Quantum entanglement using trapped atomic spins

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We propose an implementation for quantum logic and computing using trapped atomic spins of two different species, interacting via direct magnetic spin-spin interaction. In this scheme, the spins (electronic or nuclear) of distantly spaced trapped neutral atoms serve as the qubit arrays for quantum information processing and storage, and the controlled interaction between two spins, as required for universal quantum computing, is implemented in a three-step process that involves state swapping with a movable auxiliary spin.

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The field of quantum computing has advanced remarkably in the few years since Shor [1] presented his quantum algorithm for efficient prime factorization of very large numbers, potentially providing an exponential speed up over the fastest known classical algorithm. Because much of today's cryptography [2] relies on the presumed difficulty of factoring large numbers, Shor's discovery has important implications for data encryption technology and has stimulated much work in the field of quantum information.

Motivated by this and other theoretical developments, there is much interest in identifying and realizing experimental systems capable of generating large-scale quantum entanglement. In atomic systems, there have been several recent proposals using trapped ions or atoms and cavity QED systems [3–7]. Indeed, atomic systems capable of entangling two qubits have already been realized in some of these systems [8,9]. A common element in most of these proposals is that the qubits are stored in distinguishably trapped atoms/ions. The proposals differ principally in the nature of the atom-atom interaction (either phonons, photons, collisional, and induced electric-dipole moments) and in the way that these interactions are controlled.

In this paper, we propose an implementation of a quantum logic scheme utilizing the direct magnetic spin-spin interaction between individually trapped neutral atoms. The qubits of this system are stored in the long-lived hyperfine ground states of atoms, and coherent control of the spin-spin interactions is accomplished by controlling interatomic spacings. Our proposal is distinctive in that (1) the magnetic spin-spin interaction used to create the interatom entanglement is virtually decoherence-free and (2) atom-atom interactions are mediated via a movable ''header'' atom that serves to transfer quantum information from one qubit to another (see Fig. 1). The header atom can in fact be a different species, and hence, in contrast to Refs. [5,7,10], the atom trapping potentials are not required to be spin-dependent in order to maintain trap distinguishability for small atom separations and can be realized with far-detuned laser beams. This latter distinction is important because near-resonant laser traps are a significant source of decoherence.

We begin by considering in detail the interatomic potential between two neutral atoms separated by an internuclear distance $\vec{R}$. For the moment, we assume two spin 1/2 alkali atoms and momentarily neglect the hyperfine interactions. The potential can be written as the sum of three terms [11]:

$$V(\vec{R}) = V_T(\vec{R})\mathcal{P}_T + V_S(\vec{R})\mathcal{P}_S + V_D,$$

V_T and V_S are the (electronic) spin triplet and singlet potentials, respectively. V_D represents the long-range direct magnetic dipole interaction between two atoms. $\mathcal{P}_T$ and $\mathcal{P}_S$ are the projection operators into the total electronic subspace 1 (triplet) and 0 (singlet). The difference between V_T and V_S represents the exchange interaction, which is typically most important when R is less than the LeRoy radius R_0 ($\lesssim 40a_0$ for two identical alkali atoms). For two different atom species, the exchange interaction is considerably suppressed beyond the contact limit (a few a_0). In the long-range limit, both V_T and V_S are dominated by the van de Waals term $-C_6/R^6$ [12].

At low energies, we can re-express the first two terms of the potential by writing the spin triplet and singlet potentials in terms of the scattering lengths a_T and a_S:

$$V_\nu(\vec{R}) = \frac{4\pi\hbar^2}{M} a_\nu \delta(\vec{R}), \quad \nu = T,S,$$

and explicitly evaluating the projection operators to yield

$$V(\vec{R}) = \frac{4\pi\hbar^2}{M}\left(\frac{3}{4}a_T + \frac{1}{4}a_S\right) I\,\delta(\vec{R}) + \frac{4\pi\hbar^2}{M}(a_T - a_S)\frac{1}{4}\vec{\sigma}_1\cdot\vec{\sigma}_2\,\delta(\vec{R}) + V_D, \qquad (1)$$

where $\vec{\sigma}_v$ is electron Pauli spin operators [13].

