Two-mode coherent states for SU(1,1)⊗SU(1,1)

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We study two types of coherent states for two-mode realizations of the direct product group SU(1,1)⊗SU(1,1) [which is locally isomorphic to SO(2,2)] constructed from the coupling of two single-mode realizations of SU(1,1). The basis states for the relevant representations of SU(1,1)⊗SU(1,1) are constructed from the SU(1,1) Clebsch-Gordon coefficients. From these, Perelomov and Barut-Giradello coherent states are constructed. Various properties of the states are discussed, and methods for generating them are proposed. Some of the states can be generated by the operation of beam splitters with two-mode squeezed vacuum states or pair coherent states as inputs. We show that a competitive two-channel two-photon process gives rise to states of the Barut-Girardello type as the steady-state solutions of the associated master equation for appropriate initial conditions.

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I. INTRODUCTION

Two modes of a quantized electromagnetic field can become entangled, and may assume many other nonclassical properties through various kinds of nonlinear interactions. In fact, two-mode fields admit a large number of coherent states, many of which may be associated with low-order Lie groups such as SU(2) or SU(1,1). For example, if a state containing n photons and one containing only the vacuum are incident at the input ports of a beam splitter, the field of the output ports is an SU(2) coherent state [1], where the n photons are binomially distributed over the two modes [2]. The SU(2) coherent states may similarly be generated from the action of a frequency conversion device or of a directional coupler [3]. On the other hand, the two-mode squeezed vacuum state is just a particular example of a Perelomov [4] type of coherent state associated with a two-mode realization of SU(1,1) [5]. It is actually a special case of a class of two-mode SU(1,1) coherent states involving strong correlations between the modes [6]. The states are generated from the unitary evolution of two-mode number states driven by a nondegenerate parametric device (e.g., of the type that gives rise to down-conversion or harmonic generation). Another interesting type of two-mode state is the pair-coherent state first discussed in the context of quantum optics by Agarwal [7]. This type of state is actually an SU(1,1) coherent state according to the definition of Barut and Girardello [8]; it is a right eigenstate of the lowering operator in the su(1,1) Lie algebra. As in the Perelomov case, the Barut-Girardello (BG) coherent states for two-mode fields involve tight correlations with respect to the photon number states between the modes, and admit strong nonclassical properties such as various forms of squeezing and violations of the Cauchy-Schwarz and Bell inequalities. The pair-coherent states may be generated in a process involving the competition between nondegenerate parametric amplification and nondegenerate two-photon absorption [7].

One other class of two-mode SU(1,1) coherent states, the intelligent states, were discussed in the literature [9]. Intelligent states are those states that equalize various possible uncertainty products that can be constructed as consequences of the su(1,1) commutation relations. Pair-coherent states are a special case of intelligent states that involve equal uncertainty for the relevant operators. The two-mode squeezed pair-coherent state [10] is also an intelligent state, but with unequal uncertainties.

All of these two-mode SU(1,1) coherent states may be characterized by the fact that they may be written as superpositions of the form

$$\sum_{n=0}^{\infty} C_n|n+q\rangle_a|n\rangle_b, \tag{1.1}$$

where the modes are labeled a and b and the parameter q $(=0,1,2\ldots)$ is the (fixed) difference in the photon numbers of the two modes and where the coefficients C_n depend on the specific SU(1,1) state (see Sec. II for specific examples). The pairing of the photons in these states is due, of course, to the fact that photons are created or annihilated pairwise from *each* mode.

Of course, there are processes where photons are created or destroyed two at a time in a *single* mode. It turns out that such states are described by single-mode realizations of SU(1,1). The number states of the field can be separated, on the basis of parity (even or odd), to form two representations of SU(1,1). The single-mode squeezed vacuum and squeezed one-photon states are Perelemov SU(1,1) coherent states [5,11]. The BG coherent states for the single mode case are just the even or odd coherent states [12] which are special cases of the so-called Schrödinger cat states [13]. These states are characterized by oscillations in the photon number probability distributions, a phenomenon that can be interpreted as arising from interference in phase space [14].

There is, however, one possibility that, as far as we are aware, has not been explored for the two-mode case. There are processes, essentially two-channel processes, where there is competitive two-photon annihilation and creation *between* the two modes. Suppose, for example, that a and b represent the annihilation operators for photons of the same frequency but with orthogonal polarizations (perhaps left and right circular polarizations, respectively). In the case of a $J=0\rightarrow 1\rightarrow 0$ atomic cascade transition between the atomic states $|e\rangle$ and $|g\rangle$ with an off-resonant intermediate state, the effective Hamiltonian is

$$H_1=\hbar[G(a^2+b^2)|e\rangle\langle g|+\text{H.c.}]. \tag{1.2}$$

If, in addition, there is an external classical coherent field driving the transition described by the Hamiltonian,

$$H_{\text{ext}}=\hbar(G_0\varepsilon^2|e\rangle\langle g|e^{-2i\omega t}+\text{H.c.}), \tag{1.3}$$

then it can be shown that in the steady state, the radiation field is in an eigenstate of the operator (a^2+b^2). As we shall show, such states correspond to the Barut-Girardello coherent states for the coupling of two single-mode representations of SU(1,1), that is for the direct product group SU(1,1)⊗SU(1,1) which is locally isomorphic to SO(2,2), SO(2,2)≈SU(1,1)⊗SU(1,1). The basis states for SU(1,1)⊗SU(1,1) are very different from any of the two-mode SU(1,1) states previously considered, consisting of superpositions of the two-mode number states containing a fixed total photon number. As a result, photon probability distributions for the corresponding SU(1,1)⊗SU(1,1) coherent states are not concentrated along a line as is the case for the two-mode SU(1,1) states of the form of Eq. (1.1) as discussed above. Previously Bambah and Agarwal [15] have discussed the *four-mode* BG coherent states (the bipair coherent states) obtained by coupling together two two-mode representations of SU(1,1). As far as we are aware, the radiation fields obtained by coupling together two single-mode representations of SU(1,1) have not previously been discussed. We rectify this in the balance of this paper.

