Quantum theory of a Stern-Gerlach system in contact with a linearly dissipative environment

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We present the quantum theory of a Stern-Gerlach system in contact with a linearly dissipative environment at an arbitrary temperature. The dynamics of the reduced density matrix of the system is calculated using the path integral technique, under a general condition when the system and its environment may not be decoupled initially. We analyze the behavior of the density matrix in the long-time limit and compute the time scales of decay of the elements off-diagonal in the coordinate and the momentum space, for the cases of high temperature, zero temperature, and intermediate temperature baths.

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I. INTRODUCTION

The method of applying path integrals to the system-plus-environment, as proposed by Feynman and Vernon (FV) [1], has been found to be very useful in tackling the complex problem of quantum dynamics of a dissipative physical system. A few years back we investigated the problem of measurement in a Stern-Gerlach apparatus, i.e., a spin-$\frac{1}{2}$ particle in the presence of an inhomogeneous magnetic field, using the environment-induced decoherence model. In this model we took into account the coupling of the position of the particle to an environment of noninteracting harmonic oscillators [2–4]. The problem was treated using the FV path-integral approach, with the Caldeira-Leggett master equation [5] as the starting point, in the high-temperature (weak coupling) limit, and under the Markov approximation that is valid for times much larger than the relaxation time of the environment. This master equation could be solved analytically, and it yielded an apparent classical behavior of the system, in the sense that the reduced density matrix of the system was driven to a diagonal form over a characteristic time scale.

The Stern-Gerlach model of measurement is an interesting one, and it is worth looking at the problem without making the high temperature and the Markov approximations. In the FV approach [1], it is assumed that the system and its environment are decoupled at time $t=0$ (factorizable initial condition). Hakim and Ambegaokar [6] generalized this choice, but their method is applicable only to systems for which the total Hamiltonian has a translational invariance. In this paper, following Smith and Caldeira [7], we use a general path-integral formulation for the problem that allows for nonfactorizable initial conditions as well. The Stern-Gerlach problem is a nontrivial one due to the additional spin degree of freedom. Even from a technical point of view, the exact solution of this model problem is a worthwhile exercise. We obtain the solution for the dynamics of the reduced density matrix of the spin system in the coordinate space as well as the conjugate momentum space for a general initial condition and an arbitrary temperature of the bath.

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The paper is organized as follows. In Sec. II, we present the generalized influence functional method for the Stern-Gerlach problem and evaluate the propagators from the influence functionals. In Sec. III, we obtain the reduced density matrix of the system in the coordinate space and the momentum space for separable as well as nonseparable initial conditions, for an arbitrary temperature. In Sec. IV, we analyze the behavior of the density matrix elements in the long-time limit and compute the time scales of decay of the elements off-diagonal in the coordinate and the momentum space, for the cases of high temperature, zero temperature, and intermediate temperature baths. Finally in Sec. V, we summarize the results.

II. GENERALIZED PROPAGATOR AND THE INFLUENCE FUNCTIONAL

We consider a model [2–4] for the Stern-Gerlach measurement of spin. The spin (system) of a particle of mass m is measured by monitoring the position/momentum (apparatus) of the particle, and the particle is also coupled to an environment of a collection of harmonic oscillators via its position. The Hamiltonian of the entire system (considered in one space dimension) has the following pieces:

$$H^S+H^A+H^{SA}=\lambda\sigma_x+\frac{p^2}{2m}+\epsilon x\sigma_x, \tag{1}$$

$$H^{AE}+H^E=x\sum_k g_kq_k+\sum_k \frac{[p_k^2+(m_k\omega_kq_k)^2]}{2m_k}+\frac{x^2}{2}\sum_k\frac{g_k^2}{m_k\omega_k^2}, \tag{2}$$

where the superscripts S, A, and E stand for system, apparatus and environment, respectively. Here σ_x, p, and x denote the x components of spin, momentum and position of the particle; m_k, ω_k, p_k, and q_k are the mass, frequency, momentum and position of the kth environmental oscillator. The first term in the right-hand side (RHS) of Eq. (1) gives the coupling of the spin to a uniform field, the second gives the kinetic energy of the particle, and the third gives the coupling of the particle to an inhomogeneous magnetic field

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(ϵ being the product of the field gradient and the magnetic moment of the particle), for which the direction of the force on the particle depends on the direction of the spin. The first term in the RHS of Eq. (2) gives the coupling of the particle position x to a set $\{q_k\}$ of the environmental oscillators, the second term is the Hamiltonian of the environment, and the last term (''counter term'') is included so that the bare potential of the system does not shift due to the coupling [6]. The set of environmental harmonic oscillators at a temperature T has the spectral function

$$\mathcal{J}(\omega)=\frac{\pi}{2}\sum_k \frac{g_k^2}{m_k\omega_k}\delta(\omega-\omega_k). \tag{3}$$

