Canonical formalism for Lagrangians with nonlocality of finite extent

R. P. Woodard*
Department of Physics, University of Florida, Gainesville, Florida 32611
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I consider Lagrangians which depend nonlocally in time but in such a way that there is no mixing between times differing by more than some finite value Δt. By considering these systems as the limits of ever higher derivative theories, I obtain a canonical formalism in which the coordinates are the dynamical variable from t to $t+\Delta t$. A simple formula for the conjugate momenta is derived in the same way. This formalism makes apparent the virulent instability of this entire class of nonlocal Lagrangians. As an example, the formalism is applied to a nonlocal analog of the harmonic oscillator.

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I. INTRODUCTION

The traditional goal of fundamental physics is to infer the rules by which the ''present state'' of a system's dynamical variables determines their future state. Since Newton's time, most attention has been given to models for which the ''present state'' of a system's dynamical variables means their values at some instant in time and possibly also the values of their first time derivatives. This restriction corresponds to equations of motion that are local in time and contain no more than second time derivatives. It has not proved useful so far in describing the physical universe on the most fundamental level, to invoke equations of motion that are either nonlocal in time or that even possess more than two time derivatives.

The deep reason behind this surprising simplification of fundamental theory seems to be the result obtained by the 19th century physicist Ostrogradski [1]. He showed that Lagrangians which possess a finite number of higher time derivatives and are not degenerate in the highest one must give rise to Hamiltonians which are *linear* in essentially half of the canonical variables. This is a nonperturbative result. Further, it cannot be altered by quantization since the instability occurs over a large volume of the canonical phase space. I will review Ostrogradski's construction in Sec. II of this paper. For now it suffices to note that the instability must apply as well to nonlocal theories which can be represented as the limits of ever higher derivative ones.

Much of the interest in nonlocal quantum field theories has been motivated by the close connection between ultraviolet divergences and local interactions [2]. Of course it does no good to avoid divergences by introducing an infinite number of instabilities against which there is not even any barrier to decay. It is therefore of interest to know when a nonlocal Lagrangian possesses a higher derivative representation and, consequently, the Ostrogradskian instability. The higher derivative representation does seem to be valid for cases such as string field theory, where the nonlocality enters through entire functions of the derivative operator and the Lagrangian cannot be made local by a field redefinition [3]. On the other hand, the higher derivative representation is certainly not valid for the inverse differential operators which result from integrating out a local field variable. It also fails for ''maximal nonlocality'' in which the action is a nonlinear function of local actions [4]. The purpose of this paper is to demonstrate by construction that the higher derivative representation is valid for ''nonlocality of finite extent'' in which the Lagrangian connects no times differing by more than some constant Δt.

Although the results of this paper apply as well to field theories, I will work in the context of a one-dimensional, point particle whose position as a function of time is $q(t)$. A nonlocal Lagrangian of finite extent Δt is one which definitely depends upon (and mixes) $q(t)$ and $q(t+\Delta t)$, and potentially depends as well upon $q(t')$ for $t<t'<t+\Delta t$. An example would be the following nonlocal generalization of the harmonic oscillator:

$$L[q](t)=\tfrac{1}{2}m\dot{q}^2(t+\Delta t/2)-\tfrac{1}{2}m\omega^2 q(t)q(t+\Delta t). \quad (1)$$

The deterministic way of viewing such theories is that the equations of motion give the dynamical variable at the latest time —$q(t+\Delta t)$— as a function of earlier times in the range $t-\Delta t\leq t'<t+\Delta t$. In our example, the equation of motion is

$$\int_0^{\Delta t} dr\,\frac{\delta L[q](t-r)}{\delta q(t)} = -m\{\ddot{q}(t)+\tfrac{1}{2}\omega^2 q(t+\Delta t)+\tfrac{1}{2}\omega^2 q(t-\Delta t)\}=0, \quad (2)$$

and its deterministic interpretation is

$$q(t+\Delta t)=-q(t-\Delta t)-\frac{2}{\omega^2}\,\ddot{q}(t). \quad (3)$$

This paper is organized as follows. Section II is devoted to a review of Ostrogradski's result for local Lagrangians depending upon N time derivatives. My canonical formalism is presented in Sec. III and shown to correctly realize the dynamics of nonlocal Lagrangians of finite extent. This formalism is applied in Sec. IV to the Lagrangian (1) discussed above. The connection with Ostrogradski's formalism is demonstrated in Sec. V. My conclusions comprise Sec. VI.

