Equally distant, partially entangled alphabet states for quantum channels

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Each Bell state has the property that by performing just *local* operations on one qubit, the complete Bell basis can be generated. That is, states generated by local operations are totally distinguishable. This remarkable property is due to maximal quantum entanglement between the two particles. We present a set of local unitary transformations that generate out of partially entangled two-qubit state a set of four maximally distinguishable states that are mutually equally distant. We discuss quantum dense coding based on these alphabet states.

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I. INTRODUCTION

Two parties (Alice & Bob) who share a pure two-qubit state $|\Psi_1\rangle_{AB}$ can generate three other states $|\Psi_j\rangle_{AB}$ ($j=2,3,4$) such that the four states form a basis in the Hilbert space of two qubits. In general, the two parties have to perform operations on *both* qubits to generate the orthogonal states $|\Psi_j\rangle_{AB}$. Nevertheless, there is an exception—if the original state $|\Psi_1\rangle_{AB}$ is one of the four Bell states [1] then by performing unitary transformations on just *one* of the two qubits (let us assume Alice is the operations) the other three Bell states that form the Bell basis of the two-qubit system can be generated. Specifically, let us assume the system is initially in the Bell state

$$|\Psi_1\rangle_{AB}=\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B+|1\rangle_A|1\rangle_B), \tag{1.1}$$

where $|0\rangle_X$ and $|1\rangle_X$ ($X=A,B$) are basis vectors in the Hilbert space $\mathcal{H}_X$ of the qubit X (in what follows we will use the shorthand notation $|00\rangle=|0\rangle|0\rangle$ and where clear we will omit subscripts indicating the subsystem). Now we introduce four *local* (single-qubit) operations

$$\begin{aligned}
\hat{S}_1&=\hat{\mathbb{1}}=(|0\rangle\langle 0|+|1\rangle\langle 1|);\\
\hat{S}_2&=\hat{\sigma}_x=(|0\rangle\langle 1|+|1\rangle\langle 0|);\\
\hat{S}_3&=\hat{\sigma}_y=i(|0\rangle\langle 1|-|1\rangle\langle 0|);\\
\hat{S}_4&=\hat{\sigma}_z=(|0\rangle\langle 0|-|1\rangle\langle 1|),
\end{aligned} \tag{1.2}$$

where $\hat{\sigma}_\mu$ ($\mu=x,y,z$) are three Pauli operators. When the operators S_k act on the first (Alice's) qubit of the Bell state (1.1) we find

$$\begin{aligned}
|\Psi_1\rangle&=\hat{S}_1\otimes\hat{\mathbb{1}}|\Psi_1\rangle=\frac{1}{\sqrt{2}}|00\rangle+|11\rangle);\\
|\Psi_2\rangle&=\hat{S}_2\otimes\hat{\mathbb{1}}|\Psi_1\rangle=\frac{1}{\sqrt{2}}|10\rangle+|01\rangle);\\
|\Psi_3\rangle&=\hat{S}_3\otimes\hat{\mathbb{1}}|\Psi_1\rangle=\frac{i}{\sqrt{2}}|01\rangle-|10\rangle);\\
|\Psi_4\rangle&=\hat{S}_4\otimes\hat{\mathbb{1}}|\Psi_1\rangle=\frac{1}{\sqrt{2}}|00\rangle-|11\rangle).
\end{aligned} \tag{1.3}$$

We see that the four states given by Eq. (1.3) are indeed four Bell states [1]. This means that by performing just *local* operations, the two-qubit states are changed globally in such a way that the four outcomes are perfectly distinguishable (i.e., the four Bell states are mutually orthogonal). In fact, we can say that the four outcomes are mutually equally (and maximally) distant, which can be expressed by their mutual overlap $\mathcal{O}_{kl}$,

$$\mathcal{O}_{kl}=|\langle\Psi_k|\Psi_l\rangle|^2=\delta_{kl}. \tag{1.4}$$

This remarkable property of Bell states is due to the quantum entanglement between the two qubits [1]. As suggested by Bennett and Wiesner [2], this property can be utilized for the *quantum dense coding*. The idea is as follows: Alice can perform locally on her qubit four operations that result in four orthogonal two-qubit states. So after she performs one of the possible operations she sends her qubit to Bob. Then Bob can perform a measurement on the two qubits and determine with the fidelity equal to unity which of the four operations has been performed by Alice. In this way, Alice has transferred two bits of information via sending just a single two-level particle. This theoretical scenario has been implemented experimentally by the Innsbruck group [3] using polarization entangled states of photons. One can conclude that the entanglement in the case of two qubits can double a capacity of the quantum channel.

Recently, Barenco and Ekert [4], Hausladen *et al.* [5], and Bose *et al.* [6] have discussed how the channel capacity depends on the degree of entanglement between the two qubits.

