

## Localization in a random phase-conjugating medium

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We theoretically study reflection and transmission of light in a one-dimensional disordered phase-conjugating medium. Using an invariant imbedding approach a Fokker-Planck equation for the distribution of the probe light reflectance and expressions for the average probabilities of reflection and transmission are derived. A new crossover length scale for localization of light is found, which depends on the competition between phase conjugation and disorder. For weak disorder, our analytical results are in good agreement with numerical simulations.

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Over the last two decades scattering of light from random optical media has received a lot of attention [1]. In passive random media many interesting multiple-scattering effects were discovered, such as enhanced backscattering of light [2], intensity correlations in reflected and transmitted waves [3], and Anderson localization [4]. Also absorbing or amplifying random optical media have been investigated. In the latter, the combination of coherent amplification and confinement by Anderson localization leads to amplified spontaneous emission and laser action without using mirrors, which have been observed in laser dyes [5] and semiconductor powders [6], respectively. These being linear random media, it is interesting to ask what happens in a *nonlinear* active random medium, such as a disordered phase-conjugating medium (PCM). A PCM consists of a nonlinear optical medium with a large third-order susceptibility  $\chi^{(3)}$ ; see Fig. 1. The medium is pumped by two intense counterpropagating laser beams of frequency  $\omega_0$ . When a probe beam of frequency  $\omega_0 + \delta$  is incident on the material, a fourth beam will be generated due to the nonlinear polarization of the medium. This conjugate wave has frequency  $\omega_0 - \delta$  and travels with the reversed phase in the opposite direction to the probe beam [7]. The medium thus acts as a ‘‘phase-conjugating mirror.’’ Depending on the characteristics of the PCM, the reflected beam is either stronger or weaker than the incoming one, while the transmitted probe beam is always amplified [8]. It has been shown that phase conjugation also occurs in *disordered*  $\chi^{(3)}$  media [9]. This raises several interesting questions with respect to reflection and transmission of light at such a disordered medium. (1) How are the amplifying properties of a transparent PCM affected in the presence of disorder? (2) What are the fundamental similarities and differences between a nonlinear random phase-conjugating medium and a linear amplifying or absorbing random medium? (3) Is there a regime in which Anderson localization occurs, and what are the requirements for observing this? These questions and their answers form the subject of this paper.

Our starting point is the wave equation describing a one-dimensional (1D) disordered PCM [10]

$$\begin{pmatrix} \frac{\partial^2}{\partial x^2} + k_p^0 [1 + \epsilon(x)] & \gamma \\ \gamma^* & \frac{\partial^2}{\partial x^2} + k_c^0 [1 + \epsilon(x)] \end{pmatrix} \Psi(x) = 0. \quad (1)$$

Here  $k_{p,c}^0 \equiv (\omega_0 \pm \delta)/c$  and  $\Psi(x) \equiv (\mathcal{E}_p(x), \mathcal{E}_c^*(x))$ , with  $\mathcal{E}_p(x)$  and  $\mathcal{E}_c^*(x)$  the slowly varying amplitudes of the probe and conjugate electric fields, respectively. The off-diagonal parameter  $\gamma \equiv \gamma_0 e^{i\phi} = (6\omega_0^2/\epsilon_0 c^2) \chi^{(3)} \mathcal{E}_1 \mathcal{E}_2$  is the pumping-induced coupling strength between the probe and conjugate waves in the PCM, with  $\mathcal{E}_1, \mathcal{E}_2$  the electric-field amplitudes of the two pump beams. The disorder is modeled by a randomly fluctuating part  $\epsilon(x)$  in the relative dielectric constant [11].