*Email address: woodard@phys.ufl.edu

II. OSTROGRADSKI'S CONSTRUCTION

Consider a Lagrangian $L(q,\dot{q},\ldots,q^{(N)})$ which depends upon the first N derivatives of the dynamical variable $q(t)$. I shall assume only that the Lagrangian is *nondegenerate*, i.e., that the equation

$$P_N = \frac{\partial L}{\partial q^{(N)}} \tag{4}$$

can be inverted to solve for $q^{(N)}$ as a function of P_N, q, and the first $N-1$ derivatives of q. This just means that the action's dependence upon $q^{(N)}$ cannot be eliminated by partial integration, so the equation of motion,

$$\sum_{I=0}^{N} \left(-\frac{d}{dt}\right)^I \frac{\partial L}{\partial q^{(I)}} = 0, \tag{5}$$

contains $q^{(2N)}$.

Since the equation of motion determines $q^{(2N)}$ as a function of q and its first $2N-1$ derivatives, one can obviously specify the initial values of these $2N$ variables. The canonical phase space must accordingly contain N coordinates and N conjugate momenta. In Ostrogradski's construction [1] the Ith coordinate is just the $(I-1)$th derivative of q,

$$Q_I \equiv q^{(I-1)}. \tag{6}$$

The momentum canonically conjugate to Q_I is

$$P_I = \sum_{J=I}^{N} \left(-\frac{d}{dt}\right)^{J-I} \frac{\partial L}{\partial q^{(J)}}. \tag{7}$$

A consequence of nondegeneracy is that the derivatives $q^{(N+I)}$ can be determined from $P_{N-I}, P_{N-I+1}, \ldots, P_N$ and the Q_J's. In particular, $q^{(N)}$ involves only P_N and the Q_J's,

$$q^{(N)} = \mathcal{Q}(\vec{Q}, P_N). \tag{8}$$

Ostrogradski's Hamiltonian is

$$H = \sum_{I=1}^{N} P_I \dot{Q}_I - L, \tag{9}$$

$$= \sum_{I=1}^{N-1} P_I Q_{I+1} + P_N \mathcal{Q}(\vec{Q}, P_N) - L(\vec{Q}, \mathcal{Q}(\vec{Q}, P_N)), \tag{10}$$

and his canonical equations are the ones suggested by the notation

$$\dot{Q}_I = \frac{\partial H}{\partial P_I}, \quad \dot{P}_I = -\frac{\partial H}{\partial Q_I}. \tag{11}$$

It is straightforward to check that the various canonical evolution equations reproduce the equation of motion and the structure of the canonical formalism: $\dot{Q}_I$ gives the canonical definition (6) for Q_{I+1}, $\dot{Q}_N$ gives the canonical definition for P_N in its inverse form (8), $\dot{P}_{I+1}$ gives the canonical definition (7) for P_I, and $\dot{P}_1$ gives the equation of motion (5). So there is no doubt that Ostrogradski's Hamiltonian generates time evolution. When the Lagrangian is free of explicit time dependence, H is also the conserved current associated with time translation invariance.

The instability consequent upon H's linearity in $P_1, P_2, \ldots, P_{N-1}$ explains why higher derivative theories have not been of use in describing physics on the fundamental level. Note the generality of the problem. It does not depend upon any approximation scheme, nor upon any feature of the Lagrangian except nondegeneracy. Further, it must continue to afflict the theory after quantization because the instability is not confined to a small region of the classical phase space. If a fully nonlocal Lagrangian can be represented as the limit of such higher derivative Lagrangians, it must inherit their instability.

The limit of infinite N is facilitated by regarding Ostrogradski's formalism as the result of constraining a larger system with an extra pair of canonical variables,

$$Q_{N+1} \equiv q^{(N)}, \quad P_{N+1} \approx 0. \tag{12}$$

The Hamiltonian is

$$H = \sum_{I=1}^{N} P_I Q_{I+1} - L(\vec{Q}, Q_{N+1}), \tag{13}$$

and requiring that P_{N+1} remains zero imposes the canonical definition of P_N as another constraint,

$$\dot{P}_{N+1} = -\frac{\partial H}{\partial Q_{N+1}} = -P_N + \frac{\partial L}{Q_{N+1}} \approx 0. \tag{14}$$

