Scheme for direct measurement of the Weyl characteristic function for the motion of a trapped ion

Shi-Biao Zheng,[1,2] Xi-Wen Zhu,[1] Mang Feng,[1] and Lei Shi[1]
[1]Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Science, Wuhan 430071, People's Republic of China
[2]Department of Electronic Science and Applied Physics, Fuzhou University, Fuzhou 350002, People's Republic of China*
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We propose a scheme for the reconstruction of the motional state of an ion trapped in a one-dimensional harmonic potential. In the scheme the ion is multichromatically excited by three lasers. Then the measurement of the population of the lower internal state directly yields the Weyl characteristic function for the motional state. The scheme is easily generalized to the two-dimensional case. The scheme operates in the Lamb-Dicke limit.

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In recent years there has been much interest in the reconstruction of quantum states. Numerous schemes have been proposed for quantum-state measurement for both a cavity field [1] and a trapped ion [2]. Recently, experimental reconstruction of the motional quantum state of a trapped ion has been reported [3]. However, most of these schemes involve a complex data analysis.

A scheme for direct observation of the Weyl characteristic function of a cavity field was proposed by Wilkens and Meystre [4]. In a recent paper, Kim *et al.* [5] have made a similar proposal applicable to both the cavity field and ion motion. Lutterbach and Davidovich [6] have presented an alternative scheme for direct measurement of the Wigner function, again in both cavity QED and ion traps. More recently, closely following the scheme for the generation of motional Schrödinger cat states of a trapped ion [7], Bardroff *et al.* [8] have proposed a simple and fast scheme to measure the motional state of a trapped ion. The scheme involves three laser pulses. In this paper we propose an alternative scheme for the direct measurement of the Weyl characteristic function of the motional state of a trapped ion. Our scheme consists of only one laser pulse. Furthermore, our scheme can be easily generalized to reconstruct two-mode entangled motional states. The scheme works in the Lamb-Dicke limit.

We consider a two-level ion trapped in a one-dimensional (1D) harmonic potential and driven by three laser beams tuned to the carrier, first lower and upper vibrational sidebands, respectively. In the rotating-wave approximation, the Hamiltonian for this system is given by

$$H=\nu\hat{a}^{\dagger}\hat{a}+\omega_0\hat{S}_z+[\lambda E^{+}(\hat{x},t)\hat{S}^{+}+\mathrm{H.c.}], \tag{1}$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators for the vibrational mode, $\hat{S}^{+},\hat{S}^{-}$, and $\hat{S}_z$ are the raising, lowering, and inversion operators for the two-level ion, ν is the vibrational frequency, and ω_0 and λ are the transition frequency and coupling constant characterizing the transition for the two-level ion. $E^{+}(\hat{x},t)$ is the positive part of the classical driving fields

$$E^{+}(\hat{x},t)=E_0e^{-i(\omega_0t-k_0\hat{x}+\phi_0)}+E_1e^{-i[(\omega_0-\nu)t-k_1\hat{x}+\phi_1]}+E_2e^{-i[(\omega_0+\nu)t-k_2\hat{x}+\phi_2]}, \tag{2}$$

where E_l,ϕ_l, and $k_l(l=0,1,2)$ are the amplitudes, phases, and wave vectors of the driving fields, respectively. The position operator $\hat{x}$ can be expressed by $\hat{x}=\sqrt{1/(2\nu M)}(\hat{a}+\hat{a}^{\dagger})$, with M being the mass of the trapped ion.