In the case of Ohmic dissipation,

$$\mathcal{J}(\omega)=2m\gamma\omega\Theta(\Omega-\omega), \tag{4}$$

where γ is the relaxation coefficient, Ω is a cutoff frequency, much larger than the natural frequencies of motion of the system of interest, and $\Theta(x)$ is a step function that is 1 for $x>0$ and is 0 otherwise. The reduced density operator of the system at time t in the coordinate representation is written as

$$\begin{aligned}\tilde{\rho}(x,y,t)&=\int d\mathbf{R}\langle x\mathbf{R}|\rho(t)|y\mathbf{R}\rangle\\ &=\int\int\int\int\int dx'dy'd\mathbf{R}d\mathbf{Q}'d\mathbf{R}'\\ &\quad\times K(x,\mathbf{R},t;x',\mathbf{R}',0)K^*(y,\mathbf{R},t;y',\mathbf{Q}',0)\\ &\quad\times\langle x'\mathbf{R}'|\rho(0)|y'\mathbf{Q}'\rangle,\end{aligned} \tag{5}$$

where $\mathbf{R}$, $\mathbf{R}'$, and $\mathbf{Q}'$ are arbitrary configurations (N-dimensional vectors) of the environmental oscillators and K is the propagator of composite system of the particle and its environment.

The reduced density operator of the particle at time t depends on the total density operator at time $t=0$. In the FV approach [1], it is assumed that the system and its environment are decoupled at $t=0$ (factorizable initial condition). Following Smith and Caldeira [7], we consider a general initial condition for the problem that allows for nonfactorizable initial conditions as well:

$$\rho_0(x',y';\mathbf{R}',\mathbf{Q}')=\tilde{\rho}_0^S(x',y')\overline{\rho_{eq}^{SE}}(x',y';\mathbf{R}',\mathbf{Q}'), \tag{6}$$

where $\overline{\rho_{eq}^{SE}}$ is the equilibrium density operator of the universe, and $\tilde{\rho}_0^S(x',y')$ is chosen in such a way that $\mathrm{tr}\rho_0=1$.

From Eq. (5) and Eq. (6), we can write [7]

$$\tilde{\rho}(x,y,t)=\int\int dx'dy'J(x,y,t;x',y',0)\tilde{\rho}_0^S(x',y'), \tag{7}$$

where the function J is the ''generalized propagator'' for the reduced density operator of the system,

$$\begin{aligned}J(x,y,t;x',y',0)&=\int_{x'}^{x}\int_{y'}^{y}Dx(t')Dy(t')\\ &\quad\times\exp\left[\frac{i}{\hbar}\tilde{S}^S[x(t')]\right]\exp\left[-\frac{i}{\hbar}\tilde{S}^S[y(t')]\right]\\ &\quad\times F([x],[y],x',y'),\end{aligned} \tag{8}$$

$S^S[\ ]$ is the action of the system of interest, F is the ''new influence functional,''

$$\begin{aligned}F=&\int\int\int d\mathbf{R}d\mathbf{Q}'d\mathbf{R}'\overline{\rho_{eq}^{SE}}(x',\mathbf{R}';y',\mathbf{Q}')\\ &\times G([x],[y],\mathbf{R},\mathbf{R}',\mathbf{Q}'),\end{aligned} \tag{9}$$

and G is a standard path integral involving product of propagators of forced harmonic oscillators,

$$\begin{aligned}G=&\int_{\mathbf{R}}^{\mathbf{R}'}\int_{\mathbf{Q}'}^{\mathbf{R}}D\mathbf{R}(t')D\mathbf{Q}(t')\\ &\times\exp\left[\frac{i}{\hbar}\{S^E[\mathbf{R}(t')]+S^{SE}[\mathbf{R}(t'),x(t')]\}\right]\\ &\times\exp\left[-\frac{i}{\hbar}\{S^E[\mathbf{Q}(t')]+S^{SE}[\mathbf{Q}(t'),y(t')]\}\right].\end{aligned} \tag{10}$$

Here $S^E[\ ]$ is the action of the environment, and $S^{SE}[,]$ is the action of the coupling between the particle and the environment. The variables within brackets are paths connecting the appropriate end points (variables without brackets) and D (variable) is the properly normalized variation of those paths. The tilde upon the action S^S means that the counter term is included therein.

Now, in the Stern-Gerlach system governed by the Hamiltonian (1) + (2), the eigenstates of σ_x are represented as $|\uparrow\rangle$ and $|\downarrow\rangle$ with eigenvalues $s=+1$ and -1. The four elements of the propagator (8) in the spin space (corresponding to subscripts $\uparrow\uparrow$, $\downarrow\downarrow$, $\uparrow\downarrow$, $\downarrow\uparrow$) are given as

$$\begin{aligned}J_{ss'}&(x_f,y_f,t;x_i,y_i,0)\\ &=\int_{x_i}^{x_f}\int_{y_i}^{y_f}DxDy\,\exp\left[\frac{i}{\hbar}\left\{S_1[x,y]-\lambda(s-s')t\right.\right.\\ &\quad\left.\left.-\epsilon\left(s\int_0^t dt'x(t')-s'\int_0^t dt'y(t')\right)\right\}\right]\\ &\quad\times\exp\left[-\frac{1}{\hbar}S_2[x,y]\right],\end{aligned} \tag{11}$$