The paper is organized as follows. In Sec. II, we review various aspects of the group SU(1,1) and the Lie algebra su(1,1) relevant to problems in quantum optics. In Sec. III we discuss the basis states of the coupled two-mode SU(1,1) representation obtained from those of the two single-mode representations through the SU(1,1) Clebsch-Gordon coefficients. In Sec. IV we discuss two types of coherent states for the coupled states, namely, the Perelomov coherent states and the Barut-Girardello coherent states. Some properties of the states are given, and methods to generate them are also discussed. We conclude the paper in Sec. V with some remarks on possible directions for further work. In an Appendix, we derive, in our notation, the relevant SU(1,1) Clebsch-Gordon coefficients.

II. REVIEW OF RELEVANT ASPECTS OF SU(1,1)

We will be mostly concerned with the su(1,1) Lie algebra rather than the SU(1,1) group so we start by introducing the elements of that algebra, the operators K_0 and $K_\pm$ satisfying the commutation relations

$$[K_0,K_\pm]=\pm K_\pm, \quad [K_+,K_-]=-2K_0. \tag{2.1}$$

The operator K_0 is a generator of compact SU(1,1) transformations, whereas the combinations $K_1=(K_+ + K_-)/2$ and $K_2=(K_+ - K_-)/2i$ are generators of noncompact SU(1,1) transformations. The Casimir operator

$$C=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+) \tag{2.2}$$

commutes with all the elements of the Lie algebra. The relevant unitary irreducible representations are the positive discrete series [16], whose bases, which are eigenstates of K_0 and C, we denote as $|k,m\rangle$ where k is the so-called Bargmann index taking on the values $k=1/2,1,3/2,2,\ldots,$ and $m=0,1,2\ldots,\infty$. These states satisfy the relations

$$K_0|k,m\rangle=(m+k)|k,m\rangle, \tag{2.3a}$$

$$C|k,m\rangle=k(k-1)|k,m\rangle, \tag{2.3b}$$

$$K_+|k,m\rangle=[(m+1)(m+2k)]^{1/2}|k,m+1\rangle, \tag{2.3c}$$

$$K_-|k,m\rangle=[m(m+2k-1)]^{1/2}|k,m-1\rangle. \tag{2.3d}$$

The states $|k,m\rangle$ may be generated from the ''ground'' state $|k,0\rangle$ according to

$$|k,m\rangle=\left[\frac{\Gamma(2k)}{m!\,\Gamma(2k+m)}\right]^{1/2}(K_+)^m|k,0\rangle. \tag{2.4}$$

This positive discrete series is denoted D^k. An alternative labeling of the states frequently encountered in the literature is $|J,M\rangle$, where the eigenvalue of C is now $J(J+1)$ and that of K_0 is M, where now $M=-J,-J+1,-J+2,\ldots,$ in analogy to the angular momentum states. We prefer to use the notation $|k,m\rangle$, where here m is always a positive integer or zero and the Bargmann index k contains the fractional part of the spectrum of K_0. This turns out to be an advantage in deriving the Clebsh-Gordon coefficients for the coupling of certain nonstandard representations not labeled by the k values listed above. An example of such a nonstandard representation is the realization for a single-mode field, which we now discuss.

For a single-mode field described by the annihilation and creation operators a and $a^\dagger$, respectively, the su(1,1) Lie algebra is realized by the operators

$$K_0=\tfrac{1}{2}(a^\dagger a+\tfrac{1}{2}), \quad K_-=\tfrac{1}{2}a^2, \quad K_+=\tfrac{1}{2}a^{\dagger 2}. \tag{2.5}$$

The Casimir operator for this realization takes on the value $C=-\frac{3}{16}$, which means that the allowed values of the Bargmann index are $k=\frac{1}{4}$ and $\frac{3}{4}$. Note that these values do not belong to the list of allowed values given above, and so the associated representations may in some sense be interpreted as ''continuations'' of the standard positive discrete series. The correspondence between the usual number states of the single-mode field, which we denote as $|n\rangle$, and the SU(1,1) basis states $|k,m\rangle$ from Eq. (2.5) is

$$|n\rangle \Leftrightarrow |k,m\rangle \quad \text{for } n=2(m+k)-1/2. \tag{2.6}$$

Several types of SU(1,1) coherent states may be constructed, but here we shall mention only two of them: the