In the resolved sideband limit the vibrational frequency ν is much larger than other characteristic frequencies of the problem. Then the interactions of the ion with lasers can be treated using the nonlinear Jaynes-Cummings model [9,10]. In this case the Hamiltonian for such a system, in the interaction picture, is given by

$$\hat{H}_i=e^{-\eta^2/2}\sum_{j=0}^{\infty}\left\{\frac{(i\eta)^{2j}}{(j!)^2}\Omega_0e^{-i\phi_0}\hat{a}^{\dagger j}\hat{a}^{j}+\frac{(i\eta)^{2j+1}}{j!(j+1)!}[\Omega_1e^{-i\phi_1}\hat{a}^{\dagger j}\hat{a}^{j+1}+\Omega_2e^{-i\phi_2}\hat{a}^{\dagger j+1}\hat{a}^{j}]\right\}\hat{S}^{+}+\mathrm{H.c.}, \tag{3}$$

where $\Omega_l=\lambda E_l$ are the Rabi frequencies of the respective lasers and the Lamb-Dicke parameter η is defined by $\eta=k/\sqrt{2\nu M}$ assuming $k_0\simeq k_1\simeq k_2=k$.

We consider the behavior of the ion in the Lamb-Dicke regime, $\eta\ll 1$. In this limit we can expand the Hamiltonian $\hat{H}_i$ of Eq. (3) up to the first order in η. Furthermore, small Lamb-Dicke parameters lead to $e^{-\eta^2/2}\simeq 1$. Then the Hamiltonian can be simplified to

$$\hat{H}_i=(\Omega_0e^{-i\phi_0}+i\eta\Omega_1e^{-i\phi_1}\hat{a}+i\eta\Omega_2e^{-i\phi_2}\hat{a}^{\dagger})\hat{S}^{+}+\mathrm{H.c.} \tag{4}$$

*Mailing address.

We choose the amplitudes and phases of the lasers appropriately so that $\Omega_1=\Omega_2$, $\phi_0=0$, and $\phi_1+\phi_2=\pi$. Then we obtain

$$\hat{H}_i=\hat{O}(\hat{S}^+ + \hat{S}^-), \tag{5}$$

where

$$\hat{O}=\Omega_0+i\eta\Omega_1 e^{-i\phi_1}\hat{a}-i\eta\Omega_1 e^{i\phi_1}\hat{a}^\dagger. \tag{6}$$

Then the time evolution operator of the system can be expressed in the form of a 2×2 matrix with respect to the atomic basis [11],

$$\hat{U}(\tau)=\begin{pmatrix} \cos(\hat{O}\tau) & -i\sin(\hat{O}\tau) \\ -i\sin(\hat{O}\tau) & \cos(\hat{O}\tau) \end{pmatrix}. \tag{7}$$

Assume that the initial density operator of the whole system is

$$\hat{\rho}(0)=|g\rangle\langle g|\hat{\rho}_m, \tag{8}$$

where $\hat{\rho}_m$ is the unknown density operator for the ion motion and $|g\rangle$ is the ground electronic state. Then after an interaction time τ the density operator for the whole system is

$$\hat{\rho}(\tau)=[|g\rangle\cos(\hat{O}\tau)-i|e\rangle\sin(\hat{O}\tau)]\hat{\rho}_m \times[\cos(\hat{O}\tau)\langle g|+i\sin(\hat{O}\tau)\langle e|]. \tag{9}$$

We now detect the internal state of the ion. The probability of measuring the ion in the ground state $|g\rangle$ is

$$P_g=\frac{1}{2}+\frac{1}{2}\mathrm{Tr}\{\cos(2\hat{O}\tau)\hat{\rho}_m\}. \tag{10}$$

Substituting Eq. (6) into Eq. (10), we obtain

$$P_g(\alpha,\varphi)=\frac{1}{2}+\frac{1}{2}\mathrm{Re}[e^{i\varphi}\mathrm{Tr}\{\hat{D}(\alpha)\hat{\rho}_m\}], \tag{11}$$

where $\hat{D}(\alpha)$ is the displacement operator

$$\hat{D}(\alpha)=e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}. \tag{12}$$