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Specifically, these authors have analyzed the situation when initially Alice and Bob share a two-qubit system in a state

$$|\psi_1\rangle=\alpha|00\rangle+\beta|11\rangle. \tag{1.5}$$

Then Alice is performing locally one of the four unitary operations $\hat{S}_k$ given by Eq. (1.2). As a result of these operations four possible states $|\phi_k\rangle$ can be generated:

$$\begin{aligned}
|\phi_1\rangle&=\hat{S}_1\otimes\hat{\mathbb{1}}|\psi_1\rangle=(\alpha|00\rangle+\beta|11\rangle);\\
|\phi_2\rangle&=\hat{S}_2\otimes\hat{\mathbb{1}}|\psi_1\rangle=(\alpha|10\rangle+\beta|01\rangle);\\
|\phi_3\rangle&=\hat{S}_3\otimes\hat{\mathbb{1}}|\psi_1\rangle=-i(\alpha|10\rangle-\beta|01\rangle);\\
|\phi_4\rangle&=\hat{S}_4\otimes\hat{\mathbb{1}}|\psi_1\rangle=(\alpha|00\rangle-\beta|11\rangle).
\end{aligned} \tag{1.6}$$

These states represent the ''alphabet'' that is used in the given communication channel between Alice and Bob. Not all of these alphabet states are mutually orthogonal. If we evaluate the overlap $\mathcal{O}_{kl}$ we find

$$\mathcal{O}_{kl}=|\langle\phi_k|\phi_l\rangle|^2=\begin{cases}1 & \text{if } \; k=l,\\ \Delta^2 & \text{if } \; kl=12,21,34,43,\\ 0 & \text{else,}\end{cases} \tag{1.7}$$

where $\Delta=|\alpha|^2-|\beta|^2$. The fact that not all alphabet states are mutually orthogonal leads to a decrease of the channel capacity that in the present case is less than two. Nevertheless, for at least partially entangled qubits, the channel capacity is still larger than unity.

As seen from Eq. (1.7), the four states $|\phi_k\rangle$ are not mutually equally distant. Some of them are mutually orthogonal, but some of them have a nonzero overlap. The main goal of this paper is to find a set of *local* unitary operations $\hat{U}_k$ that generate out of the state $|\psi_1\rangle$ given by Eq. (1.5), the alphabet $|\psi_k\rangle=\hat{U}_k|\psi_1\rangle$ with the elements that are equally distant, that is the mutual overlaps of these four states are equal, and simultaneously we require that they are as small as possible. In other words, the states are mutually as distinguishable as possible. Formally, we are looking for transformations $\hat{U}_k$ such that

$$\mathcal{O}_{kl}=|\langle\psi_1|\hat{U}_k^\dagger\hat{U}_l|\psi_1\rangle|^2=\begin{cases}1 & \text{for } \; k=l\\ \mathcal{O} & \text{for } \; k\neq l\end{cases} \tag{1.8}$$

with $\mathcal{O}$ being as small as possible. In addition, the transformations under consideration have to fulfill the *Bell limit*, that is for $\Delta\to 0$, when $|\psi_1\rangle\to|\Psi_1\rangle$, they have to generate four maximally entangled mutually orthogonal two-qubit states. We note that this set of states is not necesarily equal to standard Bell states given by Eq. (1.3). In our case, the explicit form of these states is given by Eq. (2.21) with $\alpha=\beta=1/\sqrt{2}$.

II. EQUALLY DISTANT STATES

It is well known that any pure bipartite state can be written in the Schmidt basis [7] as given by Eq. (1.5). In order to find the four transformations $\hat{U}_k$ that fulfill the condition (1.8), we remind ourselves that the most general unitary transformation on a two-dimensional Hilbert space is an element from a four-parametric group $U(2)$

$$\hat{U}_k=e^{i\varphi_k}[\cos\psi_k\hat{\mathbb{1}}+i\sin\psi_k(\vec{n}_k\hat{\vec{\sigma}})], \tag{2.1}$$

where $\vec{n}_k=(\sin\theta_k\cos\phi_k,\sin\theta_k\sin\phi_k,\cos\theta_k)$ is a normalized vector around which the rotation is performed by an angle ψ_k.

From the condition (1.8) it follows that we have to solve the following set of equations

$$\begin{aligned}\mathcal{O}_{kl}&=|\langle\psi_1|\hat{W}_{kl}|\psi_1\rangle|^2=||\alpha|^2\langle 0|\hat{W}_{kl}|0\rangle+|\beta|^2\langle 1|\hat{W}_{kl}|1\rangle|^2\\ &=\mathcal{O}=\text{minimum},\end{aligned} \tag{2.2}$$

where we have introduced a notation

$$\hat{W}_{kl}=\hat{U}_k^\dagger\hat{U}_l. \tag{2.3}$$

Taking into account the relation between Pauli operators $\hat{\sigma}_\mu\hat{\sigma}_\nu=\delta_{\mu\nu}\hat{\mathbb{1}}-i\varepsilon_{\mu\nu\kappa}\hat{\sigma}_\kappa$ and the relation