Perelomov [17] coherent state and the Barut-Girardello [18] coherent state. The former state is generated but the action of the operator

$$S(z)=\exp(zK_{+}-z^{*}K_{-}), \tag{2.7}$$

on the ground state $|k,0\rangle$. Here z is a complex number usually parametrized as $z=-(\theta/2)\exp(-i\phi)$, where θ is a hyperbolic angle ($0\leqslant\theta<\infty$), and ϕ is an azimuthal angle ($0\leqslant\phi\leqslant 2\pi$). The resulting coherent state is

$$|\xi,k\rangle=(1-|\xi|^{2})^{k}\sum_{m=0}^{\infty}\left[\frac{\Gamma(2k+m)}{m!\Gamma(2k)}\right]^{1/2}\xi^{m}|k,m\rangle, \tag{2.8}$$

where $\xi=-\tanh(\theta/2)\exp(-i\phi)$. For $k=\frac{1}{4}$ the state is just the squeezed vacuum state, and for $k=\frac{3}{4}$ it is the squeezed one-photon state. The operator $S(z)$ of Eq. (2.7) is just the familiar squeeze operator when the generators are realized as in Eq. (2.5) [17]. On the other hand, the Barut-Girardello coherent states, which we denote as $|\zeta,k\rangle$, are defined as right eigenstates of the su(1,1) lowering operator K_{-},

$$K_{-}|\zeta,k\rangle=\zeta|\zeta,k\rangle, \tag{2.9}$$

where the eigenvalue ζ is an arbitrary complex number. The solution to Eq. (2.9) is of the form

$$|\zeta,k\rangle=N_{k}\sum_{m=0}^{\infty}\frac{\zeta^{m}}{[m!\Gamma(2k+m)]^{1/2}}|k,m\rangle, \tag{2.10}$$

where N_k is the normalization factor given by

$$N_{k}=[\Gamma(2k)|\zeta|^{-2k+1}I_{2k-1}(2|\zeta|)]^{-1/2}, \tag{2.11}$$

I_{2k-1} being a modified Bessel function. For the realization of Eq. (2.5) the BG coherent states correspond to the even and odd coherent states for $k=\frac{1}{4}$ and $\frac{3}{4}$, respectively. As we stated in Sec. I, both the Perelomov and the BG coherent states for a single-mode field are characterized by oscillating photon number probability distributions, interpreted as the result of interference in phase space [14].

The standard two-mode realization of the su(1,1) Lie algebra is given by

$$K_{0}=\tfrac{1}{2}(a^{\dagger}a+b^{\dagger}b+1),\quad K_{+}=a^{\dagger}b^{\dagger},\quad K_{-}=ab, \tag{2.12}$$

where the operators a and b are the Bose operators of the two independent modes. The Casimir operator for this realization can be written as

$$C=\tfrac{1}{4}(\Delta^{2}-1), \tag{2.13}$$

where the operator $\Delta=a^{\dagger}a-b^{\dagger}b$ is just the difference between the number of photons in the two modes. With the eigenvalue of $|\Delta|$ denoted by the positive (or zero) integer q, the Bargmann index is given by $k=(1+q)/2$, where the degeneracy parameter $q=0,1,2\ldots$. Assuming that the a mode has q more photons than the b mode, the number states of the two modes, $|n_a\rangle\otimes|n_b\rangle=|n_a,n_b\rangle$, organize themselves

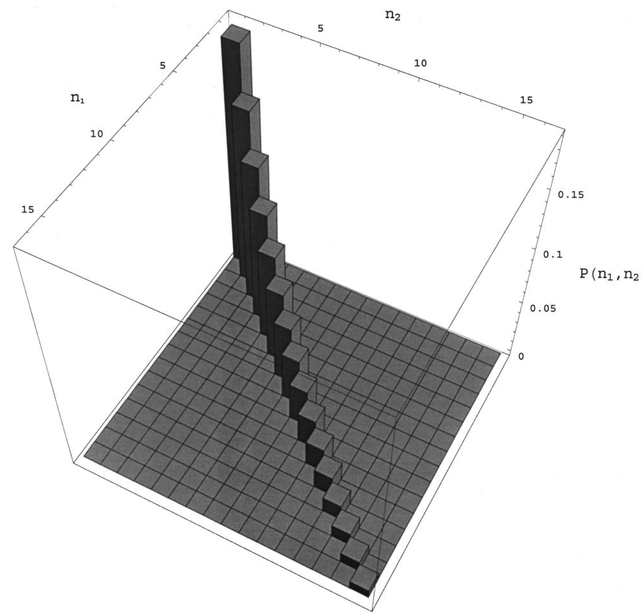

FIG. 1. The joint photon number probability distribution $P(n_1,n_2)$ vs n_1 and n_2 for the two-mode squeezed vacuum state ($q=0$) given by Eq. (2.14) for $|\xi|=0.9$.

into the sets where the su(1,1) bases are given by $|k,m\rangle=|n+q,n\rangle$, $n=0,1,2\ldots,\infty$, k is given above, and from Eqs. (2.3a) and (2.12) we have $m=n$.

The corresponding Perelomov coherent state for the two-mode field, from Eq. (2.7), but written in terms of the two-mode number states, is given by [6]

$$\left|\xi,\frac{1}{2}(1+q)\right\rangle=\sum_{n=0}^{\infty}C_{n}^{q}|n+q,n\rangle, \tag{2.14a}$$

where

$$C_{n}^{q}=(1-|\xi|^{2})^{(1+q)/2}\left[\frac{(n+q)!}{n!q!}\right]^{1/2}\xi^{n}. \tag{2.14b}$$

For the special case $q=0$ this is the two-mode squeezed vacuum state. For $q\neq 0$ it is the state obtained by the action of the two-mode squeeze operator on the number state $|q,0\rangle$ [6]. The probability of finding n_1 photons in mode a and n_2 photons in mode b is given by

$$P(n_{1},n_{2})=(1-|\xi|^{2})^{1+q}\times\left|\sum_{n=0}^{\infty}\left(\frac{(n+q)!}{n!q!}\right)^{1/2}\xi^{n}\delta_{n_{1},n+q}\delta_{n_{2},n}\right|^{2}. \tag{2.15}$$

Plotted against n_1 and n_2, this probability distribution is nonzero only along a line determined by the value of q, indicating the tight correlations between the modes. An example of such a distribution is pictured in Fig. 1 for the case with $q=0$.