The parameters φ and α are given by

$$\varphi=2\Omega_0\tau, \tag{13}$$

$$\alpha=2\eta\Omega_1\tau e^{i\phi_1}. \tag{14}$$

For an interaction time τ, the parameter φ is controllable by the Rabi frequency Ω_0. The modulus of α is controlled by Ω_1, and the phase by ϕ_1. We can rewrite Eq. (11) as

$$P_g(\alpha,\varphi)=\frac{1}{2}+\frac{1}{2}\mathrm{Re}[e^{i\varphi}\chi(\alpha)], \tag{15}$$

with $\chi(\alpha)$ being the Weyl characteristic function

$$\chi(\alpha)=\mathrm{Tr}\{\hat{\rho}_m\hat{D}(\alpha)\}. \tag{16}$$

From Eq. (15) we obtain

$$\chi(\alpha)=2P_g(\alpha,0)-1-i\left[2P_g\left(\alpha,\frac{\pi}{2}\right)-1\right]. \tag{17}$$

Thus a measurement of $P_g(\alpha,\varphi)$ for two phases $\varphi=0,\pi/2$ directly yields the Weyl characteristic function of the initial motional state at the point α.

In order to detect the electronic state, we employ an electronic V scheme [12,13], where the upper levels $|e\rangle$ and $|r\rangle$ couple to the common ground level $|g\rangle$. $|e\rangle\rightarrow|g\rangle$ is a weak electronic transition, while $|r\rangle\rightarrow|g\rangle$ is a strong one. After the interaction of the ion with the above mentioned three lasers, a laser on resonant with the transition $|r\rangle\rightarrow|g\rangle$ is used to detect the fluorescence. The presence of fluorescence is correlated with the ion being in the electronic state $|g\rangle$, while the absence of fluorescence is correlated with the ion in the state $|e\rangle$.

We note that the method can be generalized to the two-dimensional (2D) case. Raymer *et al.* [14] have proposed a scheme for the reconstruction of a two-mode running field by using balanced homodyne detection. In a more recent paper, Kim *et al.* [15] have suggested a scheme to reconstruct a two-mode entangled cavity-field state via the interaction of a V-type three-level atom with the field displaced by resonant classical sources. In a very recent paper, Solano *et al.* [16] have proposed a scheme for the measurement of the Wigner function of two trapped ions with center of mass and relative motion modes along their alignment direction. We show here how we can directly measure the two-mode Weyl characteristic function for the 2D motion of a trapped ion.

We consider a two-level ion confined in a 2D trap with vibrational frequencies ν_x and ν_y along the X and Y axis. We drive the ion with three laser beams of frequencies ω_0, $\omega_0-\nu_x$, and $\omega_0+\nu_x$ propagating along the X axis, and two of frequencies $\omega_0-\nu_y$ and $\omega_0+\nu_y$ propagating along the Y axis. In the resolved sideband and Lamb-Dicke regime, the Hamiltonian, in the interaction picture, is

$$\hat{H}_i=(\Omega_0 e^{-i\phi_0}+i\eta_x\Omega_1 e^{-i\phi_1}\hat{a}+i\eta_x\Omega_2 e^{-i\phi_2}\hat{a}^\dagger +i\eta_y\Omega_3 e^{-i\phi_3}\hat{b}+i\eta_y\Omega_4 e^{-i\phi_4}\hat{b}^+)\hat{S}^+ +\mathrm{H.c.}, \tag{18}$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators for the motional modes in the X and Y axes, η_x and η_y are the corresponding Lamb-Dicke parameters, and Ω_j and ϕ_j $(j=0,1,2,3,4)$ are the Rabi frequencies and phases of the respective lasers. We choose the amplitudes and phases of the lasers appropriately so that $\Omega_1=\Omega_2$, $\Omega_3=\Omega_4$, and $\phi_0=0$, $\phi_1+\phi_2=\pi$, $\phi_3+\phi_4=\pi$. We obtain

$$\hat{H}_i=(\Omega_0+i\eta_x\Omega_1 e^{-i\phi_1}\hat{a}-i\eta_x\Omega_1 e^{i\phi_1}\hat{a}^\dagger+i\eta_y\Omega_3 e^{-i\phi_3}\hat{b} -i\eta_y\Omega_3 e^{i\phi_3}\hat{b}^+)(\hat{S}^+ +\hat{S}^-). \tag{19}$$