At this juncture, we point out that the recent Innsbruck proposal [10] employs the close-range part of the potential represented in the first line of Eq. (1) in a type of ''controlled

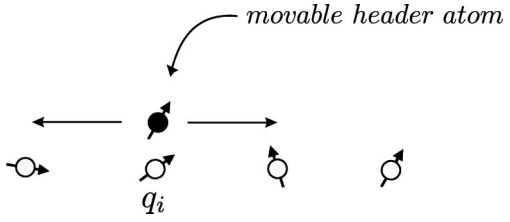

FIG. 1. A one-dimensional illustration. The q-type atom array are trapped in a red periodic CO_2 laser lattice. The movable header atom is trapped in a blue lattice.

collision.'' In our scheme, we will use the spin-dependent interaction, i.e., the long-range atomic magnetic interaction represented in the last term of Eq. (1). The proposal of Brennen, *et al.* [7] relies on the near-resonant electric dipole interaction (not present here in the ground-state Hamiltonian).

It is convenient to re-express the second term of Eq. (1) by assuming that the two interacting atoms (denoted by subscripts q and h) are harmonically bound in cylindrically symmetric traps with characteristic radial and axial sizes a_{qr}, a_{hr}, a_{qz}, and a_{hr} and furthermore that the atoms occupy the ground states of their respective traps $|0\rangle = |0\rangle_q|0\rangle_h$. In this case, we obtain

$$\mathcal{J}_E = \langle 0| \frac{4\pi\hbar^2}{M}(a_T - a_S)\,\delta(\vec{r}_q - \vec{r}_h - z_0\hat{z})|0\rangle$$

$$= \frac{4}{\sqrt{2\pi}}(a_T - a_S)\frac{a^2}{a_r^2}\frac{\hbar\omega}{a_z}e^{-z_0^2/2a_z^2},$$

for a reference harmonic trap frequency $\hbar\omega$ with ground-state size a. We have used $\vec{r}_q$ and $\vec{r}_h$ for the nuclear coordinates of the atoms with respect to their own trap centers, which are displaced by $\vec{d} = (0,0,z_0)$, and we have defined $a_\nu = \sqrt{a_{q\nu}^2 + a_{h\nu}^2}, (\nu = r,z)$. It is important to recognize that $\mathcal{J}_E$ decays exponentially with the nominal atom-atom separation z_0.

The last term in Eq. (1), V_D, contains three separate terms corresponding to electron-electron, electron-nuclear, and nuclear-nuclear magnetic dipole interactions. Between alkali atoms, the strongest is the electron dipole interaction

$$V_D^{ee} = \frac{\mu_e^2}{R^3}[\vec{\sigma}_q\cdot\vec{\sigma}_h - 3(\hat{R}\cdot\vec{\sigma}_q)(\vec{\sigma}_h\cdot\hat{R})],$$

where μ_e is the Bohr magneton. The strength of this interaction is

$$\gamma_e(R) = \frac{\mu_e^2}{R^3} \approx 5\times 10^{11}\left(\frac{a_0}{R}\right)^3 \quad \text{Hz},$$

while $\gamma_{en}(R)$ (electron-nuclear) and $\gamma_n(R)$ (nuclear-nuclear) are about 10^{-3} and 10^{-6} times smaller, respectively. Therefore one may effectively write the spin-dependent interaction Hamiltonian as

$$\mathcal{H} = \mathcal{J}_E(z_0)\vec{\sigma}_q\cdot\vec{\sigma}_h + \gamma_e(R)[\vec{\sigma}_q\cdot\vec{\sigma}_h - 3(\hat{R}\cdot\vec{\sigma}_q)(\vec{\sigma}_h\cdot\hat{R})].$$

Typically, we will have $\gamma_e(R) > \mathcal{J}_E(z_0)$ for $R > 1000a_0$ between two identical atoms.