One further point we want to make about the states in Eq. (2.14) is that the two modes are, in general, entangled. To obtain a measure of the degree to which they are entangled we may introduce the density operator $\rho_{ab}=|\xi,\frac{1}{2}(1+q)\rangle\langle\xi,1/2(1+q)|$ of the complete two-mode system from which we obtain the density operator of the a-mode subsystem as

$$\rho_a=\mathrm{Tr}_b\rho_{ab}=\sum_{n=0}^{\infty}|C_n^q|^2|n+q\rangle_{aa}\langle n+q|, \tag{2.16}$$

which is a statistical mixture. For the case of the two-mode squeezed vacuum, $q=0$, it has been shown that the state described by Eq. (2.16) has thermal-like noise [19]. Taking the trace over the a mode of ρ_a^2, we find that

$$\mathrm{Tr}_a\rho_a^2=\sum_{n=q}^{\infty}|C_n^q|^4<1 \tag{2.17}$$

for $0<|\xi|<1$. Thus the states of the form of Eq. (2.14) are entangled.

As for the corresponding two-mode BG coherent state, it is the pair-coherent state given by [7]

$$\left|\zeta,\frac{1}{2}(1+q)\right\rangle=\sum_{n=0}^{\infty}A_n^q|n+q,n\rangle, \tag{2.18a}$$

where

$$A_n^q=N_q\frac{\xi^n}{[n!(n+q)!]^{1/2}}, \tag{2.18b}$$

and where the normalization factor N_q is given by

$$N_q=[q!|\zeta|^{-q}I_q(2|\zeta|)]^{-1/2}. \tag{2.18c}$$

The photon number probability distribution is given by

$$P(n_1,n_2)=|N_1|^2\left|\sum_{n=0}^{\infty}\frac{\zeta^n}{[n!(n+q)!]^{1/2}}\delta_{n_1,n+q}\delta_{n_2,n}\right|^2. \tag{2.19}$$

Though this has a different shape than the Perelomov case, the distribution is still concentrated along a line determined by the value of q as illustrated in Fig. 2. Other two-mode SU(1,1) coherent states such as the intelligent states [9] also have tightly correlated photon probability distributions. Again, the state in Eq. (2.18) is an entangled state and, as in the previous case, it can be shown that

$$\mathrm{Tr}_a\rho_a^2=\sum_{n=q}^{\infty}|A_n^q|^4<1 \tag{2.20}$$

for $0<\zeta<\infty$.

III. COUPLED BASIS FOR SU(1,1)⊗SU(1,1)

We consider two modes of the quantized field, denoted the a and b modes, out of which we may construct two single-mode realizations of SU(1,1). The SU(1,1) operators of these individual modes are realized according to

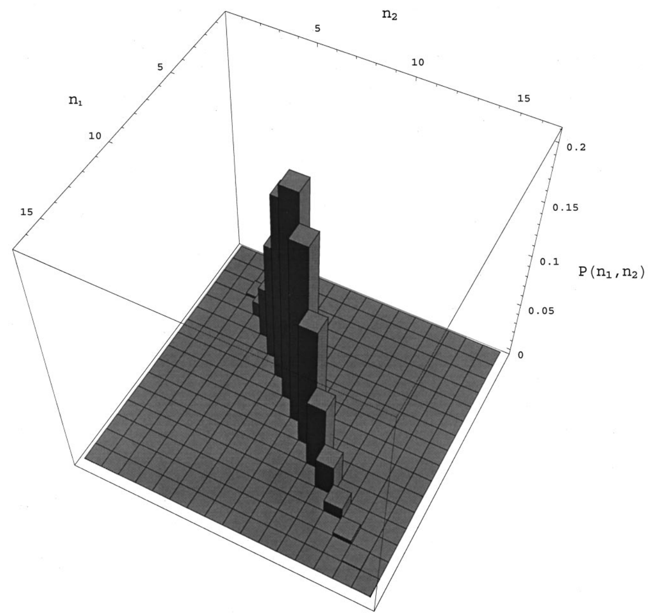

FIG. 2. The same as Fig. 1, but for the pair coherent state of Eq. (2.18) for $|\zeta|=7.5$.

$$K_0^{(1)}=\tfrac{1}{2}(a^\dagger a+\tfrac{1}{2}),\quad K_+^{(1)}=\tfrac{1}{2}a^{+2},\quad K_-^{(1)}=\tfrac{1}{2}a^2 \tag{3.1}$$

and

$$K_0^{(2)}=\tfrac{1}{2}(b^\dagger b+\tfrac{1}{2}),\quad K_+^{(2)}=\tfrac{1}{2}b^{+2},\quad K_-^{(2)}=\tfrac{1}{2}b^2, \tag{3.2}$$

where, of course, both sets of operators satisfy the su(1,1) Lie algebra of Eq. (2.1). The operators

$$K_0:=K_0^{(1)}+K_0^{(2)}=\tfrac{1}{2}(a^\dagger a+b^\dagger b+1), \tag{3.3a}$$

$$K_+:=K_+^{(1)}+K_+^{(2)}=\tfrac{1}{2}(a^{+2}+b^{+2}), \tag{3.3b}$$

$$K_-:=K_-^{(1)}+K_-^{(2)}=\tfrac{1}{2}(a^2+b^2) \tag{3.3c}$$

also satisfy the su(1,1) Lie algebra of Eq. (2.1), and generate the direct product group SU(1,1)⊗SU(1,1). The Casimir operator of this group is calculated from Eq. (2.2) and in terms of the operators for the individual modes has the form

