\mathcal{H}_{1/2}$ can be identified with the tensor product of two two-dimensional representation spaces $\mathcal{H}_{1/2}(\boldsymbol{\tau})$ and $\mathcal{H}_{1/2}(\mathbf{S})$, respectively, of $\mathcal{A}({}_{1/2}\boldsymbol{\tau})$ and $\mathcal{A}({}_{1/2}\mathbf{S})$,

$$\mathcal{H}_{1/2}=\mathcal{H}_{1/2}(\boldsymbol{\tau})\otimes\mathcal{H}_{1/2}(\mathbf{S}), \tag{42}$$

which coincides with Eq. (34) for $j=1/2$ and $n_j=2$. According to what has been illustrated above, this factorization has far reaching physical consequences: if all measurement processes are limited to (Hermitian) elements of $\mathcal{A}_{DF}$, then a state $\rho=|\psi\rangle\langle\psi|\otimes\rho_{\mathbf{S}}$, which is the product of a pure state in $\mathcal{H}_{1/2}(\boldsymbol{\tau})$ and an arbitrary density matrix in $\mathcal{H}_{1/2}(\mathbf{S})$, is a pure state of the physical algebra $\mathcal{A}_{DF}$. If in particular the initial state has this structure [possibly with $\rho_{\mathbf{S}}$ being itself a pure state of $\mathcal{A}({}_{1/2}\mathbf{S})$, this corresponding to an arbitrary pure state in $\mathcal{H}_{1/2}$], then, in spite of the decoherence of $\rho_{\mathbf{S}}$ (or equivalently the entanglement with the environment if this is not traced out) produced by the coupling of the environment to the pseudospin operators, the state maintains phase coherence as to the physical algebra, which is then DF. This means that the considered three qubit array encodes a DF logical qubit, compared to the four qubits needed within the conventional approach [12].

As a further example, consider now a four qubit array, whose Clebsch-Gordan decomposition is

$$\mathcal{H}_S=\mathcal{D}_2\oplus\mathcal{D}_1\oplus\mathcal{D}_1\oplus\mathcal{D}_1\oplus\mathcal{D}_0\oplus\mathcal{D}_0=\mathcal{H}_2\oplus\mathcal{H}_1\oplus\mathcal{H}_0. \tag{43}$$

In this case, while the factorization is trivial and useless for the $S^2=6(j=2)$ representation, it is still trivial but fruitful for the carrier space $\mathcal{H}_0$ of the two degenerate $S^2=0$ representations, where it gives rise to the DF states already considered in the literature. To be specific it can be checked that the $\mathcal{H}_0$ representations of invariant operators

$$\begin{aligned}{}_0\tau_1&\doteq({}_0b_{14}+{}_0b_{23}-{}_0b_{12}-{}_0b_{34})/(4\sqrt{3}),\\ {}_0\tau_2&\doteq({}_0E_{234}+{}_0E_{124}-{}_0E_{134}-{}_0E_{123})/(8\sqrt{3}),\\ {}_0\tau_3&\doteq-({}_0b_{14}+{}_0b_{12}+{}_0b_{13})/3\end{aligned} \tag{44}$$

obey the same relations as their analogs in Eqs. (39) and (40), whose enveloping algebra once again is the operator algebra of a DF logical qubit. As represented in $\mathcal{H}_0$ the DF subalgebra coincides with the total operator algebra, the representation of the total pseudospin algebra being the trivial (scalar) one.

For the four qubit array, apart from the reproduction of a DF qubit of vanishing pseudospin, the present approach also gives rise to a DF qutrit. Consider in fact the nine-dimensional $S^2=2(j=1)$ eigenspace $\mathcal{H}_1$ containing three degenerate three-dimensional representations. It can be checked, for instance, that the $\mathcal{H}_1$ representation of invariant operators

$$\begin{aligned}{}_1\tau_1&\doteq{}_1E_{134}/(-2\sqrt{3}),\\ {}_1\tau_2&\doteq({}_1E_{134}-3\,{}_1E_{124})/(4\sqrt{6}),\\ {}_1\tau_3&\doteq({}_1E_{234}+{}_1E_{123})/(4\sqrt{2})\end{aligned} \tag{45}$$

obey the usual commutation rules of $su(2)$ generators as in Eq. (39), while ${}_1\tau^2\equiv\Sigma_{j=1}^{3}\tau_j^2=8\hat{\mathbf{1}}$. In this case the nine-dimensional state space ${}_1\mathcal{H}$ can be identified with the product of the three-dimensional irreducible representations of the DF algebra and the total pseudospin algebra. In perfect analogy to what was said for the three qubit array one can arrange in the considered $S^2=2$ eigenspace a DF qutrit, namely a tridimensional state space of the DF algebra. Of course in this case the whole representation algebra ${}_1\mathcal{A}_{DF}$

cannot be produced by linear combination of the $sl(2,C)$ generators (and the identity) only, but products of two of them must be included too.

V. CONCLUSION

In conclusion what has been shown can be of use both with reference to the considered examples and more generally as a method to identify for given systems several alternative DF spaces, which can give rise to more chances for finding physically viable realizations of quantum computing. In particular the possibility to test DF qubit encoding in arrays of just three physical qubits may represent a substantial bonus in the near future.

More generally a different viewpoint about decoherence is advocated and shown to be effective. It is shown that the very notion of decoherence should be defined in more physical terms starting from the notion of physical algebra. Before asking if a state of a given system is pure or not we should preliminarily fix the operator algebra with respect to which we are defining the state. The main result of the paper is that if pureness is not defined in an abstract setting, starting from the operator algebra of the whole universe, but on the contrary from the operator algebra generated by the actual measurements that the experimentalist is going to perform, a thoroughly new and promising perspective appears. This result is relevant not only with reference to quantum computing but even to the foundations of quantum mechanics and the analysis of open quantum systems in general. In particular the approach in terms of representations of DF algebras may shed some light on the physical relevance of quantum coherence which, in principle, due to the structure of the Hamiltonian, could be present in unexpected situations if system algebras can be factored as the product of uncoupled collective algebras, one of them decoupled from the environment too.

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[16] Neither Eq. (7) nor Eq. (8) depend on the choice of specific representatives $A\in\tilde{A}, B\in\tilde{B}$. In fact $f([E^*+\bar{\lambda}F^*][E+\lambda F]) \geqslant 0 \forall \lambda \in C; E,F \in \mathcal{A} \Rightarrow |f(F^*E)|^2 \leqslant f(E^*E)f(F^*F)$, by which that independence follows if E and F are meant to be differences between elements of $\tilde{B}$ and $\tilde{A}$, respectively.
[17] One can easily check that for the three qubit array, in the product basis

$$|\psi\rangle \equiv \frac{1}{\sqrt{2}}(|-1,-1,1\rangle - |1,-1,-1\rangle) \Rightarrow S^2|\psi\rangle = 3|\psi\rangle,$$

$$\tau_3|\psi\rangle = -|\psi\rangle, \quad \frac{\tau_1 - i\tau_2}{2}|\psi\rangle = 0,$$

where τ operators are defined as in Eqs. (38) without restriction to the $j=\frac{1}{2}$ subspace. In this case one can explicitly construct the representation by the raising operator, namely in terms of the basis

$$|\psi\rangle, \quad \frac{\tau_1 - i\tau_2}{2}|\psi\rangle = \frac{1}{\sqrt{6}}(2|-1,1,-1\rangle - |-1,-1,1\rangle - |1,-1,-1\rangle),$$

the matrix representation of the operators τ_1, τ_2, τ_3 is given by the Pauli matrices.