We will now discuss how this interaction Hamiltonian can be used for logic gates. We first point out that this interaction resembles the quantum gate implementation using the Heisenberg spin (exchange) interaction [15] $H_J = J(t)\vec{\sigma}_1\cdot\vec{\sigma}_2$, which is known to be universal. For $\int_0^T dt(J/\hbar) = \pi/4 (\mathrm{mod}\, 2\pi)$, its unitary evolution operator creates a swap gate

$$U_{\mathrm{swap}}(T)|i\rangle_1|j\rangle_2 = \exp\left(-i\frac{\pi}{4}\right)|j\rangle_1|i\rangle_2,$$

which in turn can be use to generate XOR (controlled-NOT) gates by incorporating single bit operations [15]. However, our interaction Hamiltonian includes an anisotropic term. Fortunately, we can borrow a decoupling technique developed in NMR [16] to effect the conversion of $\vec{\sigma}_q\cdot\vec{\sigma}_h \rightarrow \sigma_{qz}\cdot\sigma_{hz}$, which is also universal. In fact, the phase gate [15] in terms of these operators is simply

$$U_{\mathrm{phase}} = e^{i(\pi/4)\sigma_{1z}\sigma_{2z}}e^{i(\pi/4)\sigma_{1z}}e^{i(\pi/4)\sigma_{2z}},$$

from which U_{XOR} can be easily made [9,17]. Furthermore, the swap gate, which we will require, can be made according to

$$U_{\mathrm{swap}}(1\leftrightarrow 2) = U_{\mathrm{XOR}}(1,2)U_{\mathrm{XOR}}(2,1)U_{\mathrm{XOR}}(1,2),$$

where $U_{\mathrm{XOR}}(i,j)$ denotes a c-NOT gate with i as the control bit operating on j. The necessary decoupling is achieved through a ''stirring'' radio-frequency field acting only on the h atom [16], and is most easily discussed in the context of the following model Hamiltonian:

$$H_S(t) = \hbar\omega_1\sigma_{1z} + \hbar\omega_2\sigma_{2z} + \Omega_S(\sigma_{2+}e^{-i\omega_S t} + \mathrm{H.c.}) + \gamma_e(R)[\vec{\sigma}_1\cdot\vec{\sigma}_2 - 3\sigma_{1z}\sigma_{2z}(\hat{R}\cdot\hat{z})^2], \quad (2)$$

where ω_S is the frequency of the stirring field, and Ω_S is the Rabi frequency of the stirring field. By analyzing this system in the rotating frame defined by $U_{\mathcal{R}} = e^{i\omega_S t\sigma_{2z}}$, we obtain [16] $U_{\mathcal{R}}^+\sigma_{2\pm}U_{\mathcal{R}} \rightarrow \sigma_{2+}e^{\pm i\omega_L t}$, and invoking a rotating wave approximation, the desired result is obtained:

$$H_S^{\mathrm{eff}} \approx \gamma_e(R)[1 - 3(\hat{R}\cdot\hat{z})^2]\sigma_{1z}\sigma_{2z} + \hbar\omega_1\sigma_{1z} + \hbar(\omega_2 - \omega_S)\sigma_{2z} + \Omega_S(\sigma_{2+} + \sigma_{2-}). \quad (3)$$

Although there are unwanted single atom terms in the second line of this Hamiltonian, they can be easily compensated with one-bit rotations. For our system, a similar procedure yields the following effective Hamiltonian:

$$\mathcal{H}_{\mathrm{eff}}(\vec{R}) \approx [\mathcal{J}_E(z_0) + \gamma_e(R) - 3\gamma_e(R)(\hat{R}\cdot\hat{z})^2]\sigma_{qz}\sigma_{hz} = J_E(z_0)\sigma_{qz}\sigma_{hz}. \quad (4)$$

Interestingly, we note that the spin and spatial dependence of the operators factorizes. This implies that coherent spin-spin interactions only require that the motional states of the atoms remain unchanged—the atoms are not necessarily required to be in the ground state $|0\rangle$ of their respective trapping potential. This particular feature of our proposal will be discussed in detail elsewhere. At $z_0 > R_0$, the effective spin-spin interaction strength is

$$J_E(z_0) \approx \langle \gamma_e(R)[1 - 3(\hat{z}\cdot\hat{R})^2]\rangle,$$

which for atoms in the ground states $|0\rangle$ previously described is readily evaluated:

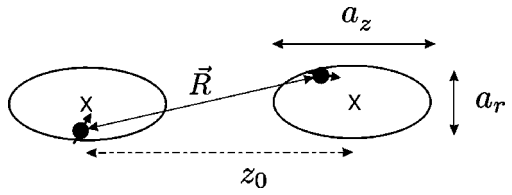

FIG. 2. The geometry of interacting header atom and qubit atom pair. The large ellipses denote trap ground states with trap centers crossed and separated by z_0. Solid circles with arrow heads denote electron spins separated by $\vec{R}$.

$$\left\langle \frac{1}{R^3}[1-3(\hat{z}\cdot\hat{R})^2] \right\rangle = \frac{1}{\sqrt{2\pi a_z^2}} \frac{1}{2a_r^4} \int_{-\infty}^{\infty} dz \exp\left(-\frac{(z-z_0)^2}{2a_z^2} \right) \times \left[2|z| - (a_r^2+z^2) \frac{\sqrt{2\pi}}{a_\rho} \exp\left(\frac{z^2}{2a_r^2} \right) \mathrm{erfc}\left(\frac{|z|}{\sqrt{2}a_r} \right) \right],$$

where erfc$(\cdot)$ is the complementary error function. The geometry of the system of two interacting spins is illustrated in Fig. 2. The result of the effective interaction is shown in Fig. 3, and we note that $J_E(z_0)$ is in the kHz range for a distance of $1000a_0$ ($\sim$50 nm), which will be more than adequate for gate operations for atoms trapped in far off-resonant optical lattices.

The principle challenge in implementing this scheme is in providing the appropriate confining potentials for the atoms. On one hand, the trapping potentials for the individual atoms need always be distinguishable in order to maintain identifiable qubits. On the other hand, as we can see from Fig. 2, the atoms need to be in close proximity ($\sim$50 nm) in order for an appreciable interaction rate even for this ''long-range'' potential. Previous proposals also requiring small interatomic spacings have suggested spin-dependent traps created by optical lattices with polarization gradients [5,7,10]. Because of

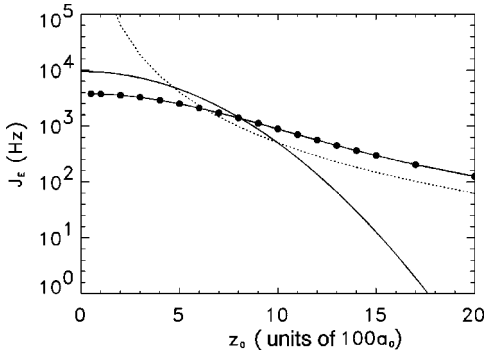

FIG. 3. The solid line denotes the exchange interaction $\mathcal{J}_E$ assuming an absolute difference of the $|a_S - a_T| = 100\,(a_0)$, while the dots are numerical results of the averaged spin dipole interaction strength J_E for $a_{qr} = a_{qz} = 400a_0$ and $a_{hr} = a_{hz} = 100a_0$. The dotted line represents the simple $1/z_0^3$ dependence. As pointed out in the text, the exchange interaction will be significantly suppressed for the case where the two atoms are two different species.

TABLE I. Parameters for different alkali-metal atoms inside a CO_2 lattice with intensity $I = 10^6$ (W/cm^2). For the ''red'' CO_2 lattice, the maximum level shift is $V_{\max} = \alpha(0)E_0^2/4$. At an intensity of $\sim 10^6$ (W/cm^2), the single photon scattering rate can provide decoherence times of many minutes. Assuming a harmonic approximation, the oscillation frequency $\nu_{\rm osc}$ inside the CO_2 trap is $\nu_{\rm osc} = 2\sqrt{V_{\max} E_R^{\rm CO_2}}$ with $E_R^{\rm CO_2} = h/(2M\lambda_{\rm CO_2}^2)$, the recoil energy (in Hz) for emitting or absorbing a CO_2-photon. The Lamb-Dicke parameter $\eta_0 = k_0 a_{\rm osc} = \sqrt{E_R^0/\nu_{\rm osc}}\,(\eta_{\rm CO_2})$ measures the trap ground-state size $a_{\rm osc}$ in terms of the resonant wavelength λ_0 (CO_2 laser $\lambda_{\rm CO_2}$).