Since the Poisson bracket with P_{N+1} gives the second derivative of the Lagrangian with respect to Q_{N+1}, nondegeneracy implies that the two constraints are second class. The resulting Dirac brackets are

$$\{Q_I, Q_J\}_D = (-\delta_{IN+1}\delta_{JN} + \delta_{IN}\delta_{JN+1}) \left[\frac{\partial^2 L}{\partial Q_{N+1}^2}\right]^{-1}, \tag{15}$$

$$\{Q_I, P_J\}_D = \delta_{IJ} - \delta_{IN+1} \left[\frac{\partial^2 L}{\partial Q_{N+1}^2}\right]^{-1} \frac{\partial^2 L}{\partial Q_J \, \partial Q_{N+1}}, \tag{16}$$

$$\{P_I, P_J\}_D = 0. \tag{17}$$

Note that there is not even any difference between Dirac brackets and Poisson brackets provided one avoids the highest Q, that is, Q_{N+1}.

III. MY CONSTRUCTION FOR FINITE NONLOCALITY

I define a nonlocal Lagrangian $L[q](t)$ of finite extent Δt as one which potentially depends upon the dynamical variable from time t to time $t+\Delta t$, with guaranteed mixing between $q(t)$ and $q(t+\Delta t)$. The requirement of mixing is the generalization of nondegeneracy and it implies

$$\frac{\delta^2 L[q](t)}{\delta q(t)\,\delta q(t+\Delta t)} \neq 0. \tag{18}$$

I shall also require that the Lagrangian contain no derivatives of either $q(t)$ or $q(t+\Delta t)$.

I label the canonical variables by a continuum parameter $0 \leqslant s \leqslant \Delta t$. They are defined as follows:

$$Q(s,t) \equiv q(s+t), \tag{19}$$

$$P(s,t) \equiv \int_s^{\Delta t} dr \frac{\delta L[q](s+t-r)}{\delta q(s+t)}. \tag{20}$$

Note that Eq. (20) implies the constraint $P(\Delta t,t) \approx 0$. Note also that whereas the s and t derivatives of $Q(s,t)$ are identical, those of $P(s,t)$ are not,

$$\frac{d}{ds}P(s,t) = \frac{d}{dt}P(s,t) - \frac{\delta L[q](t)}{\delta q(s+t)}. \tag{21}$$

Since $L[q](t)$ involves the dynamical variable from $q(t)$ up to $q(t+\Delta t)$, we see that $P(s,t)$ involves $q(s+t-\Delta t)$ up to $q(t+\Delta t)$. So decreasing s allows one to reach back further before time t, all the way to time $t-\Delta t$ at $s=0$.

Note that the equation of motion is $P(0,t)=0$. This emerges as an additional constraint from surface variations of the canonical Hamiltonian

$$H(t) \equiv \int_0^{\Delta t} dr\, P(r,t) \frac{d}{dr} Q(r,t) - L[Q](t). \tag{22}$$

We can find the canonical equations of time evolution from the fact that the only nonzero Poisson bracket is

$$\{Q(r,t),P(s,t)\} = \delta(r-s). \tag{23}$$

The result for $Q(s,t)$ is straightforward,

$$\frac{d}{dt}Q(s,t) = \{Q(s,t),H(t)\}, \tag{24}$$

$$= \int_0^{\Delta t} dr\, \delta(r-s) \frac{d}{dr} Q(r,t), \tag{25}$$

$$= \frac{d}{ds} Q(s,t). \tag{26}$$

A partial integration is necessary for $P(s,t)$ and one must be careful about the resulting surface terms,

$$\frac{d}{dt}P(s,t) = \{P(s,t),H(t)\}, \tag{27}$$

$$= -\int_0^{\Delta t} dr\, P(r,t) \frac{d}{dr}\delta(r-s) + \frac{\delta L[Q](t)}{\delta Q(s,t)}, \tag{28}$$

$$= \frac{d}{ds}P(s,t) + \frac{\delta L[Q](t)}{\delta Q(s,t)} - \delta(r-s)P(r,t)\Big|_0^{\Delta t}. \tag{29}$$

For $0 < s < \Delta t$ this simply reproduces Eq. (21), and hence the canonical definition of $P(s,t)$.