where using Eq. (4),

$$S_1[x,y]=m\int_0^t dt'\left(\frac{1}{2}(\dot{x}^2-\dot{y}^2)-\gamma\,(x\dot{x}-y\dot{y}+x\dot{y}-y\dot{x})\right), \tag{12}$$

and

$$S_2[x,y]=\frac{m\gamma}{\pi}\int_0^{\Omega}d\omega\ \omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\int_0^t dt'\times\int_0^t dt''\{x(t')-y(t')\}\cos[\omega(t'-t'')]\times\{x(t'')-y(t'')\}+\frac{m\gamma}{\pi}\int_0^{\Omega}d\omega\frac{\omega}{\omega^2+4\gamma^2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\times\left[(x_i-y_i)^2-(x_i-y_i)\int_0^t dt'\{x(t')-y(t')\}\times[2\omega\sin(\omega t')-4\gamma\cos(\omega t')]\right], \tag{13}$$

for the case of nonseparable initial condition (6), k_B being the Boltzmann constant [8]. The second term on the RHS of Eq. (13) vanishes for the case of separable initial condition.

With a change of coordinates $q=(x+y)/2$ and $\xi=x-y$, from Eq. (11), we get the components $J_{\uparrow\uparrow}$, $J_{\downarrow\downarrow}$ (J_d, i.e., diagonal in spin space) and $J_{\uparrow\downarrow}$, $J_{\downarrow\uparrow}$ (J_{od}, i.e., off-diagonal in spin space) as

$$J_d(q,\xi,t;q',\xi',0)=I_0(t)\int_{q'}^{q}\int_{\xi'}^{\xi}Dq(t')D\xi(t')\times\exp\left[\frac{i}{\hbar}\left\{S_1\mp\epsilon\int_0^t dt'\,\xi(t')\right\}\right]\times\exp\left[-\frac{1}{\hbar}S_2\right], \tag{14}$$

$$J_{od}(q,\xi,t;q',\xi',0)=I_0(t)\int_{q'}^{q}\int_{\xi'}^{\xi}Dq(t')D\xi(t')\times\exp\left[\frac{i}{\hbar}\left\{S_1\mp 2\lambda t\mp 2\epsilon\int_0^t dt'\,q(t')\right\}\right]\times\exp\left[-\frac{1}{\hbar}S_2\right], \tag{15}$$

where the upper (lower) signs correspond to $J_{\uparrow\uparrow}$ ($J_{\downarrow\downarrow}$) in J_d and $J_{\uparrow\downarrow}$ ($J_{\downarrow\uparrow}$) in J_{od},

$$I_0(t)=\frac{m\gamma e^{\gamma t}}{2\pi\hbar\sinh(\gamma t)}, \tag{16}$$

$$S_1=\int_0^t dt'\,m(\dot{\xi}(t')\dot{q}(t')-2\gamma\dot{q}(t')\xi(t')), \tag{17}$$

and

$$S_2=\frac{m\gamma}{\pi}\int_0^{\Omega}d\omega\ \omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\int_0^t dt'\int_0^t dt''\xi(t')\xi(t'')\cos[\omega(t'-t'')]+\frac{m\gamma}{\pi}\int_0^{\Omega}d\omega\frac{\omega}{\omega^2+4\gamma^2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\left[\xi'^2-\xi'\int_0^t dt'\,\xi(t')\{2\omega\sin(\omega t')-4\gamma\cos(\omega t')\}\right], \tag{18}$$

for the case of nonseparable initial condition (6). The second term on the RHS of Eq. (18) vanishes for the case of the separable initial condition.

We now solve for J_d and J_{od} in the standard way and obtain for the case nonseparable initial conditions

$$J_d(q,\xi,t;q',\xi',0)=I_0(t)\exp\left[\frac{i}{\hbar}\{L_-(t)q\xi+L_+(t)q'\xi'-M(t)q\xi'-N(t)q'\xi\mp\epsilon X(t)\xi\mp\epsilon Z(t)\xi'\}\right]\times\exp\left[-\frac{1}{\hbar}\{A(t)\xi^2+B(t)\xi\xi'+C(t)\xi'^2\}\right] \tag{19}$$

and

$$J_{od}(q,\xi,t;q',\xi',0)=I_0(t)\exp\left[\frac{i}{\hbar}\left\{L_-(t)q\xi+L_+(t)q'\xi'-M(t)q\xi'-N(t)q'\xi\mp 2\lambda t\pm\frac{\epsilon}{\gamma}(q-q')\mp 2q\epsilon t\right\}\right]\times\exp\left[-\frac{1}{\hbar}\left\{A(t)\xi^2+B(t)\xi\xi'+C(t)\xi'^2\pm\frac{2\epsilon}{m\gamma}D(t)\xi\pm\frac{2\epsilon}{m\gamma}E(t)\xi'+\frac{\epsilon^2}{\pi m\gamma}\int_0^{\Omega}d\omega\ \omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\times\int_0^t dt'\int_0^t dt''t't''\cos\{\omega(t'-t'')\}\pm\frac{\epsilon}{m\gamma}Y(t)\xi'\right\}\right], \tag{20}$$