$$(\vec{n}_k\cdot\hat{\vec{\sigma}})(\vec{n}_l\cdot\hat{\vec{\sigma}})=\vec{n}_k\cdot\vec{n}_l\hat{\mathbb{1}}-i[\vec{n}_k\times\vec{n}_l]\cdot\hat{\vec{\sigma}}, \tag{2.4}$$

we can rewrite $\hat{W}_{kl}$ as

$$\begin{aligned}\hat{W}_{kl}=&\hat{\mathbb{1}}(\cos\psi_k\cos\psi_l+\vec{n}_k\cdot\vec{n}_l\sin\psi_k\sin\psi_l)\\ &-i\hat{\vec{\sigma}}\cdot[\vec{n}_k\times\vec{n}_l]\sin\psi_k\sin\psi_l-i\hat{\vec{\sigma}}(\sin\psi_k\cos\psi_l\vec{n}_k\\ &-\sin\psi_l\cos\psi_k\vec{n}_l).\end{aligned} \tag{2.5}$$

Due to the fact that quantum states are determined up to global phase we can omit phase factors φ_k in Eq. (2.5).

From Eq. (2.2) we see that only diagonal elements of the operators $\hat{W}_{kl}$ are relevant. Taking into account that only $\hat{\sigma}_z$ has nonvanishing diagonal elements, we obtain

$$\begin{aligned}\langle 0|\hat{W}_{kl}|0\rangle&=a_{kl}-ib_{kl},\\ \langle 1|\hat{W}_{kl}|1\rangle&=a_{kl}+ib_{kl},\end{aligned} \tag{2.6}$$

where we use the notation

$$\begin{aligned}a_{kl}&=\cos\psi_k\cos\psi_l+\vec{n}_k\cdot\vec{n}_l\sin\psi_k\sin\psi_l,\\ b_{kl}&=\sin\psi_k\sin\psi_l[\vec{n}_k\times\vec{n}_l]_z+\sin\psi_k\cos\psi_l(\vec{n}_k)_z\\ &\quad-\sin\psi_l\cos\psi_k(\vec{n}_l)_z\end{aligned} \tag{2.7}$$

and

$$\vec{n}_k\cdot\vec{n}_l=\cos\theta_k\cos\theta_l+\sin\theta_k\sin\theta_l\cos(\phi_l-\phi_k);$$
$$[\vec{n}_k\times\vec{n}_l]_z=\sin\theta_k\sin\theta_l\sin(\phi_l-\phi_k). \tag{2.8}$$

It follows from Eq. (2.2) that

$$|\langle\psi_k|\psi_l\rangle|^2=\mathcal{O}_{kl}=a_{kl}^2+b_{kl}^2\Delta^2. \tag{2.9}$$

This overlap has to be minimized and made state-independent (i.e., $\mathcal{O}_{kl}=\mathcal{O}=minimal$).

In order to solve the problem, we choose $\hat{U}_1=\hat{\mathbb{1}}$ and explicitly rewrite the condition (2.9) for $k=1$ and $l=2,3,4$:

$$|\langle\psi_1|\psi_l\rangle|^2=\cos^2\psi_l+\Delta^2\sin^2\psi_l\cos^2\theta_l. \tag{2.10}$$

From Eqs. (2.9) and (2.10) it follows that in order to fulfill the Bell limit, when $\mathcal{O}=0$, two following conditions have to be valid:

$$\cos^2\psi_k=0$$
$$\vec{n}_k\cdot\vec{n}_l=\delta_{kl}. \tag{2.11}$$

Taking into account these constraints we rewrite Eqs. (2.9) and (2.10) as

$$\mathcal{O}=[\vec{n}_k\times\vec{n}_l]_z^2\Delta^2,$$
$$\mathcal{O}=\cos^2\theta_k\Delta^2, \tag{2.12}$$

respectively. Because the overlap $\mathcal{O}$ is supposed to be the same for all pairs of states, we can introduce a notation $F=\cos^2\theta_k$ and we compare the right-hand sides of Eqs. (2.12) which gives us the following equation:

$$(1-F^2)^2\sin^2(\phi_k-\phi_l)-F^2=0, \tag{2.13}$$

where we have used the relation (2.8). From the condition $\vec{n}_k\cdot\vec{n}_l=0$ for $k\neq l$ [see Eq. (2.11)] we write the constraint for the parameter F:

$$(1-F^2)\cos(\phi_l-\phi_k)\pm F^2=0, \tag{2.14}$$

where $\mp=\text{sgn}(\cos\theta_k\cos\theta_l)$. The two constraints (2.13) and (2.14) are fulfilled when $F^2=1/3$.