$$C=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+)=\tfrac{1}{4}(a^\dagger a+b^\dagger b+1)^2 -\tfrac{1}{8}(\{a^{+2},a^2\}+\{b^{+2},b^2\}+2a^{+2}b^2+2a^2b^{+2}), \tag{3.4}$$

where {,} is an anticommutator. We denote the basis states of the two individual modes in the obvious way, $|k_1,m_1\rangle$ and $|k_2,m_2\rangle$ for modes a and b, respectively. We denote the basis of the coupled representation, i.e., of SU(1,1)⊗SU(1,1), as $|K,M;k_1,k_2\rangle$, which we sometimes abbreviate as simply $|K,M\rangle$. The labels K and M are, of course, related to the eigenvalues of C and K_0 according to $K(K-1)$ and $M+K$, respectively. That is, the coupled states $|K,M\rangle$ satisfy Eqs. (2.3) and (2.4) with the obvious replacements $k\rightarrow K$ and $m\rightarrow M$. The Kronecker product of two positive discrete series reduces to the sum of positive discrete series according to the Clebsch-Gordon decomposition

$$D^{k_1}\otimes D^{k_2}=\sum_{K=k_1+k_2}^{\infty} D_K, \tag{3.5}$$

where the sum proceeds in integer steps. In other words, the allowed values of K are $K=k_1+k_2+l$ where $l=0,1,2,\ldots,\infty$.

The SU(1,1)⊗SU(1,1) basis of the coupled modes, $|K,M;k_1,k_2\rangle$, may be constructed out of the products of the individual states of the two modes, $|k_1,m_1\rangle|k_2,m_2\rangle$ according to the series

$$|K,M;k_1,k_2\rangle=\sum_{m=0}^{l+M} C(k_1,k_2,K;m,l+M-m,M)|k_1,m\rangle \times|k_2,l+M-m\rangle, \tag{3.6a}$$

where the numbers $C(k_1,k_2,K;m,l+M-m,M)$ are SU(1,1) Clebsch-Gordon coefficients. The last equation could equally well be written as

$$|K,M;k_1,k_2\rangle=\sum_{m=0}^{l+M} C(k_1,k_2,K;,l+M-m,m,M) \times|k_1,l+M-m\rangle|k_2,m\rangle, \tag{3.6b}$$

although we shall stay with the former convention throughout the balance of the paper. Numerous derivations of the SU(1,1) Clebsch-Gordon coefficients have been given in the literature [20] mostly using the angular-momentum-like notation. We derive and present explicit formulas for these coefficients in the present notation in the Appendix.

Note that, according to Eq. (2.6), our state of Eq. (3.6) is a superposition of the product states where mode a contains $n_1=2(k_1+m)-\frac{1}{2}$ photons, and mode b contains $n_2=2(k_2+l+M-m)-\frac{1}{2}$ photons. Thus we may write Eq. (3.6) in terms of the number states as

$$|K,M;k_1,k_2\rangle =\sum_{m=0}^{l+M} C(k_1,k_2,K;m,l+M-m,M) \times|2(k_1+m)-1/2\rangle|2(k_2+l+M-m)-1/2\rangle. \tag{3.7}$$

Note that for these basis states the total number of photons in the two modes, for a given M and K, is $2(M+K)-1$.

We consider a few examples. For the case with $k_1=k_2=\frac{1}{4}$ and with $K=k_1+k_2=\frac{1}{2}$ $(l=0)$, the first few coupled basis states, in terms of the number states for each mode, work out to be

$$|K=\tfrac{1}{2},M=0\rangle=|0,0\rangle, \tag{3.8a}$$

$$|K=\tfrac{1}{2},M=1\rangle=\frac{1}{\sqrt{2}}(|2,0\rangle+|0,2\rangle), \tag{3.8b}$$

$$|K=\tfrac{1}{2},M=2\rangle=\sqrt{\frac{3}{8}}(|4,0\rangle+|0,4\rangle)+\frac{1}{2}|2,2\rangle, \tag{3.8c}$$

etc. Note that the total number of photons in each of the components of the coupled basis is just $2M$. It is easy to check these results by repeatedly applying the raising operator [Eq. (3.3b)] to the ground state $|0,0\rangle$. In fact, we may dispense with Eq. (3.6) containing the Clebsch-Gordon coefficients, and obtain a simple general formula for the states of the coupled representation in terms of the two-mode number states by writing

$$|K=\tfrac{1}{2},M\rangle=N_{2M}(a^{+2}+b^{+2})^M|0,0\rangle =N_{2M}\sum_{l=0}^{M}\frac{[(2l)!(2M-2l)!]^{1/2}}{(M-l)!l!}|2l,2M-2l\rangle, \tag{3.9}$$

where we have used the binomial expansion on the operator expression, and where a factor of $M!$ has been absorbed into the normalization factor N_{2M}. This normalization factor may be evaluated as

$$N_{2M}=\left[\sum_{l=0}^{M}\frac{(2l)!(2M-2l)!}{[(M-l)!L!]^2}\right]^{-1/2}=\frac{1}{2^M}, \tag{3.10}$$

and thus we finally have

$$|K=\tfrac{1}{2},M\rangle=\frac{1}{2^M}\sum_{l=0}^{M}\frac{[(2l)!(2M-2l)!]^{1/2}}{(M-l)!l!}|2l,2M-2l\rangle. \tag{3.11}$$

We note that only even states of the field are excited, which is as expected since it is the two even sets of states from each mode that are being coupled to form the new basis. It is worth remarking here that the coupled basis states for the other allowed values of K could be derived the same way Eq. (3.11) was, by replacing the ground state $|0,0\rangle$ by the ''ground''state of the corresponding coupled representation. However, it should be remembered that these ground states are generally not product states as, for example, for the state in Eq. (3.14a) below.