where

$$L_{\pm}(t)=m\gamma\{\coth(\gamma t)\pm 1\}, \tag{21}$$

$$X(t)=\frac{e^{-\gamma t}}{\sinh(\gamma t)}\int_0^t dt'\sinh(\gamma t')e^{\gamma t'}, \tag{22}$$

$$Z(t)=\frac{1}{\sinh(\gamma t)}\int_0^t dt'\sinh\{\gamma(t-t')\}e^{\gamma t'}, \tag{23}$$

$$M(t)=\frac{m\gamma e^{\gamma t}}{\sinh(\gamma t)}, \tag{24}$$

$$N(t)=\frac{m\gamma e^{-\gamma t}}{\sinh(\gamma t)}, \tag{25}$$

and $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $Y(t)$ are of the form

$$f(t)=\frac{m\gamma}{\pi}\int_0^{\Omega}d\omega\;\omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)f_{\omega}(t), \tag{26}$$

with

$$A_{\omega}(t)=\frac{e^{-2\gamma t}}{\sinh^2(\gamma t)}\int_0^t dt'\int_0^t dt''\;\sinh(\gamma t')\times\cos\{\omega(t'-t'')\}\sinh(\gamma t'')e^{\gamma(t'+t'')}, \tag{27}$$

$$\begin{aligned}B_{\omega}(t)=&\frac{2e^{-\gamma t}}{\sinh^2(\gamma t)}\int_0^t dt'\int_0^t dt''\;\sinh(\gamma t')\\&\times\cos\{\omega(t'-t'')\}\sinh\{\gamma(t-t'')\}e^{\gamma(t'+t'')}\\&+\frac{4\gamma e^{-\gamma t}}{(\omega^2+4\gamma^2)\sinh(\gamma t)}\int_0^t dt'\sinh(\gamma t')\cos(\omega t')e^{\gamma t'}\\&-\frac{2\omega e^{-\gamma t}}{(\omega^2+4\gamma^2)\sinh(\gamma t)}\int_0^t dt'\sinh(\gamma t')\sin(\omega t')e^{\gamma t'},\end{aligned} \tag{28}$$

$$\begin{aligned}C_{\omega}(t)=&\frac{1}{\sinh^2(\gamma t)}\int_0^t dt'\int_0^t dt''\;\sinh[\gamma(t-t')]\\&\times\cos\{\omega(t'-t'')\}\sinh\{\gamma(t-t'')\}e^{\gamma(t'+t'')}\\&+\frac{4\gamma}{(\omega^2+4\gamma^2)\sinh(\gamma t)}\int_0^t dt'\\&\times\sinh\{\gamma(t-t')\}\cos(\omega t')e^{\gamma t'}\\&-\frac{2\omega}{(\omega^2+4\gamma^2)\sinh(\gamma t)}\int_0^t dt'\\&\times\sinh\{\gamma(t-t')\}\sin(\omega t')e^{\gamma t'}+\frac{1}{\omega^2+4\gamma^2},\end{aligned} \tag{29}$$

$$D_{\omega}(t)=\frac{e^{-\gamma t}}{\sinh(\gamma t)}\int_0^t dt'\int_0^t dt''\;\sinh(\gamma t')\cos\{\omega(t'-t'')\}t''e^{\gamma t'} \tag{30}$$

$$E_{\omega}(t)=\frac{1}{\sinh(\gamma t)}\int_0^t dt'\int_0^t dt''\;\sinh\{\gamma(t-t')\}\times\cos\{\omega(t'-t'')\}t''e^{\gamma t'}, \tag{31}$$

$$Y_{\omega}(t)=\frac{1}{\omega^2+4\gamma^2}\int_0^t dt'\,t'[2\omega\sin(\omega t')-4\gamma\cos(\omega t')]. \tag{32}$$

For the case of separable initial conditions, J_d and J_{od} are the same as above but with $Y(t)=0$, and with only the first term present in the expressions for $B_{\omega}(t)$ and $C_{\omega}(t)$.