Now we put our results together. We have found that transformations $\hat{U}_k$ are characterized by the following parameters:

$$\cos^2\psi_k=0,$$
$$\cos^2\theta_k=\frac{1}{3}, \tag{2.15}$$
$$\cos(\phi_l-\phi_k)=\frac{\mp F^2}{1-F^2}=\mp\frac{1}{2}.$$

If we choose $\cos\theta_2=\cos\theta_3=1/\sqrt{3}$ then in order to have three distinct transformations we have to take $\cos\theta_4=-1/\sqrt{3}$. Our result (2.15) also implies that

$$\cos(\phi_2-\phi_3)=-1/2,$$
$$\cos(\phi_2-\phi_4)=\cos(\phi_3-\phi_4)=1/2, \tag{2.16}$$

which can be obtained when $\phi_2=\phi,\phi_3=\frac{2}{3}\pi+\phi,\phi_4=\frac{\pi}{3}+\phi$. As we can see, there is still some freedom in a choice of the phase ϕ. Just for convenience we take $\phi=0$. This finishes our explicit construction of a set of *local* unitary transformations $\hat{U}_k$ for which we have obtained the expressions (here we have assumed that in Eq. (2.1) for the operators $\hat{U}_k$ the phase factors φ_k are taken to be equal to $\varphi_k=-\pi/2$)

$$\hat{U}_1=\hat{\mathbb{1}};$$
$$\hat{U}_k=\vec{n}_k\cdot\hat{\vec{\sigma}};\quad k=2,3,4, \tag{2.17}$$

where the unit vectors $\vec{n}_k$ are given by the expressions

$$\vec{n}_2=\left(\frac{2}{\sqrt{6}},0,\frac{1}{\sqrt{3}}\right);$$
$$\vec{n}_3=\left(-\frac{1}{\sqrt{6}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}}\right); \tag{2.18}$$
$$\vec{n}_4=\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{3}}\right).$$

These vectors not only fulfill the condition $\vec{n}_k\cdot\vec{n}_l=\delta_{kl}$ but also

$$[\vec{n}_k\times\vec{n}_l]=-\varepsilon_{klm}\vec{n}_m, \tag{2.19}$$

from which it follows that we can rewrite the operator $\hat{W}_{kl}=\hat{U}_k^\dagger\hat{U}_l$ for $k,l=2,3,4$ as

$$\hat{W}_{kl}=\delta_{kl}\hat{\mathbb{1}}+\varepsilon_{klm}\hat{U}_m. \tag{2.20}$$

The operators $\hat{U}_k$ generate from the reference state $|\psi_1\rangle$ via local transformations the alphabet with most distant states. We stress that for a chosen Schmidt basis the local operators $\hat{U}_k$ do not depend on the states to be rotated. In this sense, these operators are *universal*. The alphabet states in the basis $\{|00\rangle,|10\rangle,|01\rangle,|11\rangle\}$ read

$$|\psi_1\rangle=(\alpha,0,0,\beta); \tag{2.21}$$

$$|\psi_2\rangle=\left(\frac{1}{\sqrt{3}}\alpha,\sqrt{\frac{2}{3}}\alpha,\sqrt{\frac{2}{3}}\beta,-\frac{1}{\sqrt{3}}\beta\right);$$

$$|\psi_3\rangle=\left(\frac{1}{\sqrt{3}}\alpha,\frac{-1-i\sqrt{3}}{\sqrt{6}}\alpha,\frac{-1+i\sqrt{3}}{\sqrt{6}}\beta,-\frac{1}{\sqrt{3}}\beta\right);$$

$$|\psi_4\rangle = \left(-\frac{1}{\sqrt{3}}\alpha, \frac{1-i\sqrt{3}}{\sqrt{6}}\alpha, \frac{1+i\sqrt{3}}{\sqrt{6}}\beta, \frac{1}{\sqrt{3}}\beta \right).$$

By construction, the mutual overlap between these states is minimal and equal to

$$\mathcal{O} = \frac{1}{3}\Delta^2. \tag{2.22}$$

Comment 1. We note that the universal transformations we have derived generate a set of four states $|\psi_k\rangle$ for the state $|\psi_1\rangle$. In fact, these transformations generate the same set of states if generated from any state from this set (that is, we observe a specific permutation invariance in the set). To prove this property it is enough to observe that

$$\hat{U}_k \hat{U}_l = \delta_{kl}\hat{\mathbb{1}} + \varepsilon_{klm}\hat{U}_m, \tag{2.23}$$

which means that the state $|\phi_k\rangle = \hat{U}_k|\psi_l\rangle$ is equal to one of the states $|\psi_m\rangle = \hat{U}_m|\psi_1\rangle$ given by Eqs. (2.21).