It is interesting to note that the states of Eq. (3.8) are precisely those generated from the action of a 50/50 beam splitter with the number states $|M,M\rangle$ $M=0,1,2\ldots$ at the input ports [1]. That is to say, using the angular-momentum formalism with which a 50/50 beams splitter may be described by the rotation operator $U_{BS}=\exp[-i(\pi/2)J_1]$ [1], where J_1 is an angular-momentum operator given in the Schwinger [21] realization as $J_1=(a^{\dagger}b+ab^{\dagger})/2$, it can be shown [1] that

$$U_{BS}|M,M\rangle=(-i)^M|K=\tfrac{1}{2},M\rangle. \tag{3.12}$$

In fact, the state of Eq. (3.8b) has been generated by a beam splitter from the input state $|1{,}1\rangle$ obtained from the output of a parametric down-converter in an experiment on two-photon interference by Hong *et al.* [22]. More recently, the state of Eq. (3.8c) has been generated by a beam splitter from the state input state $|2{,}2\rangle$, again obtained from parametric down-conversion, in an experiment on four-photon interference by Ou, Rhee, and Wang [23].

The joint probability of finding n_1 photons in mode a and n_2 in mode b for the coupled basis states is given by

$$P(n_1,n_2)=|\langle n_1,n_2|K,M;k_1,k_2\rangle|^2 = \left|\sum_{m=0}^{l+M} C(k_1,k_2,K;m,l+M-m,M) \times \delta_{n_1,2(k_1+m)-1/2}\delta_{n_2,2(k_2+l+M-m)-1/2}\right|^2. \tag{3.13}$$

In Fig. 3 we collectively plot $P(n_1,n_2)$ versus n_1 and n_2 for the case with $l=0$ for various M's. The important point is that the basis of SU(1,1)⊗SU(1,1) consists of superpositions of product states along *perpendiculars* to the diagonal in the in the n_1-n_2 plane where M increases *along* the diagonal. Unlike the standard two-mode realization of SU(1,1) as discussed in Sec. II, the numbers of photons in each of the modes are not tightly correlated, although the total number of photons in the two modes for a given K and M, in fact the number $2M$, is fixed, as already noted. The photon probability distribution for the case with $k_1=\frac{3}{4}$ and $k_2=\frac{1}{4}$, and with $K=1(l=0)$, is shown in Fig. 4. In this case only the odd states of the a mode are occupied, and only the even states of the b mode.

One important point to be made here is that the ''ground'' states for the coupled representation in the cases for which $l=0$ are just the product of the ground states of the two modes, i.e., $|K{,}0\rangle=|k_1{,}0\rangle|k_2{,}0\rangle$, with $K=k_1+k_2$. Such states are obviously not entangled. However, for cases where $l>0$, the corresponding ground states *are* entangled. For example, for the case with $k_1=k_2=\frac{1}{4}$ and $l=1$, such that $K=\frac{3}{2}$, in terms of the photon number states the ground state is

$$|K=\tfrac{3}{2},M=0\rangle=\frac{1}{\sqrt{2}}(|0{,}2\rangle-|2{,}0\rangle). \tag{3.14a}$$

The first and second excited states are given as

$$|K=\tfrac{3}{2},M=1\rangle=\frac{1}{\sqrt{2}}(|0{,}4\rangle-|4{,}0\rangle) \tag{3.14b}$$

and

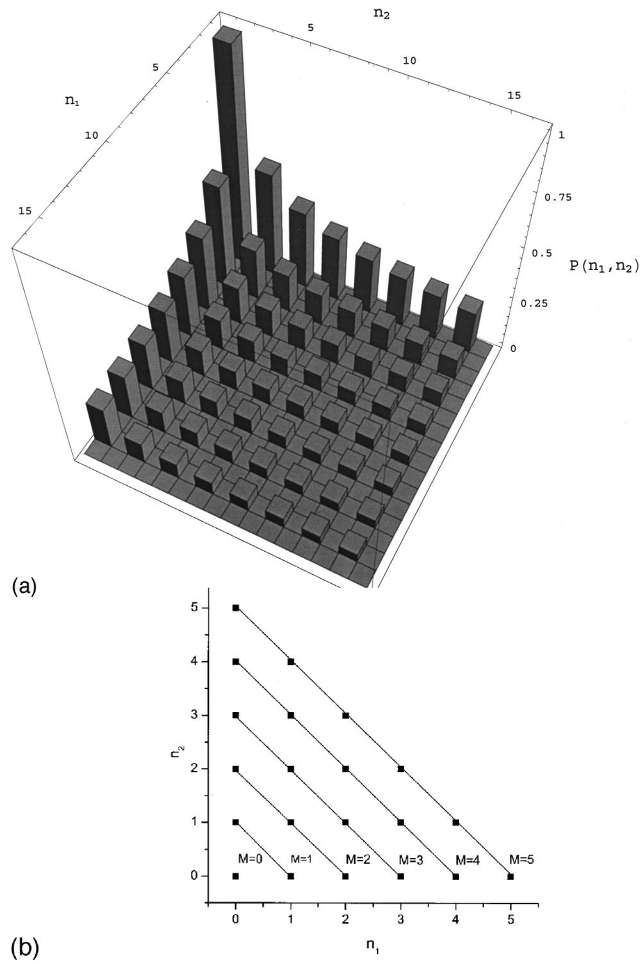

FIG. 3. (a) Joint photon number probability distribution from Eq. (3.13) for the coupled basis states of Eq. (3.7) for the cases $k_1=k_2=\frac{1}{4}$, and $l=0$; hence $K=\frac{1}{2}$. The states shown are for $M=0,1,2,3,\ldots$. The value of M increases *along* the diagonal, and thus the basis states for a given M are along lines perpendicular to the diagonal as indicated in (b), which is a two-dimensional version of (a), suppressing $P(n_1,n_2)$, indicating the photon number states comprising the basis states for the $K=\frac{1}{2}$ coupled basis.