III. REDUCED DENSITY MATRIX

We obtain the reduced density operator of the spin system $\tilde{\rho}_d(q,\xi,t)$ as given by Eq. (7):

$$\tilde{\rho}_d(q,\xi,t)=\int\int dq'\,d\xi'\,J_d(q,\xi,t;q',\xi',0)\tilde{\rho}_0^S(q',\xi',0), \tag{33}$$

where the initial density matrix corresponding to a Gaussian wave packet of width σ and mean momentum $\overline{p}$ is taken as

$$\tilde{\rho}_0^S(q',\xi',0)=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left[\left(\frac{i\overline{p}\xi'}{\hbar}\right)-\left(\frac{q'^2+\xi'^2}{8\sigma^2}\right)\right]. \tag{34}$$

From Eq. (33), we obtain

$$\tilde{\rho}_d(q,\xi,t)=I_0(t)\sqrt{\frac{\pi\hbar^2}{\hbar C_1(t)+2\sigma^2L_+^2(t)}}\exp\left[\frac{-M^2(t)}{4[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\left(q-\frac{\overline{p}}{M(t)}\right)^2\right]$$
$$\times\exp\left[-\left\{\frac{A(t)}{\hbar}+\frac{2\sigma^2N^2(t)}{\hbar^2}-\frac{[4\sigma^2N(t)L_+(t)-\hbar B(t)]^2}{4\hbar^2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\right\}\xi^2\right]$$
$$\times\exp\left[\frac{i}{\hbar}\left\{L_-(t)q\xi-\frac{[4\sigma^2N(t)L_+(t)-\hbar B(t)]M(t)}{2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\left(q-\frac{\overline{p}}{M(t)}\right)\xi\right\}\right]$$
$$\times\exp\left[\mp\left\{\frac{\epsilon M(t)Z(t)}{2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\left(q-\frac{\overline{p}}{N(t)}\right)\right\}\right]\exp\left[\frac{-\epsilon^2Z^2(t)}{4[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\right]$$
$$\times\exp\left[\mp\frac{i\epsilon}{\hbar}\left\{X(t)+\frac{Z(t)[4\sigma^2N(t)L_+(t)-\hbar B(t)]}{2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\right\}\xi\right],\tag{35}$$

where $C_1(t)=C(t)+\hbar/8\sigma^2$. $I_0(t)$ given in Eq. (16) satisfies the normalization condition of the reduced density operator (35).

The diagonal elements of the density matrix in coordinate space are found by putting $q=x$ and $\xi=0$ in the above equation (35):

$$\tilde{\rho}_d(x,0,t)=I_0(t)\sqrt{\frac{\pi\hbar^2}{\hbar C_1(t)+2\sigma^2L_+^2(t)}}\times\exp\left[\frac{-M^2(t)}{4[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\times\left\{x-\frac{\overline{p}}{M(t)}\pm\frac{\epsilon}{2m\gamma}\left(t-\frac{1}{2\gamma}+\frac{e^{-2\gamma t}}{2\gamma}\right)\right\}^2\right].\tag{36}$$

We now calculate the difference in the widths of the wave packets for the different initial conditions as

$$\sigma_S^2(t)-\sigma_{NS}^2(t)=\frac{2\hbar}{M^2(t)}[C_S(t)-C_{NS}(t)].\tag{37}$$

Here subscript S stands for separable initial conditions and NS stands for nonseparable initial conditions. We find that in the case of high temperature, the difference in the widths goes as $-k_BT/4m\gamma^2$ in the long-time limit whereas in the case of zero temperature this goes as $(\hbar/2m\pi\gamma)\{\ln[(\Omega^2+4\gamma^2)/4\gamma^2]+\ln(2\gamma t)\}$. It is because of this time independence (in the case of high temperature) or logarithmic (slow) time dependence (in the case of zero temperature) that the wave packets show similar behavior in the long-time limit for separable as well as non- separable initial conditions for all temperatures.

We next consider the density matrix in the momentum space, u and v being conjugate to x and y. With $P=u-v$ and $p=(u+v)/2$,

$$\tilde{\rho}_d(P,p,t)=\int\int dq\,d\xi\,\tilde{\rho}_d(q,\xi,t)e^{i(Pq+p\xi)}$$
$$=\sqrt{\frac{\pi}{a(t)+c(t)}}\exp\left[\frac{iP\overline{p}}{M(t)}\right]\exp\left[\frac{-P^2}{2\alpha(t)M(t)}\right]$$
$$\times\exp\left[\mp i\epsilon\frac{Z(t)}{M(t)}P\right]\exp\left[\frac{e^2(t)P^2}{4[a(t)+c(t)]}\right]$$
$$\times\exp\left[-\frac{e(t)[b(t)+d(t)+g(t)+ip]P}{2[a(t)+c(t)]}\right]$$
$$\times\exp\left[-\frac{1}{4[a(t)+c(t)]}\times\left\{p+\frac{L_-(t)\overline{p}}{\hbar M(t)}\mp\frac{\epsilon}{\hbar}\left(X(t)+\frac{Z(t)L_-(t)}{M(t)}\right)\right\}^2\right],\tag{38}$$

where

$$a(t)=\frac{A(t)}{\hbar}+\frac{2\sigma^2N^2(t)}{\hbar^2}-\frac{[4\sigma^2N(t)L_+(t)-\hbar B(t)]^2}{4\hbar^2[\hbar C_1(t)+2\sigma^2L_+^2(t)]},\tag{39}$$

