Metastable states of a coupled pair on a repulsive barrier

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Resonance penetration of two coupled particles through a repulsive barrier is considered. It is shown that a local minimum of the total potential generates metastable bound states, and their spectrum determines the position of resonances in the penetration probability. It is pointed out that the probabilities of tunneling of two interacting particles from the false vacuum can be essentially higher than has been assumed earlier.

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In a paper by Saito and Kayanuma [1], it was pointed out that there exists a new quantum phenomenon—resonance transparency of a single repulsive barrier for a coupled pair of particles. To consider this effect, a one-dimensional rectangular repulsive barrier and an infinite one-dimensional rectangular potential well coupling the pair were chosen. Since the interactions were simple, it was possible to solve the initial two-dimensional Schrödinger equation by reducing it to a system of one-dimensional equations by means of projection onto seven eigenfunctions. However, there still remained the question of how this effect manifests itself in other systems.

This paper continues the study of the effect of resonance transparency of two type one-dimensional barriers for a pair of identical particles coupled by the oscillatory interaction. This pair interaction allows us to reduce the problem of three-dimensional scattering of a three-dimensional oscillator to the solution of a two-dimensional equation analogous to the equation derived in Ref. [1]. Besides, it is just this sort of pair interaction that is used in literature [2] devoted to the probability of induced decay of the false vacuum in collisions of high-energy particles (see, for instance, [3,4]). It was pointed out there that it is possible to describe the processes of transition from the false vacuum on the basis of quantum-mechanical tunneling of a pair of particles through the barrier; but the study was performed for a system where only one of the oscillator particles interacts with the barrier. Here, we show that, when the two particles interact with the barrier, there arises the same effect of resonance transparency as in Ref. [1].

The first potential barrier we study is taken in the Gaussian form from Ref. [2] in order to show that it is possible to drastically increase the probability of induced decay of the false vacuum. The second potential barrier of the Coulomb form is investigated in order to draw attention to the fact that the resonance tunneling of the barrier is feasible in the problems of fusion of heavy nuclei. The method of investigation is based on the numerical solution of the two-dimensional Schrödinger equation without any further simplifications.

Consider the penetration of a pair of identical particles with masses $m_1=m_2=m$, and coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ coupled by an oscillatory interaction through the potential barrier $V_0(x_1)+V_0(x_2)$. The Hamiltonian of this system ($\hbar=1$)

$$-\frac{1}{4m}\Delta_R-\frac{1}{m}\Delta_r+\frac{m\omega^2}{4}r^2+V_0(\mathbf{R}-\mathbf{r}/2)+V_0(\mathbf{R}+\mathbf{r}/2),$$

written in coordinates of the center of inertia of the pair $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2$ and in an internal coordinate of the relative motion $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ describes the three-dimensional motion of a three-dimensional oscillator with the frequency of vibrations ω. Since the potential barrier depends only on one variable, and the oscillatory interaction is additive in $\mathbf{r}$ projections, the wave function is factorized, and its nontrivial part describing the process of scattering depends only on two variables. It is convenient to represent these variables in the dimensionless form

$$x=\sqrt{\frac{m\omega}{2}}(x_1-x_2),\quad y=\sqrt{\frac{m\omega}{2}}(x_1+x_2).$$

The Schrödinger equation in these variables is of the form

$$[-\partial_x^2-\partial_y^2+x^2+V(x-y)+V(x+y)-E]\Psi=0, \tag{1}$$

where the energy E is written in units $\omega/2$, and the potential barrier $V(x\pm y)=(2/\omega)V_0[(x\pm y)/\sqrt{2m\omega}]$ in what follows will be written in a form convenient for us. Equation (1) should be supplemented with boundary conditions. Let the process of scattering proceed from left to right, and the initial state of the oscillator be the state n. Then the boundary conditions are written in the form

$$\Psi|_{y\to-\infty}=\exp(ik_ny)\varphi_n(x)-\sum_{j\leq N}S_{nj}\exp(-ik_jy)\varphi_j(x),$$

$$\Psi|_{y\to+\infty}=\sum_{j\leq N}R_{nj}\exp(ik_jy)\varphi_j(x), \tag{2}$$

$$\Psi|_{x\to\pm\infty}=0.$$

The wave functions of the oscillator $\varphi_j(x)$ obey the Schrödinger equation

$$(-\partial_x^2+x^2-\varepsilon_i)\varphi_i=0 \tag{3}$$

with the energy $\varepsilon_j=2j+1$ $(j=0,1,2,\dots)$, momenta $k_j=\sqrt{E-\varepsilon_j}$, and the number N of the last open channel ($E-\varepsilon_{N+1}<0$). In what follows, we consider an oscillator com-

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posed of bosons, whose spectrum is convenient to number from 1. So, hereafter, $\varepsilon_j = 4j-3$ $(j=1,2,\ldots)$.