$$|K=\tfrac{3}{2},M=2\rangle=\frac{1}{\sqrt{32}}(|2{,}4\rangle-|4{,}2\rangle) + \sqrt{\left(\frac{15}{32}\right)}(|0{,}6\rangle-|6{,}0\rangle), \tag{3.14c}$$

respectively. These states are clearly entangled and we shall return to this point below. Note that the total number of photons in the basis states for this case is $2M+2$, and further note that the number state $|2M+2{,}2M+2\rangle$ does not occur in the basis for this case of $K=\frac{3}{2}$. In Fig. 5 we plot the photon number probability distribution for some of the basis states for the coupled states with $K=\frac{3}{2}$. It is evident that states along the diagonal of the n_1-n_2 plane are not populated. The suppression of these states can certainly be construed as a result of interference. Here we shall not consider cases for higher values of K.

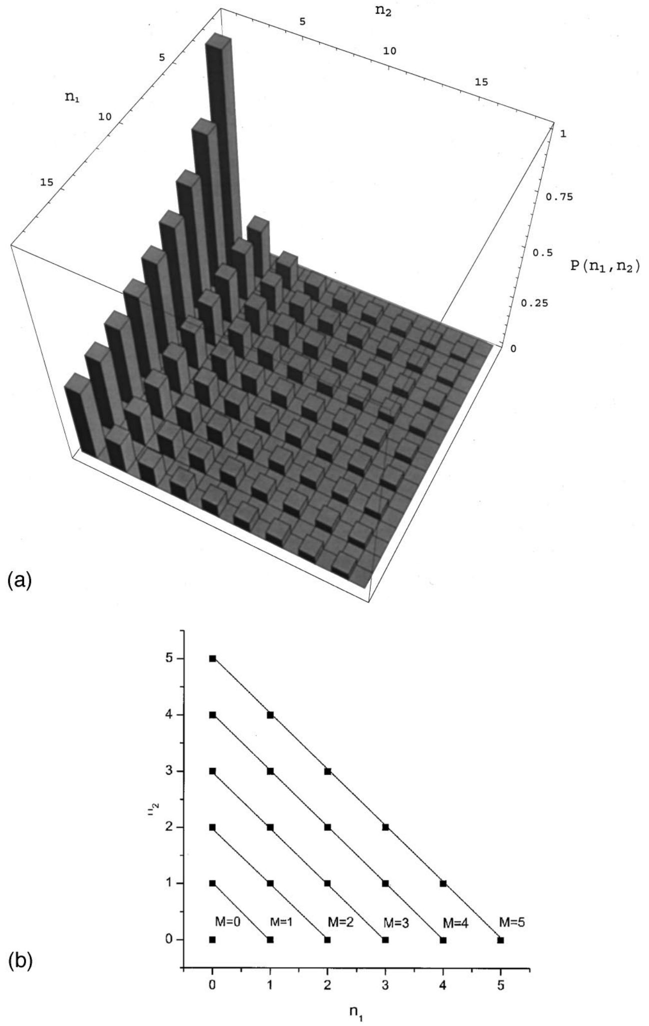

FIG. 4. Same as Fig. 3, but with $k_1=\frac{3}{4}$, $k_2=\frac{1}{4}$, and $l=0$; hence $K=1$.

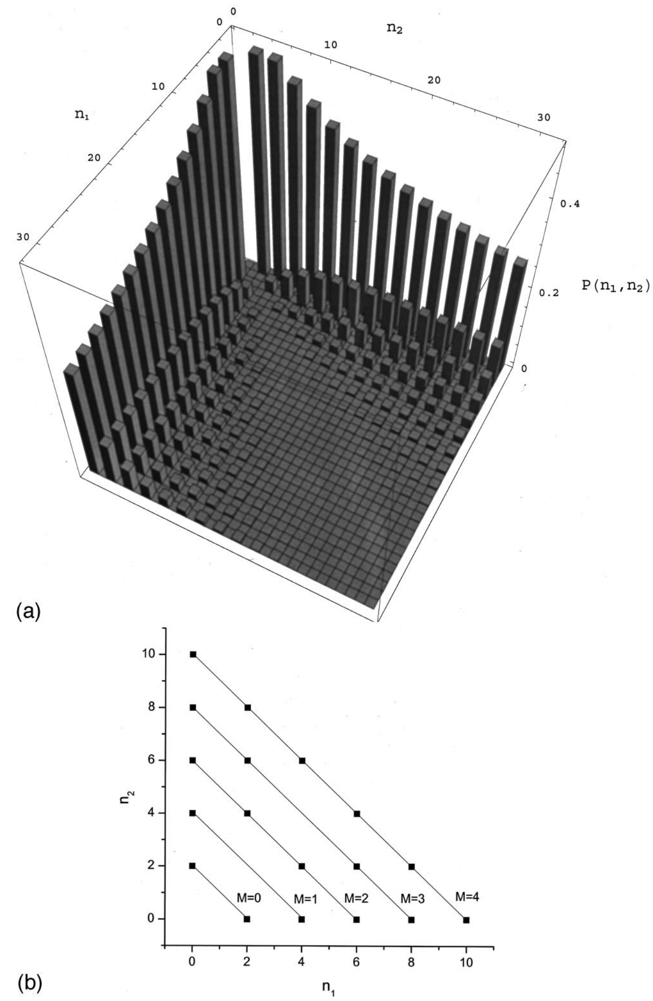

FIG. 5. Same as Fig. 3, but with $k_1=k_2=\frac{1}{4}$, and $l=1$; hence $K=\frac{3}{2}$.