Since the two surface terms cannot be canceled by anything else, they must be imposed as constraints,

$$P(s,t) \approx 0, \quad s = 0,\Delta t. \tag{30}$$

Requiring that they be preserved under time evolution implies two additional constraints,

$$\frac{d}{ds}P(s,t) + \frac{\delta L[Q](t)}{\delta Q(s,t)} \approx 0, \quad s=0,\Delta t. \tag{31}$$

Nondegeneracy — and the absence in $L[q](t)$ of derivatives of $q(t)$ and/or $q(t+\Delta t)$—guarantees that the four constraints are second class.

Note that the $H(t)$ is conserved when $L[q](t)$ is free of explicit time dependence,

$$\frac{dH}{dt}(t) = \int_0^{\Delta t} ds \left\{ \frac{dP}{dt}(s,t)\frac{dQ}{ds}(s,t) + P(s,t)\frac{d^2Q}{ds^2}(s,t) \right\} - \frac{d}{dt}L[Q](t), \tag{32}$$

$$= \int_0^{\Delta t} ds \frac{d}{ds}\left[P(s,t)\frac{d}{ds}Q(s,t) \right] + \int_0^{\Delta t} ds \frac{\delta L[Q](t)}{\delta Q(s,t)} \frac{d}{ds}Q(s,t) - \frac{d}{dt}L[Q](t), \tag{33}$$

$$= P(s,t)\frac{d}{ds}Q(s,t)\Big|_0^{\Delta t}, \tag{34}$$

$$\approx 0. \tag{35}$$

Note also that the Hamiltonian has inherited the Ostrogradskian instability. After eliminating the constraints, it must be linear in all the $P(s,t)$ except possibly $(d/ds)P(s,t)$ at $s=0$ and at $s=\Delta t$.

IV. A SIMPLE EXAMPLE

It is useful to see how the general construction given in the preceding section applies to the Lagrangian (1) presented in Sec. I. Of course the canonical coordinates are always $Q(s,t)=q(s+t)$ for $0 \leqslant s \leqslant \Delta t$. To find the canonical momenta, note that the functional derivative of the Lagrangian is

$$\frac{\delta L[q](t)}{\delta q(s+t)} = -m\dot{q}\left(t+\frac{\Delta t}{2}\right)\delta'\left(s-\frac{\Delta t}{2}\right) - \frac{1}{2}m\omega^2[q(t+\Delta t)\,\delta(s) + q(t)\,\delta(s-\Delta t)]. \tag{36}$$

Substituting in Eq. (20) gives

$$P(s,t)=m\dot{q}\left(t+\frac{\Delta t}{2}\right)\delta\left(s-\frac{\Delta t}{2}\right)-m\ddot{q}(s+t)\,\theta\left(\frac{\Delta t}{2}-s\right)-\frac{1}{2}m\omega^2[q(s+t+\Delta t)\,\theta(-s)+q(s+t-\Delta t)\,\theta(\Delta t-s)]. \tag{37}$$

Note that $P(\Delta t,t)=0$ and that

$$P(0,t)=-m[\ddot{q}(t)+\tfrac{1}{2}\omega^2 q(t+\Delta t)+\tfrac{1}{2}\omega^2 q(t-\Delta t)] \tag{38}$$

indeed vanishes with the equation of motion.

The canonical Hamiltonian is

$$H(t)=\int_0^{\Delta t}ds\,P(s,t)\frac{dQ}{ds}(s,t)-\frac{1}{2}m\left[\frac{dQ}{ds}\left(\frac{\Delta t}{2},t\right)\right]^2+\frac{1}{2}m\omega^2 Q(0,t)Q(\Delta t,t). \tag{39}$$

The canonical evolution equations are

$$\frac{dQ}{dt}(s,t)=\frac{dQ}{ds}(s,t), \tag{40}$$

$$\frac{dP}{dt}(s,t)=\frac{dP}{ds}(s,t)-m\frac{dQ}{ds}\left(\frac{\Delta t}{2},t\right)\delta'\left(s-\frac{\Delta t}{2}\right)-\frac{1}{2}m\omega^2[Q(\Delta t,t)\,\delta(s)+Q(0,t)\,\delta(s-\Delta t)]. \tag{41}$$

It is simple to check that substituting $Q(s,t)=q(s+t)$ and relation (37) for $P(s,t)$ indeed verifies these equations.