$$b(t)=\mp\frac{i\epsilon}{\hbar}\left\{X(t)+\frac{Z(t)[4\sigma^2N(t)L_+(t)-\hbar B(t)]}{2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\right\},\tag{40}$$

$$c(t)=\frac{\theta^2(t)}{2\hbar^2\alpha(t)M(t)},\tag{41}$$

$$\alpha(t)=\frac{M(t)}{2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}, \tag{42}$$

$$\theta(t)=L_-(t)-[4\sigma^2N(t)L_+(t)-\hbar B(t)]\alpha(t), \tag{43}$$

$$d(t)=\mp\frac{i\epsilon}{\hbar}\frac{Z(t)}{M(t)}\theta(t), \tag{44}$$

$$e(t)=\frac{\theta(t)}{\hbar\alpha(t)M(t)}, \tag{45}$$

$$g(t)=\frac{iL_-(t)}{\hbar M(t)}\overline{p}. \tag{46}$$

The diagonal elements of the density matrix in momentum space are obtained by putting $P=0$ and $p=u$ in the equation for $\tilde{\rho}_d(P,p,t)$:

$$\tilde{\rho}_d(0,u,t)=\sqrt{\frac{\pi}{a(t)+c(t)}}\exp\left\{\frac{-1}{4[a(t)+c(t)]}\times\left(u+\frac{L_-(t)\overline{p}}{\hbar M(t)}\mp\frac{\epsilon}{\hbar}\left[X(t)+\frac{Z(t)L_-(t)}{M(t)}\right]\right)^2\right\}$$
$$=\sqrt{\frac{\pi}{a(t)+c(t)}}\exp\left[\frac{-1}{4[a(t)+c(t)]}\left\{u+\frac{e^{-2\gamma t}\overline{p}}{\hbar}\mp\frac{\epsilon}{2\hbar\gamma}(1-e^{-2\gamma t})\right\}^2\right]. \tag{47}$$

This has the classical Ornstein-Uhlenbeck form with the spin- dependent drift caused by the field. This momentum distribution is centered around $\mp\epsilon/2\hbar\gamma$ for the up and down spins in the limit of $t\rightarrow\infty$. Thus it is seen that the measurement of the particle momentum can determine the spin.

Now we obtain the spin-off-diagonal elements of the density matrix in the coordinate space $\tilde{\rho}_{od}(q,\xi,t)$ as

$$\tilde{\rho}_{od}(q,\xi,t)=\int\int dq'\,d\xi'\,J_{od}(q,\xi,t;q',\xi',0)\tilde{\rho}_0^S(q',\xi',0), \tag{48}$$

with the same initial density matrix $\tilde{\rho}_0^S(q',\xi',0)$ as before. Thus, for nonseparable initial conditions,

$$\tilde{\rho}_{od}(q,\xi,t)=I_0(t)\sqrt{\frac{\pi\hbar^2}{\hbar C_1(t)+2\sigma^2L_+^2(t)}}\exp\left[\frac{-\epsilon^2}{\pi\hbar m\gamma}\int_0^\Omega d\omega\;\omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\int_0^t dt'\int_0^t dt''t't''\cos\{\omega(t'-t'')\}\right]$$
$$\times\exp\left[\frac{-M^2(t)}{4[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\left(q-\frac{\overline{p}}{M(t)}\right)^2\right]\exp\left[-\left\{\frac{A(t)}{\hbar}+\frac{2\sigma^2N^2(t)}{\hbar^2}-\frac{[4\sigma^2L_+(t)N(t)-\hbar B(t)]^2}{4\hbar^2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\right\}\xi^2\right]$$
$$\times\exp\left[\frac{i}{\hbar}\left\{L_-(t)q\xi-\frac{[4\sigma^2L_+(t)N(t)-\hbar B(t)]}{2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}M(t)\left(q-\frac{\overline{p}}{M(t)}\right)\xi\right\}\right]$$
$$\times\exp\left[\frac{\epsilon^2\left\{2E(t)+Y(t)-\frac{4\sigma^2m}{\hbar}L_+(t)\right\}^2}{4m^2\gamma^2[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\right]$$
$$\times\exp\left[\frac{\mp\epsilon\left\{2E(t)+Y(t)-\frac{4\sigma^2m}{\hbar}L_+(t)\right\}[4\sigma^2L_+(t)N(t)-\hbar B(t)]\xi}{2\hbar m\gamma[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\right]$$
$$\times\exp\left[\frac{\pm i\epsilon M(t)}{2m\gamma[\hbar C_1(t)+2\sigma^2L_+^2(t)]}\left(q-\frac{\overline{p}}{M(t)}\right)\left(2E(t)+Y(t)-\frac{4\sigma^2m}{\hbar}L_+(t)\right)\right]\exp\left[\frac{\mp2\epsilon D(t)\xi}{\hbar m\gamma}\right]\exp\left[\frac{\mp2i\lambda t}{\hbar}\right]$$
$$\times\exp\left[\frac{\pm i\epsilon q}{\hbar\gamma}\right]\exp\left[\frac{-2\sigma^2\epsilon^2}{\hbar^2m\gamma^2}\right]\exp\left[\frac{\mp4\sigma^2\epsilon}{\hbar^2\gamma}N(t)\xi\right]\exp\left[\frac{\mp2i\epsilon qt}{\hbar}\right]. \tag{49}$$

For the case of separable initial conditions, $\tilde{\rho}_d(q,\xi,t)$ and $\tilde{\rho}_{od}(q,\xi,t)$ are the same as above, but with $Y(t)=0$ and with the appropriate expressions for $B(t)$ and $C(t)$ as stated after Eq. (32).