Comment 2. We have derived our transformations under the assumption that the reference state from which the other three alphabet states are generated is a pure state. When the reference state $\hat{\rho}_1$ is a statistical mixture of two qubits, which in general is characterized by 15 parameters, our transformations generate an alphabet $\hat{\rho}_k = \hat{U}_k \hat{\rho}_1 \hat{U}_k^\dagger$ such that in general $\mathrm{Tr}(\hat{\rho}_k \hat{\rho}_l) \neq const$. Nevertheless, for a large class of statistical mixtures of two qubits the transformations $\hat{U}_k$ generate equally distant alphabets. A simple example would be to consider the reference state to be a mixture of the form $\hat{\rho}_1 = s|\psi_1\rangle\langle\psi_1| + (1-s)/4\hat{\mathbb{1}}$. In this case the overlap between alphabet states is constant and equal to $\mathcal{O} = \mathrm{Tr}(\hat{\rho}_k \hat{\rho}_l) = s^2\Delta^2/3 + (1-s^2)/4$. Another example is when the reference state is taken to be $\hat{\rho}_1 = \Sigma_m \lambda_m \hat{U}_m |\psi_1\rangle\langle\psi_1| \hat{U}_m^\dagger$ and the three alphabet states are generated by the operators $\hat{U}_k$. In this case we find that the overlap between the states is constant and equal to $\mathcal{O} = \frac{1}{3}\Delta^2 \Sigma_{m,n=1}^4 \lambda_m \lambda_n$.

III. CHANNEL CAPACITY

Let us assume that Alice and Bob are using alphabet states described above for quantum communication. The capacity of the quantum channel is given by the expression [8]

$$C = \max_\pi \left[S\left(\sum \pi_k \hat{\varrho}_k \right) - \sum \pi_k S(\hat{\varrho}_k) \right], \tag{3.1}$$

where $\hat{\varrho}_k$ are the alphabet states at the output of the channel (i.e., at Bob's side of the communication channel) and π_k are the probabilities with which the alphabet states are used by Alice. In the right-hand side of Eq. (3.1), the function S is the von Neumann entropy $S(\hat{\varrho}) = -\mathrm{Tr}(\hat{\varrho}\log_2 \hat{\varrho})$.

Firstly, we analyze ideal channels and then we describe quantum capacity of noisy channels.

A. Ideal channel

In the case of the ideal channel we evaluate the capacity for two sets of alphabet states—the one used by Bose *et al.* given by Eq. (1.6) —and the other set we have derived earlier in the paper [see Eq. (2.21)]. We denote the reference pure state from which the elements of alphabets are generated as $\hat{\varrho}_{AB} = |\psi_1\rangle\langle\psi_1|$, where $|\psi_1\rangle$ is given by Eq. (1.5).

Interestingly enough, for both cases we find the capacity of the quantum channel to be the same:

$$C = 1 + S(\hat{\varrho}_A), \tag{3.2}$$

where $\hat{\varrho}_A = \mathrm{Tr}_B \hat{\varrho}_{AB}$. Obviously, in the Bell limit, when the $|\psi_1\rangle$ is equal to a Bell state and Alice's qubit is in a maximally mixed state with $S(\hat{\varrho}_A) = 1$, the capacity of the quantum channel is equal to 2.

But the question is: Why for the two alphabets discussed above is the quantum capacity mutually *equal* for an arbitrary reference state $|\psi_1\rangle$? To illuminate this problem, we remind ourselves that the two sets of the operators that generate two alphabets (1.6) and (2.21), respectively, fullfill the Bell's limit, i.e., $\vec{n}_k \cdot \vec{n}_l = \delta_{kl}$ [see Eq. (2.11)]. Therefore we concentrate our attention on those transformations (2.17) that have this property.

Because of the unitarity of these transformations the second term in Eq. (3.1) is equal to zero. The input probability that maximizes the expression for capacity is equal to $\pi_k = 1/4$. In this case the density operator $\hat{\bar{\varrho}} = \Sigma_k \hat{\varrho}_k / 4$ in the matrix form (in the basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$) reads

$$\hat{\bar{\varrho}} = \frac{1}{4}\begin{pmatrix} |\alpha|^2\left(1 + \sum_k n_k^z \cdot n_k^z\right) & |\alpha|^2 \sum_k n_k^z(n_k^x - i n_k^y) & \alpha\beta^* \sum_k n_k^z(n_k^x + i n_k^y) & \alpha\beta^*\left(1 - \sum_k n_k^z \cdot n_k^z\right) \\ |\alpha|^2 \sum_k n_k^z(n_k^x + i n_k^y) & |\alpha|^2 \sum_k (n_k^y \cdot n_k^y + n_k^x \cdot n_k^x) & \alpha\beta^* \sum_k (n_k^x + i n_k^y)(n_k^x + i n_k^y) & \alpha\beta^* \sum_k n_k^z(n_k^x - i n_k^y) \\ \alpha^*\beta \sum_k n_k^z(n_k^x - i n_k^y) & \alpha^*\beta \sum_k (n_k^x - i n_k^y)(n_k^x - i n_k^y) & |\beta|^2 \sum_k (n_k^y \cdot n_k^y + n_k^x \cdot n_k^x) & |\beta|^2 \sum_k n_k^z(n_k^x + i n_k^y) \\ \alpha^*\beta\left(1 - \sum_k n_k^z \cdot n_k^z\right) & \alpha^*\beta \sum_k n_k^z(n_k^x + n_k^y) & |\beta|^2 \sum_k n_k^z(n_k^x - i n_k^y) & |\beta|^2\left(1 + \sum_k n_k^z \cdot n_k^z\right) \end{pmatrix}, \tag{3.3}$$