The probabilities of penetration W_{ij} and reflection D_{ij} are defined as the ratio of the density of a penetrated or a reflected flux to the density of the flux of incident particles:

$$W_{ij}=|R_{ij}|^2\frac{k_j}{k_i}, \quad D_{ij}=|S_{ij}|^2\frac{k_j}{k_i}.$$

It is obvious that $\Sigma_{j\leqslant N}(W_{ij}+D_{ij})=1$.

To determine the probabilities of penetration (reflection) in the above way, it is necessary to solve the two-dimensional differential equation (1); its numerical solutions will be given below. The numerical solution was based on a three-diagonal approximation of second derivatives with constant steps in x and y: $h_x=0.025$, $h_y=0.005$, respectively. Finite dimensions $|y_{\max}|=12$ and $|x_{\max}|=7$ of the range of numerical calculations, at a given degree of discretization, provided accuracy to within the third decimal point in all the presented calculations.

As it was indicated above, in papers devoted to the induced decay of false vacuum, use was made of the model of quantum-mechanical tunneling of a pair coupled by the oscillatory interaction through a barrier [2]. The case when only one particle interacts with the barrier was considered. In the framework of the resonance tunneling, we can expect that the picture of tunneling would essentially change when the interaction of both the incident particles with the barrier is switched on. The barrier used in Ref. [2], upon being made dimensionless, is of the form

$$V(X)=\frac{2}{g^2\omega}\exp(-g^2X^2/\omega), \quad X=x\pm y, \tag{4}$$

where ω is the oscillator frequency, and $g^2\ll 1$ is the model parameter of false vacuum. In Ref. [2], the dependence of the tunneling probability was calculated in the range of g^2 from 0.09 till 0.01 and at fixed frequency $\omega=1/2$. In the present calculations, the same value of ω but greater values of g^2 are accepted. The reason is that there arise extremely (in the framework of numerical calculations) narrow resonances in the energy dependence of the tunneling probability of a coupled pair through the barrier. Therefore, in Fig. 1, we present the results of numerical calculations of Eq. (1) with barrier (4) at $g^2=0.5$, 0.3, 0.2 denoted by letters A, B, C, respectively.

In Fig. 1, we draw the probabilities of penetration of a coupled pair from the ground state into all the possible states, i.e., $W=\Sigma_{j\leqslant N}W_{1j}$ at different values of g^2. A prominent resonance dependence of the tunneling probability shows that there do exist barrier resonances under discussion. Note that the first resonance with decreasing g^2 shifts toward higher energies, but the quantity g^2E_r diminishes.

When $g^2=0.2$, the probability of resonance penetration is many times ($\sim 10^8$) as large as the probability of penetration in the nonresonance region (background). Therefore, in Fig. 1, we present the results of calculations on the logarithmic scale. They demonstrate not only the indicated exceeding but also the coincidence of the background part of the curve with the probability of penetration of a structureless particle, i.e., with the solution of Eq. (1) for the barrier potential $2V(y)$. To estimate the contribution of narrow resonances to the probability of penetration of the particle flux distributed over energy, in Fig. 1, we plot the penetrated flux

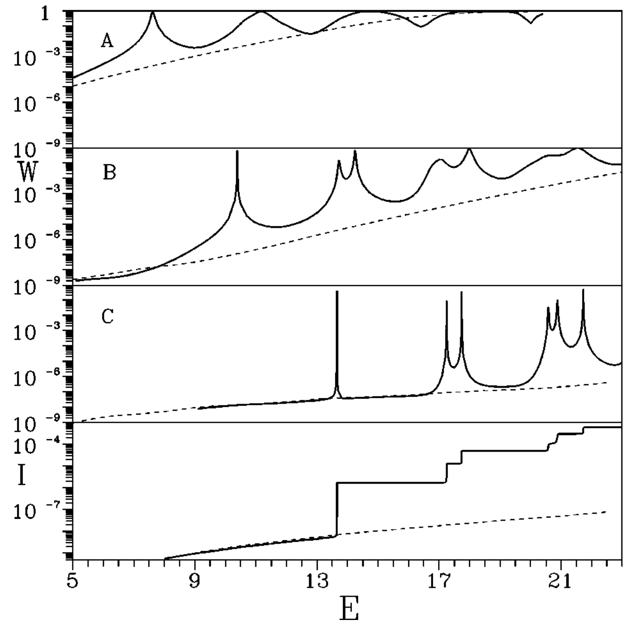

FIG. 1. Total probabilities of penetration through the barrier. The solid curve is for potential (4); the dotted one, for potential $2V(y)$. The explanation is given in the text.

$$I(E)=\frac{1}{E-E_0}\int_{E_0}^{E}W(E')dE',$$

in the case when the incident flux is distributed uniformly from E_0 till E; the quantity $E_0=5$. It is seen that the main contribution to the probability of penetration comes from resonances. At $E=23$, the difference from the background penetration amounts to four orders.