	Li	Na	K	Rb	Cs
M	6.9	23	39	87	133
$\alpha(0)$ (units of a_0^3)	159.2	162	292.8	319.2	402.2
$V_{\max}$ (MHz)	181	185	334	364	458
$\nu_{\rm osc}$ (kHz)	432	239	247	172	156
$a_{\rm osc}$ (units of a_0)	778	573	433	347	295
λ_0 (nm)	670	589	766	780	852
E_R(kHz)	64	25	8.7	3.7	2
η_0	0.39	0.32	0.19	0.15	0.11
$\eta_{\rm CO_2}$	0.025	0.018	0.014	0.011	0.009

the nature of these types of traps, the types of atomic manipulations are rather restricted, and hence scalability is difficult.

To circumvent this complication, we will use two different atomic species, one for the (stationary) quantum register, and one for the quantum header atom. Each species of atom will be separately trapped by different laser fields. By appropriate choice of atom and frequency of the trapping fields, we can make these traps essentially independent. For a concrete example, consider a quantum register consisting of an array of single atoms (type q for qubit) trapped in a three-dimensional standing wave formed by an interfering laser field of CO_2 lasers (wavelength $\lambda_{\rm CO_2} \approx 10.6$ μm) [18]. The qubits will be separated by $\lambda_{\rm CO_2}/2$, which is more than enough to allow individual addressing, and at this separation, the long range Casimir-Polder interaction is negligible [12]. The trapping details presented in Tables I and II, but we point out that the potential V is very well approximated by the dc-polarizability of the atom $\alpha(0)$ and the laser electric

TABLE II. Parameters for a ''blue'' lattice with $\Omega_L \sim 1.6 \times 10^{10}$ (Hz) ($\sim$ laser power of 10 kw/cm^2), $\gamma = 10^7$ (Hz), and $\delta_L = 2 \times 10^{12}$ (Hz). For the near-resonant ''blue'' lattice on the h-type atoms, the effective single-photon scattering rate is approximately $\gamma_{\rm eff} = \eta^2(\Omega_L^2/4\delta_L^2)\gamma$. We see as indicated in Table II the confining frequency $\nu_{\rm osc}$ is indeed much larger than that of CO_2 laser (on q-type atoms).

	Li	Na	K	Rb	Cs
M	6.9	23	39	87	133
$\nu_{\rm osc}$ (kHz)	4061	2530	1494	982	727
$a_{\rm osc}\,(a_0)$	254	176	176	145	137
η	0.13	0.1	0.076	0.06	0.05
$\gamma_{\rm eff}$ (Hz)	2.5	1.6	0.9	0.6	0.5

field amplitude E as $V=-\alpha(0)E^2/2$. The relevant parameters are tabulated for alkali atoms in Table I.

A separate laser field provides confinement for the header atom (of a different type atom, h). By choosing a trapping wavelength somewhat closer to the atomic resonance of h (and detuned to the blue of the resonance), we can provide a potential that acts principally on the h atom. The potential depth for this trap is given by $V_{\max}=\hbar\Omega_L^2/4\delta_L$, with Ω_L the Rabi frequency of the laser, and $\delta_L=\omega_L-\omega_0$ the detuning. The h atom is also affected by the far off-resonant CO_2 laser field of course, but we can arrange for the off-resonant blue detuned field to dominate the far off-resonant CO_2 laser potential by a suitable choice of atoms. Trapping parameters for this case are listed in Table II. The quantum register atoms (type q) will also be affected, at some level, by the blue lattice, but the detuning between the blue field and the q atoms will be much larger, so the potential will be dominated by the CO_2 laser field for the q atoms.