where n_k^j denotes the jth component of the vector $\vec{n}_k$. These three-dimensional vectors create a complete system in three-dimensional real vector space, i.e.,

$$\sum_{k=2}^{4} n_k^j n_k^l = \delta^{jl}, \tag{3.4}$$

for $j,l = x,y,z$. Using this property, we evaluate the operator $\bar{\varrho}$ for which we find

$$\bar{\varrho} = \frac{|\alpha|^2}{2}(|00\rangle\langle 00| + |10\rangle\langle 10|) + \frac{|\beta|^2}{2}(|01\rangle\langle 01| + |11\rangle\langle 11|). \tag{3.5}$$

The corresponding quantum capacity of the ideal channel with pure signal states then reads

$$C = 1 - |\alpha|^2 \log|\alpha|^2 - |\beta|^2 \log|\beta|^2 = 1 + S(\hat{\varrho}_A). \tag{3.6}$$

and is equal for *all* alphabets which satisfy the condition (2.11).

Comment 3. The capacity (3.6) is the biggest possible capacity of the quantum channel for alphabets that are generated by local operations from the reference state (1.5). To see this, we can imagine for a while that there exist four local unitary transformations that generate the alphabet for which the capacity is bigger than Eq. (3.6). In the case of a maximally entangled state they must fulfill the Bell limit, i.e., the alphabet is an orthogonal basis. On the other hand, we have shown that all transformations that satisfy the Bell limit have to fulfill the condition (2.11). Consequently, they have to belong to the set of our equivalent transformations, with the channel capacity (3.6). This contradicts the original assumption, which proves our statement.

1. Mixed reference state

Let us assume that the reference state $\hat{\varrho}_{AB}$ shared by Alice and Bob is a statistical mixture that is parametrized as

$$\hat{\varrho}_{AB} = \sum_{j=1}^{4} \lambda_j |\chi_j\rangle\langle\chi_j|. \tag{3.7}$$

In this spectral decomposition the orthogonal states $|\chi_j\rangle$ can be written in the same Schmidt basis for all $j = 1,2,3,4$:

$$\begin{aligned}
|\chi_1\rangle &= \alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B;\\
|\chi_2\rangle &= \beta^*|0\rangle_A|0\rangle_B - \alpha^*|1\rangle_A|1\rangle_B;\\
|\chi_3\rangle &= \gamma|0\rangle_A|1\rangle_B + \delta|1\rangle_A|0\rangle_B;\\
|\chi_4\rangle &= \delta^*|0\rangle_A|1\rangle_B - \gamma^*|1\rangle_A|0\rangle_B.
\end{aligned} \tag{3.8}$$

(note that $|\chi_1\rangle = |\psi_1\rangle$).

With this reference state the alphabet is the set of states $\hat{\varrho}_k = \hat{U}_k \hat{\varrho}_{AB} \hat{U}_k^\dagger$ generated by the set of four local transformations $\{\hat{U}_1 = \hat{1}, \hat{U}_k = \vec{n}_k \cdot \vec{\sigma}\}$, as before.

In this case the second term in the expression (3.1) for channel capacity does not vanish. Our transformations are unitary. Therefore the entropy for state $\hat{\varrho}_k$ is the same and equals

$$S(\hat{\varrho}_{AB}) = -\sum_j \lambda_j \log\lambda_j. \tag{3.9}$$

To evaluate the final expression for the channel capacity we have to find the entropy of the state

$$\bar{\varrho} = \frac{1}{4}\sum_{k=1}^{4} \hat{U}_k \hat{\varrho}_{AB} \hat{U}_k^\dagger = \sum_{j=1}^{4} \lambda_j \frac{1}{4}\sum_{k=1}^{4} \hat{U}_k |\chi_j\rangle\langle\chi_j| \hat{U}_k^\dagger. \tag{3.10}$$

The term $\frac{1}{4}\Sigma_{k=1}^4 \hat{U}_k|\chi_j\rangle\langle\chi_j|\hat{U}_k^\dagger$ is for all $j = 1,2,3,4$, diagonal as in Eq. (3.5). It means that $\bar{\varrho}$ is diagonal in the given Schmidt basis

$$\bar{\varrho} = \frac{x}{2}(|00\rangle\langle 00| + |10\rangle\langle 10|) + \frac{y}{2}(|01\rangle\langle 01| + |11\rangle\langle 11|), \tag{3.11}$$

where

$$\begin{aligned}
x &= |\alpha|^2\lambda_1 + |\beta|^2\lambda_2 + |\gamma|^2\lambda_3 + |\delta|^2\lambda_4,\\
y &= |\alpha|^2\lambda_2 + |\beta|^2\lambda_1 + |\gamma|^2\lambda_4 + |\delta|^2\lambda_3.
\end{aligned} \tag{3.12}$$