The constraints are $P(0,t)\approx 0$, $P(\Delta t,t)\approx 0$, and the apparently singular pair

$$\frac{dP}{ds}(0,t)-\tfrac{1}{2}m\omega^2 Q(\Delta t,t)\,\delta(0)\approx 0, \tag{42}$$

$$\frac{dP}{ds}(\Delta t,t)-\tfrac{1}{2}m\omega^2 Q(0,t)\,\delta(0)\approx 0. \tag{43}$$

However, the vanishing of $P(s,t)$ at the end points means that the end-point derivatives contain δ functions, so the actual constraints are the perfectly regular coefficients of $\delta(0)$,

$$P(0^+,t)-\tfrac{1}{2}m\omega^2 Q(\Delta t,t)\approx 0, \tag{44}$$

$$-P(\Delta t^-,t)-\tfrac{1}{2}m\omega^2 Q(0,t)\approx 0. \tag{45}$$

Note that these constraints are implied by Eq. (37) and, where necessary, the vanishing of Eq. (38). Note also that the constraints determine both the actual end-point values of $P(s,t)$ and its limit as the end points are approached.

Since the Lagrangian (1) has no explicit dependence upon time, the Hamiltonian should be conserved. To see that it is, first substitute $Q(s,t)=q(s,t)$ and relation (37) for $P(s,t)$ to obtain

$$\int_0^{\Delta t}ds\,P(s,t)\frac{dQ}{dq}(s,t)=m\dot{q}^2\left(t+\frac{\Delta t}{2}\right)-m\int_0^{\Delta t/2}ds\,\dot{q}(s+t)\ddot{q}(s+t)-\frac{1}{2}m\omega^2\int_0^{\Delta t}ds\,q(s+t-\Delta t)\dot{q}(s+t), \tag{46}$$

$$=\frac{1}{2}m\left[\dot{q}^2\left(t+\frac{\Delta t}{2}\right)+\dot{q}^2(t)\right]-\frac{1}{2}m\omega^2\int_0^{\Delta t}ds\,q(s+t-\Delta t)\dot{q}(s+t). \tag{47}$$

Then subtract expression (1) to determine the configuration space Hamiltonian,

$$H(t)=\frac{1}{2}m\dot{q}^2(t)+\frac{1}{2}m\omega^2 q(t)q(t+\Delta t)-\frac{1}{2}m\omega^2\int_0^{\Delta t}ds\,q(s+t-\Delta t)\dot{q}(s+t). \tag{48}$$

Now use the fact that the integrand depends upon t only through the sum $s+t$ to express the derivative of the integral as a surface term,

$$\frac{dH}{dt}(t)=m\dot{q}(t)\ddot{q}(t)+\frac{1}{2}m\omega^2[\dot{q}(t)q(t+\Delta t)+q(t)\dot{q}(t+\Delta t)]-\frac{1}{2}m\omega^2 q(s+t-\Delta t)\dot{q}(s+t)\Big|_{s=0}^{s=\Delta t}, \tag{49}$$

$$=m\dot{q}(t)\left[\ddot{q}(t)+\frac{1}{2}\omega^2 q(t-\Delta t)+\frac{1}{2}\omega^2 q(t+\Delta t)\right]. \tag{50}$$

The most straightforward way of demonstrating that the transformation to the constrained phase space is invertible is by exhibiting the inverse. Of course we always have

$$q(s+t)=Q(s,t),\quad \forall 0\leqslant s\leqslant \Delta t. \tag{51}$$

For $-\Delta t<s<0$ one recovers $q(s+t)$ from relation (37),

$$q(s+t)=-\frac{2}{m\omega^2}P(s+\Delta t,t)-\frac{2}{\omega^2}\frac{d}{ds}\left[\frac{dQ}{ds}(s+\Delta t,t)\,\theta\left(-\frac{\Delta t}{2}-s\right)\right]. \tag{52}$$

The end-point case of $s=-\Delta t$ is given by the constraint $P(0,t)\approx 0$,

$$q(t-\Delta t)=-Q(\Delta t,t)-\frac{2}{\omega^2}\frac{d^2Q}{ds^2}(0,t), \tag{53}$$

where I am of course defining differentiation in the right-handed sense,

$$\frac{df}{dx}(x)\equiv\lim_{\epsilon\to 0^+}\frac{f(x+\epsilon)-f(x)}{\epsilon}. \tag{54}$$