Now, we describe a simple scheme of arising barrier metastable states that make the barrier transparent. It is not difficult to verify that the potential energy $U(x,y)=V(x+y)+V(x-y)+x^2$ possesses a local minimum at $y=0$ (the center of mass in the middle of the barrier) and at some values of $x=\pm x_0$. A maximum is at $x=0$.

Thus, there exist two potential ''wells'' separated by the barrier. Bound states of that system split into even and odd states. The magnitude of splitting is determined by the probability of penetration through the internal barrier. When $2V(x=0,y=0)\gg 1$, this shift can be very small, and the spectrum of even states is determined by the spectrum of an isolated ''well.'' In the first approximation, the position of resonances can be described by the oscillator spectrum of bound states at $y=0$ and $x=x_0$:

$$E_{n_xn_y}=E_0+2\omega_x(1/2+n_x)+2\omega_y(1/2+n_y), \tag{5}$$

where n_x and n_y are oscillator quantum numbers; $E_0=2V(x_0)$; and frequencies ω_x and ω_y are determined by the second derivatives at the point of local minimum: $\omega_x=\sqrt{\partial_x^2 U(x,y)/2}$, $\omega_y=\sqrt{\partial_y^2 U(x,y)/2}$.

For potential (4), it is not difficult to obtain the oscillator-model parameters

$$E_0=\omega[1+2\ln(2/\omega)]/g^2,$$

$$\omega_x^2=4\ln(2/\omega), \tag{6}$$

$$\omega_y^2=\omega_x^2-1.$$

Let us compare the positions of resonances drawn in Fig. 1 at $g^2=0.2$ with the results of calculation by formulas (5) and (6) from which it is clear that there exist small but clearly seen satellite resonances. The position of the first resonance $E_r=13.65$ is well described by the oscillator energy in the ground state $E_{00}=13.92$. The second group of resonances is generated by a single excitation of oscillators either along y or along x: $E_{01}=18.18$, $E_{10}=18.63$. They are associated with the resonances at energies 17.24 and 17.74. The third group of resonances is generated by a double excitation $E_{02}=22.45$, $E_{11}=22.90$, $E_{20}=23.34$, and respectively, resonance energies are 20.58, 20.88, 21.72. So, the simple oscillator model of a metastable barrier state gives a correct qualitative picture of the origin of resonances. Comparison with Fig. 1 shows that the largest values of the tunneling probability correspond to metastable states with a minimal excitation along the coordinate of the center of inertia.

Decomposition around the point of equilibrium does not exhaust all possibilities of the oscillator model. Agreement between the resonance energy and the energy of a metastable state can be improved by a simple variational procedure. To this end, we consider the position of a minimum x_0 and frequencies ω_x and ω_y to be unknown quantities that are determined by the minimum of the average of the total Hamiltonian

$$\overline{H}=\langle\phi_x\phi_y|-\partial_x^2-\partial_y^2+x^2+V(x-y)+V(x+y)|\phi_y\phi_x\rangle$$

over normalized eigenfunctions of the oscillator in the ground state:

$$\phi_x=\left(\frac{\omega_x}{2\pi}\right)^{1/4}\exp\left(-\frac{\omega_x}{4}(x-x_0)^2\right),$$

$$\phi_y=\left(\frac{\omega_y}{2\pi}\right)^{1/4}\exp\left(-\frac{\omega_y}{4}y^2\right).$$

Varying $\overline{H}$ over x_0, ω_x, and ω_y, we can derive a system of three nonlinear equations, which will not be presented here because they are too cumbersome. The two of the three equations are solved analytically: $x_0=x_0(\omega_x,\omega_y)$, $\omega_y=\sqrt{\omega_x^2-1}$. In this way, the function of one variable $\overline{H}=\overline{H}(\omega_y)$ is to be estimated numerically; its minimum determines the variational estimate $E_{\rm var}$ for the first resonance. Note that the variational connection of ω_x and ω_y is the same as in the case of decomposition (6). In Table I, we present the comparison of $E_{\rm var}$ with positions of the first resonance E_r drawn in Fig. 1. Agreement can be considered good in the framework of the above-indicated accuracy. When $g^2\ll 1$, the variational expressions get simplified and allow the following decomposition:

TABLE I. Comparison of positions of the first resonance with the variational estimate

g^2	$E_{\rm var}$	E_r	$g^2E_r\omega/2$
0.5	7.649	7.62	1.30
0.3	10.416	10.38	0.779
0.2	13.680	13.65	0.683

$$E_{\rm var}^{\rm as}\sim E_{00}+O(g^2).$$

It coincides, with an accuracy up to $O(g^2)$, with the energy derived by a simple decomposition around the minimum of $U(x,y)$. So, the estimate of the resonance spectrum made by formulas (5) and (6) is asymptotic as $g^2\to 0$. In particular, when $g^2\to 0$, we can indicate the limiting position of the first resonance in units of g^2, i.e., the quantity $g^2E\omega/2$ used in Ref. [2]:

$$g^2E\omega/2\to\omega^2[1+2\ln(2/\omega)]/2.$$

At $\omega=1/2$, this energy tends to 0.472; this is shown in the fourth column of Table I. In Ref. [2] where the study was made of the penetration of a pair through the barrier, a smooth curve was obtained for the probability of penetration in the energy interval from 1.2 to 2. From the presented calculations it follows that, if the interaction of both particles with the potential barrier is taken into account, this curve becomes essentially nonmonotone.

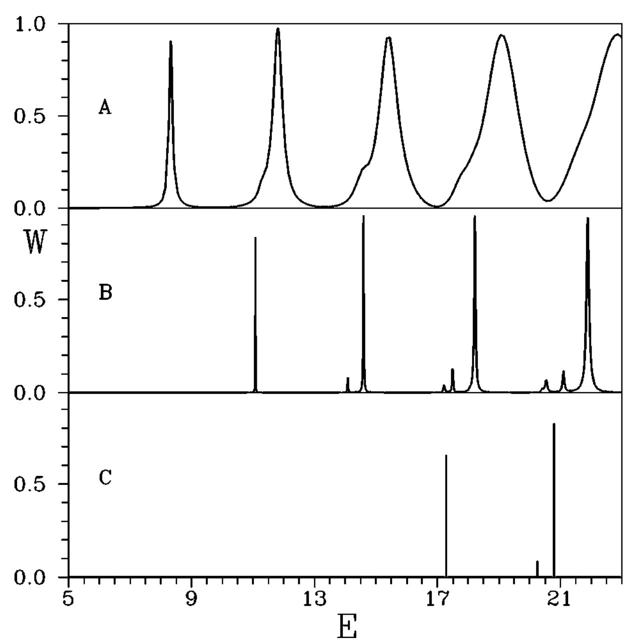

FIG. 2. Probabilities of penetration through barriers of the Coulomb type. Explanations are given in the text.

The barriers considered above are of the form of the Gaussian function. For completeness, below we present the calculations for a barrier of the Coulomb shape cutoff both at short and long distances:

$$V(X)=\begin{cases} Q/X_{\min} & :|X|<X_{\min} \\ Q/|X| & :X_{\min}\leqslant|X|\leqslant X_{\max}; \quad X=x\pm y. \\ Q/X_{\max} & :|X|>X_{\max}. \end{cases} \tag{7}$$

The cutoff at short distances was introduced for modeling the nuclear Coulomb barrier in the framework of constraints imposed by the one-dimensional scattering. With this cutoff, the notion of ''barrier height'' is meaningful for the one-dimensional model of scattering. For a greater analogy, the barrier width at $|X|=X_{\min}$ should be small in spatial units of the problem, i.e., as compared with the mean-square dimension of the oscillator. The cutoff at long distances was introduced to make it possible to use an asymptotics of the type (2). The quantity $X_{\max}$ should be larger than one for imitating the barrier of small transparency. Here we took $X_{\min}=0.1$, $X_{\max}=5$. The quantity Q determines the energy height of the barrier. In Fig. 2, we show the results of calculations for $Q=2$, 4, 10 denoted by A, B, and C, respectively.

In this case, a clear picture of the resonance tunneling of a coupled pair is also observed. We do not present the analysis of the oscillator model for the position of resonances, because the chosen potential is essentially of the model character. We only mention that satellite resonances manifest themselves clearly, and energies of principal resonances are equidistant.

The considered mechanism of the transparency of barriers for a coupled pair manifests itself for all the potential barriers chosen for the investigation. As the resonance transparency was first observed [1] for barriers of the rectangular shape, and the coupling in a pair was of the nonoscillator type, it could be assumed that the resonance transparency of barriers for composite particles could be observed for a wide class of interactions. Therefore, the effects of quantum transparency could occur in various fields of physics. In particular, when the interaction of two particles with a barrier is taken into account, the picture of induced decay of the false vacuum changes essentially.

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