Assume the initial density operator of the whole system is $|g\rangle\langle g|\hat{\rho}_m$, where $\hat{\rho}_m$ is the unknown density operator for the 2D motion. Following the previous calculations, we obtain the probability of finding the ion in the ground electronic state $|g\rangle$ after an interaction time τ,

$$P_g(\alpha,\beta,\varphi)=\frac{1}{2}+\frac{1}{2}\,\mathrm{Re}[e^{i\varphi}\mathrm{Tr}\{\hat{D}_a(\alpha)\hat{D}_b(\beta)\hat{\rho}_m\}], \quad (20)$$

where $\hat{D}_a(\alpha)$ and $\hat{D}_b(\beta)$ are the displacement operators for modes a and b. The parameters φ are given by Eq. (13), and α and β are given by

$$\alpha=2\eta_x\Omega_1\tau e^{i\phi_1}, \quad (21)$$

$$\beta=2\eta_y\Omega_3\tau e^{i\phi_3}. \quad (22)$$

We can rewrite Eq. (20) as

$$P_g(\alpha,\varphi)=\frac{1}{2}+\frac{1}{2}\,\mathrm{Re}[e^{i\varphi}\chi(\alpha,\beta)], \quad (23)$$

with $\chi(\alpha,\beta)$ being the two-mode Weyl characteristic function of the initial motional state

$$\chi(\alpha,\beta)=\mathrm{Tr}\{\hat{\rho}_m\hat{D}_a(\alpha)\hat{D}_b(\beta)\}. \quad (24)$$

From Eq. (23) we obtain

$$\chi(\alpha,\beta)=2P_g(\alpha,\beta,0)-1-i\left[2P_g\left(\alpha,\beta,\frac{\pi}{2}\right)-1\right]. \quad (25)$$

Therefore, a measurement of $P_g(\alpha,\beta,\varphi)$ for two phases $\varphi=0,\pi/2$ directly yields the two-mode Weyl characteristic function of the initial motional state at the point (α,β).

Finally, we make a comparison of the present scheme with previous ones. Like previous schemes [5,6,8], the present one also works in the Lamb-Dicke limit. However, the present scheme is not a mere extension of previous ones and has some advantages. The scheme of Ref. [6] uses dispersive coupling to achieve a rotation, with the coupling strength between the internal and external degrees of freedom proportional to the square of the Lamb-Dicke parameter η. In the present scheme the coupling strength is proportional to η and thus the time needed to complete the procedure is greatly decreased, which is of experimental importance in view of decoherence. According to the scheme of Ref. [5], the Weyl characteristic function is approximately obtained under the condition that the ion motion is first displaced by a large amount. In this case the probability of the ion in one certain internal state after a Jaynes-Cummings evolution oscillates very fast and thus even small fluctuations in the durations and intensities of the laser fields may cause fatal errors. This drawback is avoided in the present scheme. In the scheme of Ref. [8], when displacement Raman beams are applied, only the part of the motional state that is correlated with one electronic state is shifted. In order to measure $\chi(\alpha)$ at the point α, we should apply the Raman beams for a time $\tau=|\alpha|/(\eta\Omega)$. In the present scheme two parts of the motional state correlated with the corresponding electronic states undergo displacements with same amplitudes but opposite phases. In this case we need an interaction time $\tau=|\alpha|/(2\eta\Omega)$ for obtaining $\chi(\alpha)$ at the point α. Therefore, for the same parameters η and Ω the time needed to complete the procedure is only about half of that required by the scheme of Ref. [8]. Another advantage of the present scheme is that it can be easily generalized to reconstruct an entangled state of the 2D motion.

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