IV. OFF-DIAGONAL ELEMENTS OF THE DENSITY MATRIX AND THE TIME SCALES OF DECAY

The steady-state (longtime limit) density matrix is expected to be (at least almost) diagonal in the basis in which the system Hamiltonian is diagonal, independent of which system variable couples to the environment variables in the system-plus-environment interaction and the initial state of the system. This basis is called the ''preferred basis,'' and the time scale over which this near diagonalization takes place is the decoherence time [9]. With the above point in view, we examine the longtime behavior of the density matrix elements off-diagonal in the coordinate space and the momentum space, for the cases of (i) high temperature, (ii) zero temperature, and (iii) intermediate temperature baths.

A. High-temperature limit

(1) First we study the longtime behavior of the spin-off-diagonal elements of the density matrix in the coordinate space, $\tilde{\rho}_{od}(q,\xi,t)$ [Eq. (49)] in the limit $k_BT \gg \hbar\gamma$. Note that the terms to be examined in the long-time limit are the following:

$$\exp\left[\frac{-\epsilon^2}{\pi\hbar m\gamma}\int_0^{\Omega} d\omega\ \omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \times \int_0^t dt' \int_0^t dt'' t' t'' \cos\{\omega(t'-t'')\}\right],$$

and

$$\exp\left[\frac{\epsilon^2\left[2E(t)+Y(t)-\left(\frac{4\sigma^2 m}{\hbar}\right)L_+(t)\right]^2}{4m^2\gamma^2[\hbar C_1(t)+2\sigma^2 L_+^2(t)]}\right].$$

The above two terms in the long-time limit together go as

$$\exp\left[\frac{-\epsilon^2 k_BT}{6\hbar^2 m\gamma}t^3\right]=\exp\left[\frac{-\epsilon^2 D}{12\hbar^2 m^2\gamma^2}t^3\right], \tag{50}$$

and this drives the entire $\tilde{\rho}_{od}(q,\xi,t)$ to zero. Here $D=2m\gamma k_BT$ is the usual diffusion coefficient. The time scales over which this happens is

$$t_s=\left(\frac{12\hbar^2 m^2\gamma^2}{\epsilon^2 D}\right)^{1/3}. \tag{51}$$

The above result is true for separable as well as for nonseparable initial conditions. Thus, in the limit of high temperature, we recover the previous results [2–4]. For atomic scale particles of mass $m\approx 10^{-24}$ g, and with $\gamma\approx 10^{12}\ \mathrm{s}^{-1}$, temperature $T\approx 300$ K, and $\epsilon\approx 1$ eV/cm, this time t_s is about 10^{-9} s.

(2) Next, the spatial nonlocality of the spin-diagonal components is examined, i.e., the long-time behavior of the spin-diagonal elements of the density matrix $\tilde{\rho}_d(q,\xi,t)$ [Eq. (35)] in the coordinate space. We find that the term of interest is $\exp\{[-A(t)/\hbar]\xi^2\}$ which goes as $\exp[-D\xi^2/4\gamma\hbar^2]$ in the long-time limit. The off-diagonal elements of $\tilde{\rho}_d(q,\xi,t)$ in coordinate space rapidly decay over a time-scale of

$$t_r=\frac{4\hbar^2}{D\xi^2}, \tag{52}$$

though the density matrix in the coordinate space does not eventually become diagonal, the extent of nonlocality being the thermal de Broglie wavelength $\lambda_d=h/\sqrt{2\pi m k_BT}$ of the particle.

(3) Next, we analyze the longstime behavior of the spin-diagonal elements of the density matrix $\tilde{\rho}_d(P,p,t)$ [Eq. (38)] in the momentum space. The terms of interest are

$$\exp\left[\frac{-P^2}{2\alpha(t)M(t)}\right]\exp\left[\frac{e^2(t)P^2}{4[a(t)+c(t)]}\right] =\exp\left[\frac{-\hbar^2P^2a(t)}{2\hbar^2\alpha(t)M(t)a(t)+\theta^2(t)}\right]. \tag{53}$$

It can be shown that in the limit of high temperature and for long times, $\theta^2(t)\to 0$ for all initial conditions. Thus the above term of interest becomes $\exp[-P^2/2\alpha(t)M(t)]$, which goes as $\exp[-(Dt/4m^2\gamma^2-3k_BT/4m\gamma^2)P^2]$ for separable initial conditions, and $\exp[-(Dt/4m^2\gamma^2-k_BT/4m\gamma^2)P^2]$ for nonseparable initial conditions. Both tend to $\exp[-Dt/4m^2\gamma^2P^2]$ in the long-time limit.