Finally, taking into account that the reduced density operator $\hat{\varrho}_A$ has the form

$$\hat{\varrho}_A = \mathrm{Tr}_B(\hat{\varrho}_{AB}) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \tag{3.13}$$

we can express the capacity of the ideal channel as

$$\begin{aligned}
C &= \sum_j \lambda_j \log\lambda_j + 1 - x\log x - y\log y\\
&= 1 + S(\hat{\varrho}_A) - S(\hat{\varrho}_{AB}).
\end{aligned} \tag{3.14}$$

B. Pauli channel

From above, it follows alphabets that fulfill the condition (2.11) lead to the same capacity of the *ideal* quantum channel. Let us assume that the channel is noisy. We will model an imperfect channel as a Pauli channel [1] characterized by the parameters p_x, p_y, p_z and $p = p_x + p_y + p_z$. In this case, the alphabet states that are used for coding at the output can be expressed as

$$\hat{\rho}'_k = (1-p)|\psi_k\rangle\langle\psi_k| + \sum_{\mu=x,y,z} p_\mu \hat{\sigma}_\mu |\psi_k\rangle\langle\psi_k| \hat{\sigma}_\mu, \tag{3.15}$$

(here we implicitly assume that Bob's qubit is left intact). Taking into account the explicit expression for the operators $\hat{U}_k$, we find

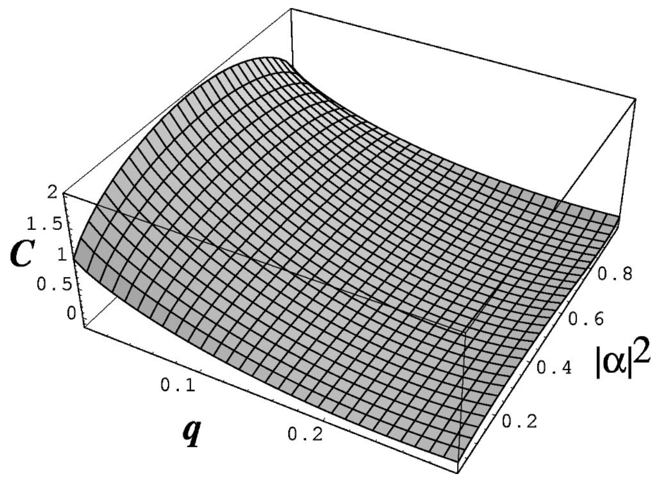

FIG. 1. We plot the capacity of the depolarizing channel as a function of the parameters q and $|\alpha|^2$ that characterize the alphabet used.

$$\hat{\sigma}_\mu \hat{U}_k = \sum_{\nu=x,y,z} n_k^{(\nu)} [\delta_{\mu\nu}\hat{\mathbb{1}} - i\varepsilon_{\mu\nu\kappa}\hat{\sigma}_\kappa]. \tag{3.16}$$

With the help of the last expression, we rewrite the density operator (3.15) as

$$\hat{\varrho}_k' = (1-p)\sum_{\mu,\nu=1}^{3} \hat{\sigma}_\mu \hat{\varrho}_0 \hat{\sigma}_\nu n_k^\mu n_k^\nu + \sum_{\mu=1}^{3} p_\mu \{(n_k^\mu)^2 \hat{\varrho}_0 + i n_k^\mu [(\vec{n}_\mu \times \vec{n}_k)\cdot \hat{\vec{\sigma}}, \hat{\varrho}_0] + [(\vec{n}_\mu \times \vec{n}_k)\cdot \hat{\vec{\sigma}}]\hat{\varrho}_0 [(\vec{n}_\mu \times \vec{n}_k)\cdot \hat{\vec{\sigma}}]\} \tag{3.17}$$

where $\hat{\varrho}_0$ denotes the reference state from which the alphabet is generated, and $\vec{n}_\mu$ is the vector defined by $\vec{n}_\mu \cdot \hat{\vec{\sigma}} = \hat{\sigma}_\mu$ for $\mu = x,y,z$. So this specifies the alphabet used. Now we want to evaluate the capacity of the channel (3.1). We assume the input probability $\pi_k = 1/4$ and in this case

$$\widehat{\varrho} = \frac{1}{4}\left(\hat{\varrho}_0 + \sum_\mu \hat{\sigma}_\mu \hat{\varrho}_0 \hat{\sigma}_\mu\right), \tag{3.18}$$

which is the same as Eq. (3.5)! So only the second term in Eq. (3.1) can be different for a different choice of Alice transformations.

Let us assume that Alice and Bob are using for communication two alphabets (2.21) and (1.6), respectively. We want to find which alphabet gives us a higher capacity of an imperefect Pauli channel.