In order to calculate the reflection and transmission coefficients  $r_p, r_c, t_p$ , and  $t_c$ , we use an invariant imbedding approach [12,13]. Following Ref. [13] we obtain the evolution equations for the probe and conjugate waves in the medium,

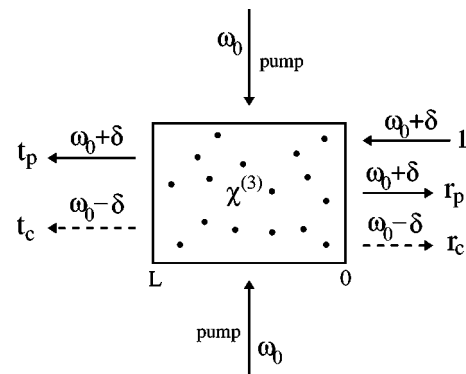


FIG. 1. A 1D disordered phase-conjugating medium, as described in the text. Solid (dashed) horizontal lines denote probe (conjugate) beams.

$$\frac{\partial \mathcal{E}_{p,(c)}^{(*)}(x)}{\partial L} = ik_0 + iA(L) + B(L) + \frac{ik_p^0}{2} \epsilon(L) \mathcal{E}_{p,(c)}^{(*)}(x), \quad (2)$$

with

$$A(L) \equiv \frac{\beta \delta \sqrt{\delta^2 + \gamma_0^2}}{\delta^2 + \gamma_0^2 \cos^2(\beta L)}, \quad (3)$$

$$B(L) \equiv \frac{\beta \gamma_0^2 \sin(\beta L) \cos(\beta L)}{\delta^2 + \gamma_0^2 \cos^2(\beta L)}, \quad (4)$$

$k_0 \equiv \omega_0/c$ , and  $\beta \equiv \sqrt{\delta^2 + \gamma_0^2}/c$ . Using the boundary conditions from Fig. 1 at  $x=0$  and  $x=L$  then yields

$$\begin{aligned} \frac{dr_p}{dL} &= \left[ ik_0 + iA(L) + B(L) + \frac{ik_p^0}{2} \epsilon(L)(1+r_p) \right] (1+r_p) \\ &\quad - ik_p^0(1-r_p), \quad r_p(0)=0 \end{aligned} \quad (5a)$$

$$\begin{aligned} \frac{dr_c}{dL} &= \left[ ik_0 + iA(L) + B(L) + \frac{ik_p^0}{2} \epsilon(L)(1+r_p) \right] r_c \\ &\quad - ik_c^0 r_c - i \frac{\gamma_0}{c}, \quad r_c(0)=0 \end{aligned} \quad (5b)$$

$$\begin{aligned} \frac{dt_p}{dL} &= \left[ ik_0 + iA(L) + B(L) + \frac{ik_p^0}{2} \epsilon(L)(1+r_p) \right] t_p, \\ t_p(0) &= 1 \end{aligned} \quad (5c)$$

$$\begin{aligned} \frac{dt_c}{dL} &= \left[ ik_0 + iA(L) + B(L) + \frac{ik_p^0}{2} \epsilon(L)(1+r_p) \right] t_c, \\ t_c(0) &= 0. \end{aligned} \quad (5d)$$

In the absence of phase conjugation, for  $\gamma_0=0$ , Eqs. (5a) and (5c) reduce to the well-known embedding equations for a linear random medium [13], and  $r_c = t_c = 0$ . In the absence of disorder, Eqs. (5b) and (5c) reduce to the evolution equations for  $r_c$  and  $t_p$  in a transparent PCM, and  $r_p = t_c = 0$ . Equations (5) satisfy the energy conservation law  $R_p + T_p - R_c - T_c = 1$ , with  $R_p \equiv |r_p|^2$  the probe reflectance, etc. They form the basis of all our results here. We first derive a Fokker-Planck (FP) equation for the probability distribution of  $R_p$ . We set  $r_p \equiv R_p e^{i\Theta_p}$ , substitute this into (5a), and subsequently into the Liouville equation