Thus we see that for both separable as well as for nonseparable initial conditions, the off-diagonal elements of the density matrix in momentum space, i.e., $\tilde{\rho}_d(P,p,t)$, $P\neq 0$, goes to zero over a time scale of

$$t_m=\frac{4m^2\gamma^2}{DP^2}, \tag{54}$$

where P is the extent of momentum space off-diagonality, and the spin-diagonal components become diagonal with time in the momentum space.

B. Zero temperature limit

(1) We study the behavior of the density matrix off-diagonal in spin space. We find that the term dominating the temporal behavior of $\tilde{\rho}_{od}(q,\xi,t)$ [Eq. (49)] for separable as well as for nonseparable initial conditions is

$$\exp\left[-\left\{\frac{\epsilon^2}{\hbar m\gamma}\int_0^{\Omega}d\omega\ \omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\int_0^t dt'\int_0^t dt''t't''\ \cos\{\omega(t'-t'')\}\right\}\right],$$

which goes as $\exp[(-\epsilon^2t^2/\pi\hbar m\gamma)\ln(\Omega t)]$ in the long-time limit. Thus even at $T=0$, $\tilde{\rho}_{od}(q,\xi,t)$ goes to zero. This happens over a time scale of

$$t_{s0}=\left(\frac{\pi\hbar m\gamma}{\epsilon^2}\right)^{1/2}, \tag{55}$$

which is about 10^{-7} s for the parameters chosen above.

(2) Next we analyze the behavior of the elements $\tilde{\rho}_d(q,\xi,t)$ [Eq. (35)], off-diagonal in the coordinate space. We find that $\exp\{[-A(t)/\hbar]\xi^2\}$ in the long-time limit goes as $\exp\{-(m\gamma/2\pi\hbar)\ln[(\Omega^2+4\gamma^2)/4\gamma^2]\xi^2\}$. Thus the off-diagonal elements of $\tilde{\rho}_d(q,\xi,t)$ decay over a time scale of

$$t_{r0}=\frac{2\pi\hbar}{m\gamma^2\ \ln\left(\frac{\Omega^2+4\gamma^2}{4\gamma^2}\right)\xi^2}, \tag{56}$$

even though the density matrix does not become diagonal.

(3) We now analyze the behavior of the elements $\tilde{\rho}_d(P,p,t)$ [Eq. (38)], off-diagonal in momentum space. We find that the term of interest $\exp[-P^2/2\alpha(t)M(t)]$ goes as $\exp[-2\sigma^2P^2]$ in the long-time limit for separable as well as nonseparable initial conditions. Thus at $T=0$, the momentum off-diagonal elements decay over a time scale of

$$t_{m0}=\frac{1}{2\gamma\sigma^2P^2}, \tag{57}$$

but do not go to zero.

C. Intermediate temperatures

This regime is quantified by the fact it is valid for all finite temperatures for times $t>\hbar/k_BT$. In this regime in the limit of long time, we get the high-temperature results as the dominant terms for all the cases treated above.

V. SUMMARY

We have solved the dynamics of the Stern-Gerlach spin in contact with a linearly dissipative environment at an arbitrary temperature. We find that even at zero temperature, the elements of the density maxtrix off-diagonal in the spin space $\tilde{\rho}_{od}(q,\xi,t)$ go to zero in the long-time limit, although at a rate slower than that in the high-temperature case. However, the spin-diagonal components $\tilde{\rho}_d(q,\xi,t)$, which are off-diagonal in coordinate space, and $\tilde{\rho}_d(P,p,t)$ off-diagonal in momentum space, do not go to zero in the long-time limit for a zero-temperature bath.

We also find that for any finite temperature for time greater than the ''crossover time'' $\hbar/k_BT$ [6], the elements of $\tilde{\rho}_{od}(q,\xi,t)$ and the off-diagonal elements of $\tilde{\rho}_d(P,p,t)$ go to zero in the long-time limit whereas the off-diagonal elements of $\tilde{\rho}_d(q,\xi,t)$ in the coordinate space do not vanish in the long-time limit, with the dominant terms being similar to the ones in the high-temperature limit. This implies that a measurement of momentum yields the spin for any finite temperature for times much greater than $\hbar/k_BT$.

We recover all the previous results in the high-temperature limit. Thus except for the extreme quantum case of zero temperature, the system shows the expected classical diffusive behavior over long times.

These ideas can possibly be tested in a spin-recombination interference experiment, in which a first Stern-Gerlach apparatus (SGA) splits the spin-half beam and a second SGA recombines these split beams in a reversed magnetic field. The decoherence of the positional wave function of the spins can be studied as a pressure of gas in the SGA.

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