Gate operations in this system can be achieved in a three-step process requiring quantum-state swapping between the quantum bits and the header atom. To execute a gate operation between two qubits q_i and q_j, we first translate the header atom h to the location of q_i and perform a state swap $q_i \leftrightarrow h$. The header atom is then translated to site q_j and the gate operation between $h(q_i)$ and q_j is performed. Finally, the header atom is translated back to q_i to and the state swap is repeated. The header atom effectively acts as a quantum bus between the qubits, and in this sense our scheme shares certain features with the quantum gear machine proposed by DiVincenzo [14].

Single-bit operations can be realized either by directly addressing the individual qubits q_i, or, alternatively, we can use the header atom as a mediator. The latter option may be easier in some cases than the direct spatial selection of q_i because the h atoms can be sparsely distributed and have different resonance level structures. The single-bit operation will again be a three-step process: (i) perform a state swap between q_i and h, (ii) perform the arbitrary qubit operation on h, and (iii) repeat the state swap between h and q_i.

In considering the ultimate scalability of this, and other lattice-based schemes, it is necessary to compare the characteristic intrinsic decoherence time of the system to the gate time *plus* the transport time of the moving atoms [20]. Additionally, the transport of the moving atoms (the h-type atom in this case) must be adiabatic such that the motional state of the atom remains unchanged. This latter condition implies constraints on the magnitude of the motion, which we can estimate using perturbation theory. Consider the Hamiltonian for a one-dimenstional harmonically trapped particle with mass M and trap frequency ω_t subjected to a force $F(t)$,

$$H=\frac{p^2}{2M}+\frac{1}{2}M\omega_t^2 q^2-qF(t), \tag{5}$$

adiabaticity condition for the header qubit translation requires its motional state wave function to be essentially unchanged. This problem is equivalent to the problem of a translating simple harmonic oscillator (SHO) potential $M\omega_t^2[q-q_0(t)]^2/2$ [with $F(t)=M\omega_t^2 q_0(t)$] up to a deterministic phase factor due to $M\omega_t^2 q_0^2(t)/2$. Calculating the probability for excitation out of the ground state is a standard textbook problem [19] and the result to first order is

$$p_1^{(1)}=\frac{1/2M(\delta v)^2}{\hbar\omega_t}\exp(-\omega_t\tau), \tag{6}$$

for a time-dependent force $F(t)=(F_0\tau/\omega)/(\tau^2+t^2)$. $1/2M(\delta v)^2$ is the energy gained from the impulse $M\delta v=\int_{-\infty}^{\infty}F(t)dt$ of the force. We see that adiabaticity is maintained even after the translating atom gains a very large speed, but satisfying the condition $\omega_t\tau\gg 1$, i.e., a force to be slowly turning on and off compared with the harmonic-trap period. Similar conclusions are reached for an initial coherent motional state wave packet. This condition effectively then puts no constraint on the header atom speed, contrary to the strong conditions as obtained in Ref. [20]. For our problem, creative pulse shape design will allow the header atom to be adiabatically transported over many qubits within the single photon-scattering coherence time.

Our discussion thus far has been limited to alkali atoms with no nuclear spin (e.g., ^{78}Rb). When the nuclear spin I is nonzero, the atomic spin takes on values $F=I\pm 1/2$ and we must include the hyperfine interaction $V_{\rm hf}\sim a_{\rm hf}\vec{\sigma}\cdot\vec{\sigma}^n$. The spin-spin interaction becomes considerably richer in detail. However, if a strong Zeeman interaction is applied using a uniform magnetic field, the resulting two manifolds of Zeeman states correspond roughly to the electronic spin up/down such that the good basis becomes $|I,S,I_z,S_z\rangle$ [21]. Alternatively, we could choose an atom with no nuclear spin such as ^{78}Rb (radioactive lifetime about 20 minutes).

In summary, we have proposed a new quantum computing implementation with trapped atomic spins. Utilizing dual optical lattices for two different types of atoms provides a novel method to control the binary interaction between any pair of qubits. In addition, our proposal, being based on the periodic structure of optical lattices, is readily scalable, and in particular, redundant parallel processing of information can be implemented using multiple h-type atoms operating on repetitive blocks of q-type atoms. This may be useful in implementing error correction [22], concatenated coding, and fault-tolerant computing [23].