$$\frac{\partial Q}{\partial L} = - \frac{\partial}{\partial R_p} \left[ Q \frac{dR_p}{dL} \right] - \frac{\partial}{\partial \Theta_p} \left[ Q \frac{d\Theta_p}{dL} \right],$$

where  $Q(R_p, \Theta_p)$  is the density of points  $(R_p, \Theta_p)$  in phase space, and average over the disorder. Assuming a Gaussian distribution for  $\epsilon(L)$ , with  $\langle \epsilon(L) \rangle = 0$  and  $\langle \epsilon(L) \epsilon(L') \rangle = g \delta(L-L')$ , where pointed brackets denote an average over disorder, yields

$$\begin{aligned} \frac{\partial W}{\partial l} &= R_p(1-R_p)^2 \frac{\partial^2 W}{\partial R_p^2} + \{1 - 2[3 + M(L)]R_p + 5R_p^2\} \\ &\quad \times \frac{\partial W}{\partial R_p} - 2[1 + M(L) - 2R_p]W, \quad W(0) = \delta(R_p). \end{aligned} \quad (6)$$

Here  $W \equiv \langle Q \rangle$ ,  $l \equiv L/\xi_0$ ,  $M(L) \equiv \xi_0 B(L)$ , and  $\xi_0^{-1} \equiv \frac{1}{2} g k_p^0$ , the inverse localization length in the absence of phase conjugation. In deriving Eq. (6) we have neglected angular variations of  $W$ , the random-phase approximation (RPA), which applies to the situation of weak disorder when  $\xi_0 \gg 1/\beta$ . Equation (6) is of the same form as the equation for the probability distribution of the reflectance at a linear active random medium [14], with the important difference that in the latter case  $M(L)$  is an  $L$ -independent constant, proportional to the imaginary part of the dielectric constant. Using this analogy, our phase-conjugating medium alternates between a linear amplifying [for  $M(L) < 0$ ] and linear absorbing [for  $M(L) > 0$ ] random medium. For  $M(L) = 0$  the well-known Fokker-Planck equation for a passive random medium  $\partial W/\partial l = \partial/\partial R_p [R_p \partial/\partial R_p (1-R_p)^2 W]$  [15] is retrieved.

Multiplying both sides of Eq. (6) by  $R_p^n$  and integrating by parts leads to a recursion relation for the moments of the probe reflectance,

$$\frac{d}{dl} \langle R_p^n \rangle = n^2 \langle R_p^{n+1} \rangle - 2n[n - M(L)] \langle R_p^n \rangle + n^2 \langle R_p^{n-1} \rangle. \quad (7)$$

For  $n=1$  and setting  $\langle R_p^2 \rangle \approx \langle R_p \rangle$  [16], integration of Eq. (7) yields for the average probe reflectance

$$\begin{aligned} \langle R_p \rangle &= [C + \alpha^2 \gamma_0^2 \cos(2\beta L) + 2\alpha\beta\gamma_0^2 \sin(2\beta L) \\ &\quad - (C + \alpha^2 \gamma_0^2) e^{-\alpha L}] / [C + (4\beta^2 + \alpha^2) \gamma_0^2 \cos(2\beta L)], \end{aligned} \quad (8)$$

with  $C \equiv (4\beta^2 + \alpha^2)(2\delta^2 + \gamma_0^2)$  and  $\alpha \equiv 1/\xi_0$ . In the absence of phase conjugation, this reduces to  $\langle R_p \rangle = 1 - e^{-L/\xi_0}$  [17] and in the absence of disorder  $\langle R_p \rangle = 0$ , as for a transparent PCM. Using Eqs. (5a) and (5b), one can directly obtain an evolution equation for the average  $Z_{n,m} \equiv \langle R_p^n R_c^m \rangle$ , which is given by

$$\begin{aligned} \frac{d}{dl} Z_{n,m} &= n(m+n)Z_{n+1,m} - [2n^2 + m(n+1)]Z_{n,m} \\ &\quad + n^2 Z_{n-1,m} + 2(m+n)M(L)Z_{n,m} + 2mM(L), \end{aligned} \quad (9)$$

and is equivalent to Eq. (7) for  $m=0$ . Solving Eq. (9) for the conjugate reflectance yields

$$\langle R_c \rangle = -1 + \frac{\delta^2 + \gamma_0^2}{\delta^2 + \gamma_0^2 \cos^2(\beta L)} e^{-\alpha L} + \langle R_p \rangle. \quad (10)$$