1. Depolarizing channel

First, let us assume a depolarizing channel, for which $p_x = p_y = p_z \equiv q$ with $0 \leqslant q \leqslant 1/3$. In this case the operators $\hat{\rho}_k'$ given by Eq. (3.15) have the same eigenvalues for both alphabets that read

$$\eta_1 = 2q|\alpha|^2; \quad \eta_2 = 2q|\beta|^2;$$

$$\eta_3 = \frac{1}{2}\left(1 - 2q + \sqrt{(1-2q)^2 - 16q|\alpha|^2|\beta|^2(1-3q)}\right); \tag{3.19}$$

$$\eta_4 = \frac{1}{2}\left(1 - 2q - \sqrt{(1-2q)^2 - 16q|\alpha|^2|\beta|^2(1-3q)}\right),$$

so that the capacity can be expressed as $C = S(\widehat{\rho}) - S(\hat{\rho}_k')$. We plot this capacity in Fig. 1. We clearly see that the larger the degree of entanglement, the greater is the capacity of the quantum channel, irrespectively, on the value of the parameter q.

From above it follows that for the depolarizing channel both alphabets provide us with the same capacity.

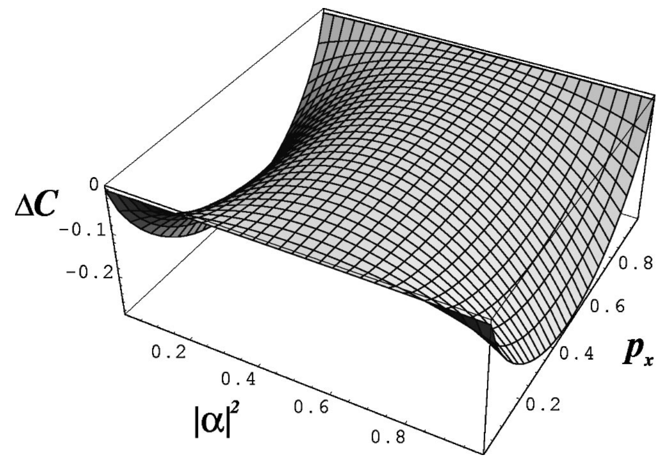

FIG. 2. We plot the difference between the capacity using the standard alphabet (1.6) and the equally distant alphabet. This difference is plotted as a function of $|\alpha|^2$ and p_x that characterizes the x-Pauli channel.

2. x-Pauli channel

Now our task is to present an example that illustrates that with the equally distant alphabet (2.11), Alice can perform better (i.e., the channel capacity is higher) than with the standard alphabet (1.6). Let us assume the channel such that $p_y = p_z = 0$ with $0 \geqslant p_x \geqslant 1$. In this case we find two nonzero eigenvalues of the output state $\hat{\varrho}_k'$ (3.17)

$$\eta_\pm = \frac{1}{2}(1 \pm 4p(1-p_x)\delta_k^2), \tag{3.20}$$

where $\delta_k^2 = 1 - |\langle\psi_k|\hat{\sigma}_x|\psi_k\rangle|^2$, $|\psi_k\rangle = \hat{U}_k \otimes \hat{\mathbb{1}}|\psi_0\rangle$ with

$$\langle\psi_k|\hat{\sigma}_x|\psi_k\rangle = 2n_k^z n_k^x(\alpha^2 - \beta^2). \tag{3.21}$$

For the standard alphabet (1.6) we find $\delta_k = 1$ for all k, while for the equally distant alphabet (2.11) $\delta_1 = 1, \delta_2^2 = 1 - 8\triangle^2/9, \delta_3^2 = \delta_4^2 = 1 - 2\triangle^2/9$.

Using these results, we directly evaluate the two capacities of our interest. We note that for both alphabets the operator $\hat{\bar{\varrho}}$ is the same [see Eq. (3.18)]. Consequently, the entropy $S(\hat{\bar{\varrho}})$ in the expression for the channel capacity is the same. Therefore, the only difference can arise from the terms $S(\hat{\varrho}_k')$. This entropy is determined by the eigenvalues $\eta_{\pm}$. Obviously, the closer the eigenvalues are to 1/2, the larger is the entropy $S(\hat{\varrho}_k')$ and the smaller is the capacity. We see that in the case of the standard alphabet the eigenvalues are closer to 1/2 than in the case of the equally distant alphabet. Therefore we conclude that the second alphabet leads to a higher channel capacity for the given x-Pauli channel. We plot the difference between the standard and the equally distant alphabet capacities in Fig. 2.

IV. CONCLUSIONS

In this paper, we have presented a set of four local unitary operators that generate from a partially entangled pure two-qubit state a set of equally distant states with a minimal overlap. We have evaluated capacity of an ideal and Pauli channels using this alphabet. We have shown that in some cases our alphabet leads to a higher channel capacity than the standard alphabet used by Bose *et al.* [6].

We conclude that in order to validate the capacity of our quantum channel Alice has to use a block coding scheme for sending a message. Bob on his end has to perform a collective measurement on the whole message rather than individual letters (alphabet states). The explicit expression for this collective decision rule is given in Ref. [8]. As shown by Holevo [8] and Hausladen *et al.* [5], in this case the information transmitted per letter can be made arbitrarily close to the channel capacity (3.1).

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