It is amusing to close the section by exhibiting the runaway solutions which are one possible consequence of the Ostrogradskian instability. Since the configuration space equation of motion,

$$\ddot{q}(t)+\tfrac{1}{2}\omega^2[q(t+\Delta t)+q(t-\Delta t)]=0, \tag{55}$$

is linear and invariant under time translation, the general solution must be a superposition of terms having the form e^{ikt}. The allowed frequencies are complex numbers k which obey

$$k^2=\omega^2\cos(k\Delta t). \tag{56}$$

The equation is transcendental but graphing both sides shows a single pair of $\pm$ real solutions. To find the remaining solutions, make the substitution

$$k=\alpha+i\beta, \tag{57}$$

and take the real and imaginary parts of the equation,

$$\alpha^2-\beta^2=\omega^2\cos(\alpha\Delta t)\cosh(\beta\Delta t), \tag{58}$$

$$2\alpha\beta=-\omega^2\sin(\alpha\Delta t)\sinh(\beta\Delta t). \tag{59}$$

Graphical analysis indicates a conjugate pair of solutions for $\alpha\Delta t$ in each 2π interval of the real line. For large integer N, these solutions have the form

$$\alpha\Delta t\approx 2\pi N-\frac{2\ln(N)}{\pi N}, \tag{60}$$

$$\pm\beta\Delta t\approx\ln\left(\frac{8\pi^2N^2}{\omega^2\Delta t^2}\right)+\left(\frac{\ln(N)}{\pi N}\right)^2. \tag{61}$$

So this system has the infinite number of solutions predicted by the Ostrogradskian analysis, and all but two of them grow or fall exponentially.

V. OSTROGRADSKIAN DERIVATION

My representation is related to the infinite N limit of Ostrogradski's through the Maclaurin series,

$$Q(s,t)=\sum_{I=0}^{\infty}\frac{s^I}{I!}Q_{I+1}(t). \tag{62}$$

Note that differentiation with respect to the Ostrogradskian coordinates is realized by the functional chain rule

$$\frac{\partial}{\partial Q_I(t)}=\int_0^{\Delta t}ds\left[\frac{\partial Q(s,t)}{\partial Q_I(t)}\right]\frac{\delta}{\delta Q(s,t)}=\int_0^{\Delta t}ds\,\frac{s^{I-1}}{(I-1)!}\frac{\delta}{\delta Q(s,t)}, \tag{63}$$

where the functional derivative is defined by

$$\frac{\delta Q(r,t)}{\delta Q(s,t)}=\delta(r-s) \tag{64}$$

and the ordinary rules of calculus. From Eq. (19) one obtains a useful formula for the higher derivative representation,

$$\frac{\partial L[q](t)}{\partial q^{(I)}(t)}=\int_0^{\Delta t}ds\,\frac{s^I}{I!}\frac{\delta L[q](t)}{\delta q(s+t)}. \tag{65}$$

The conjugate momentum $P(s,t)$ should depend linearly on the Ostrogradskian momenta,

$$P(s,t)=\sum_{I=0}^{\infty}p_I(s)P_{I+1}(t). \tag{66}$$

The combination coefficients $p_I(s)$ can be determined by enforcing the canonical Poisson bracket (23),

$$\delta(r-s)=\sum_{I=0}^{\infty}\frac{r^I}{I!}\sum_{J=0}^{\infty}p_J(s)\{Q_{I+1}(t),P_{J+1}(t)\}, \tag{67}$$

$$=\sum_{I=0}^{\infty}\frac{r^I}{I!}p_I(s). \tag{68}$$

By acting $(\partial/\partial r)^J$ and then taking $r\to 0$, one finds

$$p_J(s)=\left(-\frac{d}{ds}\right)^J\delta(s). \tag{69}$$

To obtain my formula (20) for the conjugate momenta, note first that, for infinite N, the Ostrogradskian momenta are

$$P_I(t)=\sum_{J=I}^{\infty}\left(-\frac{d}{dt}\right)^{J-I}\frac{\partial L[q](t)}{\partial q^{(J)}(t)}, \tag{70}$$

$$=\sum_{J=I}^{\infty}\left(\frac{d}{dt}\right)^{J-I}\int_0^{\Delta t}dr\frac{r^J}{J!}\frac{\delta L[q](t)}{\delta q(r+t)}. \tag{71}$$