Similarly, one obtains for the probe transmittance from Eqs. (5a) and (5c) [18]

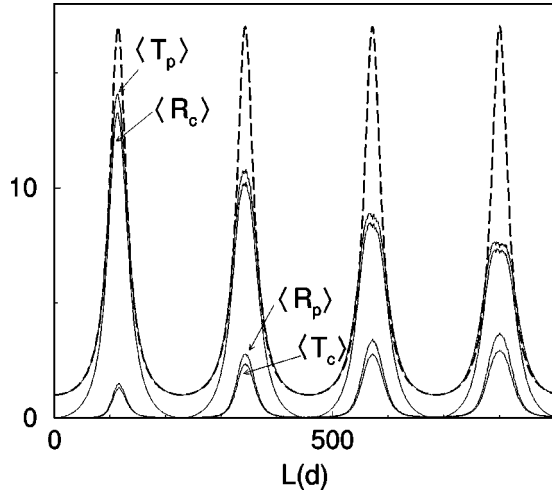


FIG. 2. Probe and conjugate reflectance and transmittance (averaged over 10 000 realizations of the disorder) at a disordered phase-conjugating medium as a function of the length  $L$  (in units of the lattice spacing  $d$ ) of the medium. The dashed curve denotes  $T_p$  in the absence of disorder. Parameters used are  $\delta=10^{10} \text{ s}^{-1}$ ,  $\gamma_0=4 \times 10^{10} \text{ s}^{-1}$  and  $\epsilon(x) \in [-0.05, 0.05]$ , corresponding to a localization length  $\xi_0=1.2 \text{ m}$ .

$$\langle T_p \rangle = \frac{\delta^2 + \gamma_0^2}{\delta^2 + \gamma_0^2 \cos^2(\beta L)} e^{-\alpha L}. \quad (11)$$

The conjugate transmittance is then given by  $\langle R_c \rangle = 0$ , through the conservation law  $\langle R_p \rangle + \langle T_p \rangle - \langle R_c \rangle - \langle T_c \rangle = 1$ .

In order to test these analytical predictions we have carried out numerical simulations. Using a transfer matrix method [19], Eqs. (1) are discretized on a 1D lattice with lattice constant  $d$ , into which disorder is introduced by letting  $\epsilon(x)$  randomly fluctuate from site to site. Figures 2–4 show the probe and conjugate reflectance and transmittance as a function of the length  $L$  of the medium for various values of the detuning  $\delta$  and disorder. In all cases we took  $d=10^{-4} \text{ m}$  and  $\omega_0=10^{15} \text{ s}^{-1}$  and typical PCM parameters.

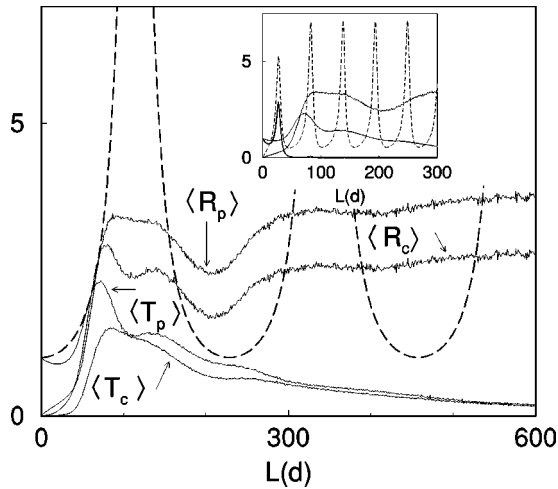


FIG. 3. Same as Fig. 2 but now for larger disorder,  $\epsilon(x) \in [-0.5, 0.5]$ , corresponding to a localization length  $\xi_0=12 \text{ mm}$ . The inset compares the analytic results (8) for  $\langle R_p \rangle$  (dashed curve) and (11) for  $\langle T_p \rangle$  (thick solid curve) with the numerical ones.