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[1] P. W. Shor, in *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science*, edited by S. Goldwasser (IEEE Computer Society Press, Los Alamitos, CA, 1994), p. 124.
[2] R. Rivest, A. Shamir, and L. Adelman, Commun. ACM **21**, 120 (1978).
[3] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[4] T. Pellizzari, S. A. Gardiner, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **75**, 3788 (1995).
[5] A. Sorensen and K. Molmer, Phys. Rev. Lett. **83**, 2274 (1999).
[6] A. Imamoglu, D. D. Awschalom, G. Burkard, D. P. DiVincenzo, D. Loss, M. Sherwin, and A. Small, Phys. Rev. Lett. **83**, 4204 (1999).
[7] Gavin K. Brennen, Carlton M. Caves, Poul S. Jessen, and Ivan H. Deutsch, Phys. Rev. Lett. **82**, 1060 (1999).
[8] Q. A. Turchette, C. S. Wood, B. E. King, C. J. Myatt, D. Leibfried, W. M. Itano, C. Monroe, and D. J. Wineland, Phys. Rev. Lett. **81**, 3631 (1998).
[9] Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi, and H. J. Kimble, Phys. Rev. Lett. **75**, 4710 (1995).
[10] D. Jaksch, H.-J. Briegel, J. I. Cirac, C. W. Gardiner, and P. Zoller1, Phys. Rev. Lett. **82**, 1975 (1999).
[11] A. J. Moerdijk, B. J. Verhaar, and T. M. Nagtegaal, Phys. Rev. A **53**, 4343 (1996).
[12] M. Marinescu and L. You, Phys. Rev. A **59**, 1936 (1999).
[13] Julio Gea-Banacloche, Phys. Rev. A **57**, R1 (1998).
[14] D. P. DiVincenzo, Phys. Rev. A **51**, 1015 (1995).
[15] D. Loss and D. P. Divincenzo, Phys. Rev. A **57**, 120 (1998).
[16] A. Abragham, *The Principles of Nuclear Magnetism* (Oxford University Press, New York, 1961), p. 527.
[17] S. Lloyd, Phys. Rev. Lett. **75**, 346 (1995); Adriano Barenco, Charles H. Bennett, Richard Cleve, David P. DiVincenzo, Norman Margolus, Peter Shor, Tycho Sleator, John A. Smolin, and Harald Weinfurter, Phys. Rev. A **52**, 3457 (1995).
[18] S. Friebel, C. D'Andrea, J. Walz, M. Weitz, and T. W. Hänsch, Phys. Rev. A **57**, R20 (1998); T. Takekoshi, J. R. Yeh, and R. J. Knize, Opt. Commun. **114**, 421 (1995); T. Takekoshi, B. M. Patterson, and R. J. Knize, Phys. Rev. Lett. **81**, 5105 (1998).
[19] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, New York, 1985), p. 352.
[20] H.-J. Briegel, T. Calarco, D. Jaksch, J. I. Cirac, and P. Zoller, J. Mod. Opt. **47**, 415 (2000).
[21] D. K. Walter, W. Happer, and T. G. Walker, Phys. Rev. A **58**, 3642 (1998).
[22] P. Shor, Phys. Rev. A **52**, R2493 (1995); R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Phys. Rev. Lett. **77**, 198 (1996); A. Steane, *ibid.* **77**, 793 (1996).
[23] P. Shor, in *Proceedings of the 37th Conference on the Foundations of Computer Science, Burlington, Vermont, 1996* (IEEE Computer Society Press, Burlington, VT, 1996), pp. 56–65; D. P. DiVincenzo and P. Shor, Phys. Rev. Lett. **77**, 3260 (1996); J. Preskill, in *Introduction to Quantum Computation*, edited by H.-K. Lo, S. Popescu, and T. P. Spiller (World Scientific, London, 1998), p. 213; A. Yu. Kitaev (e-print quant-ph/9707021).