Now substitute this and Eq. (69) into Eq. (66),

$$P(s,t)=\sum_{I=0}^{\infty}\left[\left(-\frac{d}{ds}\right)^I\delta(s)\right]\times\sum_{J=I+1}^{\infty}\left(-\frac{d}{dt}\right)^{J-I-1}\int_0^{\Delta t}dr\,\frac{r^J}{J!}\frac{\delta L[q](t)}{\delta q(r+t)}. \tag{72}$$

Simplification is achieved by exploiting the identity

$$\frac{r^J}{J!}=\int_0^r dr'\frac{(r-r')^I(r')^{J-I-1}}{I!(J-I-1)!} \tag{73}$$

to recognize the two sums as Taylor expansions of the shift operator,

$$P(s,t)=\int_0^{\Delta t}dr\int_0^r dr'\sum_{I=0}^{\infty}\frac{(r-r')^I}{I!}\left(-\frac{d}{ds}\right)^I\delta(s)\times\sum_{J=I+1}^{\infty}\frac{(r')^{J-I-1}}{(J-I-1)!}\left(-\frac{d}{dt}\right)^{J-I-1}\frac{\delta L[q](t)}{\delta q(r+t)}, \tag{74}$$

$$=\int_0^{\Delta t}dr\int_0^r dr'\,\delta(s-r+r')\,\frac{\delta L[q](t-r')}{\delta q(r+t-r')}, \tag{75}$$

$$=\int_s^{\Delta t}dr\,\frac{\delta L[q](s+t-r)}{\delta q(s+t)}. \tag{76}$$

The Hamiltonian follows similarly,

$$H(t)=\sum_{I=1}^{\infty}P_I(t)Q_{I+1}(t)-L[Q](t), \tag{77}$$

$$=\sum_{I=1}^{\infty}\int_0^{\Delta t}ds\,\frac{s^{I-1}}{(I-1)!}P(s,t)\left(\frac{d}{dr}\right)^I Q(r,t)\bigg|_{r=0}-L[Q](t), \tag{78}$$

$$=\int_0^{\Delta t}ds\,P(s,t)\frac{d}{ds}Q(s,t)-L[Q](t). \tag{79}$$

Its instability is manifest from the fact that it has been derived from Ostrogradski's result in the limit that the number of derivatives becomes infinite.

VI. DISCUSSION

I have shown that Lagrangians with nonlocality of finite extent Δt can be treated as the limits of higher derivative Lagrangians. I have also given a canonical formalism that is somewhat more natural in which the canonical variables are labeled by a continuum parameter s, for $0\leq s\leq\Delta t$. The canonical coordinates are just the dynamical variables at times $t+s$. A quantum-mechanical state in such a system would be a functional of these coordinates. The conjugate momenta (20) are given by a simple integral of a functional derivative of the Lagrangian. With the canonical coordinates, the momenta allow one to reconstruct the dynamical variables at times $t-s$.

There is no physical motivation for this exercise because all such models are virulently unstable. Indeed, the only point of the formalism is to remove any doubt about a possible phenomenological role for these Lagrangians. They have inherited the full Ostrogradskian instability: essentially half of the directions in the classical phase space access arbitrarily negative energies. There is not even any barrier to decay. This is a nonperturbative result and, because it arises from a large region of phase space, it must survive quantization.

Negative results of such power and generality seem to pose an irresistible challenge, to mathematically inclined physicists. Nothing I can honestly add is likely to much discourage further attempts to carve out a physical niche for nonlocal Lagrangians, but I do recommend that these efforts be preceded by sober reflection upon the following fact: in the long struggle of our species to understand the universe, it has *never once* proven useful to invoke a theory that is nonlocal on the most fundamental level. Yet the subset of local Lagrangians containing no more than first derivatives is a minuscule fragment of the set of all functionals of the dynamical variable. The Ostrogradskian instability offers a simple and compelling explanation for the complete dominance of this tiny subset over its much larger whole. The only alternative would seem to be coincidence on a scale that makes even the worst fine-tuning problem seem inconsequential.

Note added. Shortly after submitting this paper, I learned of important work by Llosa and Vives [5] on the problem of canonically formulating a general nonlocal Lagrangian. My work can be viewed as a specialization of their technique to the case of nonlocality of finite extent where the Euler-Lagrange equations are deterministic, where an explicit Poisson bracket structure can be determined and where the formalism can be derived from the infinite N limit of Ostrogradski's construction. (None of these features can be present in the general case.) Note should also be taken of the recent work of Gomis, Kamimura, and Llosa on canonically formulating space-time noncommutative theories [6].

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