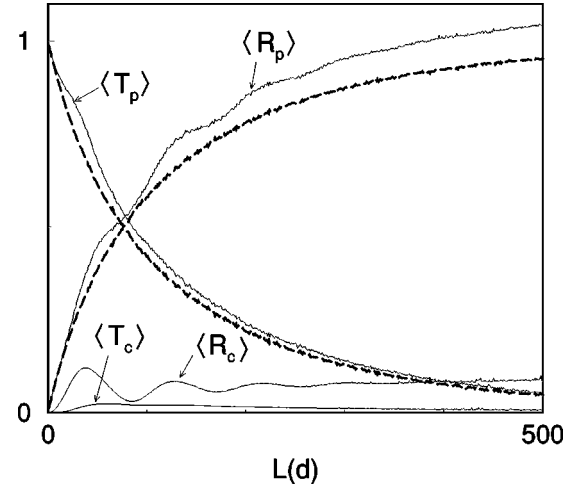


FIG. 4. Same as Fig. 3, but for  $\delta=10^{11} \text{ s}^{-1}$ . The thick dotted curves denote  $\langle R_p \rangle$  and  $\langle T_p \rangle$  in the absence of phase conjugation.

Figure 2 shows how the periodic behavior of  $\langle R_c \rangle$  and  $\langle T_p \rangle$  that is characteristic of a transparent PCM becomes “modulated” by an exponentially decaying envelope in the presence of weak disorder. Simultaneously, and with the same periodicity, some probe light is now reflected and some conjugate light transmitted, due to normal reflections in the disordered medium. When the amount of disorder is increased, the oscillatory behavior of the reflectances and transmittances is less and becomes suppressed for large  $L$ ; see Fig. 3. The reflected probe and conjugate intensities then both saturate, with  $\lim_{L \rightarrow \infty} \langle R_c \rangle = \lim_{L \rightarrow \infty} \langle R_p \rangle - 1$ , while  $\langle T_p \rangle$  and  $\langle T_c \rangle$  decay to zero (localization). For a transparent PCM the conservation law  $T_p - R_c = 1$  applies; i.e., for each pump photon scattered into the forward (probe) beam in the medium, a photon from the other pump is scattered into the backward (phase-conjugate) beam. In the localization regime of Fig. 3, on the other hand, the conservation law  $\langle R_p \rangle - \langle R_c \rangle = 1$  applies [cf. Eq. (10)]. Hence  $\langle T_p \rangle$  has exchanged roles with  $R_p$  due to disorder: all pump photons that are absorbed into probe and conjugate beams are now reflected, and despite amplification, transmitted intensities are suppressed. This suppression has also been found in linear amplifying random media [20,21]. The saturation of  $\langle R_c \rangle$  suggests that the phase-conjugate reflected beam arises in the region into which the probe beam penetrates and that amplification mostly takes place within a localization length of the point of incidence. The behavior of the transmitted intensities with increasing length of the medium is determined by two competing effects: on the one hand, enhancement occurs due to increased probability of multiple reflections; on the other hand, less light is transmitted due to increased probability of retroreflection of the incoming probe light. For small  $L$ , the latter effect dominates  $\langle T_p \rangle$  in Fig. 3. As  $L$  increases, the increasing amplification of probe light due to multiple scattering takes over, which leads to exponential increase and a maximum in  $\langle T_p \rangle$ . For larger again  $L$ , most of the probe light is reflected, and  $\langle T_p \rangle$  decreases exponentially to zero, as in a normal disordered medium [22]. The cross-over length scale  $L_c$  between exponential increase and decrease is given by the solution of  $\delta^2 + \gamma_0^2 \cos^2(\beta L) = 2\gamma_0^2 \beta \xi_0 \cos(\beta L) \sin(\beta L)$ , which for  $\delta \ll \gamma_0$  becomes

$$L_c \approx \frac{c}{\gamma_0} \left[ \frac{\pi}{2} - \arctan \left( \frac{c}{2\gamma_0 \xi_0} \right) \right]. \quad (12)$$

In the opposite limit of  $\delta \gg \gamma_0$  phase-conjugate reflection is weak (maximum value of  $R_c = 0.16$ ) and we retrieve exponential localization; see Fig. 4. Randomness now dominates over phase conjugation and has almost washed out the oscillatory behavior of  $\langle R_p \rangle$  and  $\langle T_p \rangle$ .

Comparing the numerical results with the analytic ones from Eqs. (8), (10), and (11) we find good agreement (deviations  $< 5\%$ ) for weak disorder, as in Fig. 2, and for stronger disorder and weak phase conjugation, as in Fig. 4. In the intermediate regime, for  $\xi_0 > 1/\beta \approx c/\gamma_0$  results differ considerably; see inset in Fig. 3. There the RPA and the assumption  $\langle R_p^2 \rangle \approx \langle R_p \rangle$  are not valid and a different approach is needed.

In conclusion, we have studied reflection and transmission of light at a 1D disordered phase-conjugating medium in the limit of small disorder, for  $\beta \xi_0 \gg 1$ . The predicted behavior of reflectances and transmittances arising from the interplay between amplification and Anderson localization displays

features similar to those in a linear disordered amplifying medium. The main difference is the coupling of two waves in the PCM, which leads to additional interference effects. In future work we intend to (1) investigate the strong disorder regime; there the reflection of the pump beams cannot be neglected, and a full nonlinear analysis is required; (2) study the distribution of reflection and transmission eigenvalues and the statistical fluctuations in reflectance and transmittance for a multimode 2D or 3D disordered phase-conjugating medium [23]. This is relevant to experiments, which mostly employ 3D PCM's [24], and interesting in the context of random lasers: in a linear disordered amplifying medium the average reflectance becomes infinitely large with increasing amplification, upon approaching threshold [21]. It would be interesting to investigate whether something similar occurs in a disordered PCM, this being a "naturally" amplifying medium and feasible candidate for nonlinear random lasing.

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- [1] See, e.g., *Scattering and Localization of Classical Waves in Random Media*, edited by P. Sheng (World Scientific, Singapore, 1990); *Introduction to Wave Scattering, Localization and Mesoscopic Phenomena*, edited by P. Sheng (Academic Press, New York, 1995).
- [2] Y. Kuga and J. Ishimaru, *J. Opt. Soc. Am. A* **1**, 831 (1984); M. P. van Albada and A. Lagendijk, *Phys. Rev. Lett.* **55**, 2692 (1985); P. E. Wolf and G. Maret, *ibid.* **55**, 2696 (1985).
- [3] See A. García-Márín *et al.*, *Phys. Rev. Lett.* **81**, 329 (1998), and references therein.
- [4] A. Lagendijk *et al.*, *Physica A* **140**, 183 (1986); M. Kaveh, *Philos. Mag. B* **56**, 693 (1987); R. Dalichaouch *et al.*, *Nature (London)* **354**, 53 (1991); A. Z. Genack and N. Garcia, *Phys. Rev. Lett.* **66**, 1850 (1991); D. S. Wiersma *et al.*, *Nature (London)* **390**, 671 (1997).
- [5] N. M. Lawandy *et al.*, *Nature (London)* **368**, 436 (1994).
- [6] H. Cao *et al.*, *Phys. Rev. Lett.* **82**, 2278 (1999).
- [7] *Optical Phase Conjugation*, edited by R. A. Fisher (Academic Press, New York, 1983).
- [8] The conjugate reflectance  $R_c$  (defined as the reflected conjugate intensity divided by the incoming probe intensity) and probe transmittance  $T_p$  are given by [7]  $R_c = \sin^2(\beta L) / [\cos^2(\beta L) + (\delta^2/\gamma_0^2)]$  and  $T_p = [1 + (\delta^2/\gamma_0^2)] / [\cos^2(\beta L) + (\delta^2/\gamma_0^2)]$ , with  $\beta \equiv \sqrt{\delta^2 + \gamma_0^2}/c$ .
- [9] V. E. Kravtsov *et al.*, *Phys. Rev. B* **41**, 2794 (1990); V. I. Yudson and P. Reineker, *ibid.* **45**, 2073 (1992).
- [10] D. Lenstra, in *Huygens Principle 1690–1990; Theory and Applications*, edited by H. Blok, H. A. Ferwerda, and H. K. Kuiken (North-Holland, Amsterdam, 1990).
- [11] In the presence of disorder some of the pump light will be retroreflected, but since the pumps are typically three orders of magnitude stronger than the probe, we neglect this reflection for weak disorder. Also, as long as probe and conjugate reflected and transmitted intensities are sufficiently small, pump depletion can be neglected and the pumps may be taken as constant in the medium.
- [12] R. Bellman and G. M. Wing, *An Introduction to Invariant Imbedding* (Wiley, New York, 1976).
- [13] R. Rammal and B. Doucot, *J. Phys. (Paris)* **48**, 509 (1987).
- [14] P. Pradhan and N. Kumar, *Phys. Rev. B* **50**, 9644 (1994).
- [15] A. A. Abrikosov, *Solid State Commun.* **37**, 997 (1981); V. I. Mel'nikov, *Fiz. Tverd. Tela (Leningrad)* **23**, 782 (1981) [*Sov. Phys. Solid State* **23**, 444 (1981)].
- [16] This is appropriate if  $R_p \approx 0$  or  $R_p \approx 1$ , in the regime of either weak disorder or a combination of strong disorder and weak phase conjugation.
- [17] P. W. Anderson *et al.*, *Phys. Rev. B* **22**, 3519 (1980); N. Kumar, *ibid.* **31**, 5513 (1985).
- [18] Or, more generally,  $(d/dL)Z_{n,m} = (m+n)^2 Z_{n+1,m} - [m+2mn+2n^2] Z_{n,m} + n^2 Z_{n-1,m} + 2(m+n)M(L)Z_{n,m}$ , with  $Z_{n,m} \equiv \langle R_p^n T_p^m \rangle$ . This moments equation generalizes the one obtained by Freilikher *et al.* [*Phys. Rev. Lett.* **73**, 810 (1994)] for a linear absorbing random medium.
- [19] P. K. Datta *et al.*, *Phys. Rev. B* **47**, 10 727 (1993).
- [20] J. C. J. Paasschens *et al.*, *Phys. Rev. B* **54**, 11 887 (1996); S. K. Joshi and A. M. Jayannavar, e-print cond-mat/0004132.
- [21] C. W. J. Beenakker *et al.*, *Phys. Rev. Lett.* **76**, 1368 (1996).
- [22] S. K. Joshi and A. M. Jayannavar, *Phys. Rev. B* **56**, 12 038 (1997).
- [23] The 1D invariant imbedding approach used here can straightforwardly be extended to 2D and 3D disordered PCM's by decomposing the system into one longitudinal dimension (direction of propagation) and one or two transverse dimension(s) (transverse modes).
- [24] See, e.g., M. Y. Lanzerotti *et al.*, *Appl. Phys. Lett.* **69**, 1199 (1996).