IV. COHERENT STATES FOR SU(1,1)⊗SU(1,1)

We now construct coherent states appropriate to the group SU(1,1)⊗SU(1,1). As there are several types of coherent states for SU(1,1), so there will be for SU(1,1)⊗SU(1,1). We shall restrict our attention to just two types: Perelomov and Barut-Girardello.

The Perelomov coherent state is given by

$$|\xi,K\rangle = S(z)|K,0;k_1,k_2\rangle \tag{4.1a}$$

$$=(1-|\xi|^2)^K \sum_{M=0}^{\infty}\left[\frac{\Gamma(2K+M)}{M!\,\Gamma(2K)}\right]^{1/2} \times \xi^M |K,M;k_1,k_2\rangle \tag{4.1b}$$

$$=\sum_{M=0}^{\infty}\sum_{m=0}^{l+M} A(l,M,m,\xi)|k_1,m\rangle|k_2,l+M-m\rangle, \tag{4.1c}$$

where

$$A(l,M,m,\xi)=(1-|\xi|^2)^K \xi^M \left[\frac{\Gamma(2K+M)}{M!\,\Gamma(2K)}\right]^{1/2} \times C(k_1,k_2,K;m,l+M-m,M). \tag{4.2}$$

We note that the operator $S(z)$, the squeeze operator of Eq. (2.7), from Eqs. (3.3), factors into a product of squeeze operators for the two modes, i.e.,

$$S(z)=\exp(zK_+ - z^*K_-)=S_1(z)S_2(z), \tag{4.3}$$

where $S_i(z)=\exp(zK_+^{(i)}-z^*K_-^{(i)})$ for $i=1$ and 2. These squeeze operators could be realized by a pair of degenerate parametric amplifiers acting with identical coupling strengths and phases of the classical pump fields. In terms of the photon number states, the Perelomov state may be written as

$$|\xi,K;k_1,k_2\rangle=\sum_{M=0}^{\infty}\sum_{m=0}^{l+M} A(l,M,m,\xi)|2(m+k_1)-1/2\rangle \times|2(l+M-m+k_2)-1/2\rangle. \tag{4.4}$$

With $l=0$, the ground state in Eq. (4.1) is just the two-mode product state

$$|K,0\rangle=|k_1,0\rangle|k_2,0)\rangle=|2k_1-1/2\rangle|2k_2-1/2\rangle,\quad K=k_1+k_2 \tag{4.5}$$

(group states on the left and middle and number states on the right) and thus, because of the factorization of the squeeze operator according to Eq. (4.3), it follows that the corresponding SU(1,1)⊗SU(1,1) Perelomov coherent state factors into a product of single-mode SU(1,1) coherent states, i.e. a product two single-mode squeezed vacuum states:

$$|\xi,K\rangle=|\xi,k_1\rangle\otimes|\xi,k_2\rangle,\quad K=k_1+k_2. \tag{4.6}$$

Thus there is no entanglement in this case. For the special case $k_1=k_2=\frac{1}{4}$, using Eq. (4.1) the Perelomov state takes the form

$$|\xi,K=\tfrac{1}{2}\rangle=(1-|\xi|^2)^{1/2}\sum_{M=0}^{\infty}\xi^M|K=\tfrac{1}{2},M\rangle, \tag{4.7}$$

where the coupled states $|K=\frac{1}{2},M\rangle$ are given in terms of the two-mode number states by Eq. (3.8).

The photon number probability distribution for the states of Eq. (4.1c) is given by

$$P(n_1,n_2)=|\langle n_1,n_2|\xi,K\rangle|^2$$
$$=\left|\sum_{M=0}^{\infty}\sum_{m=0}^{l+M}A(l,M,m,\xi)\,\delta_{n_1,2(m+k_1)-1/2}\times\delta_{n_2,2(l+M-m+k_2)-1/2}\right|^2. \tag{4.8}$$

This distribution is displayed for various $|\xi|$ for the important case of $k_1=k_2=\frac{1}{4}$ and for the cases $l=0$ and 1 in Fig. 6. We note that the distributions are symmetric about the diagonal, and they contain ''holes'' wherever either n_1 or n_2 (or both) is an odd integer. For the case with $l=0$, these oscillations in the probability distribution are the result of the oscillations present in the photon statistics of each of the modes separately, those states being single-mode squeezed vacuum states as contained in Eq. (4.6), with each of the single mode states being of the form of Eq. (2.8).

At this point we can demonstrate an interesting result, namely, that if a two-mode squeezed vacuum state, an entangled state, is incident at the two input ports of a 50/50 beam splitter whose action is described by the previously introduced rotation operator $U_{BS}=\exp[-i(\pi/2)J_1]$, then the output state of the beam splitter is an SU(1,1)⊗SU(1,1) Perelomov coherent state of Eq. (4.6) with $k_1=k_2=\frac{1}{4}$, which is an *unentangled* state. That this, the beam splitter disentangles the two-mode squeezed vacuum states into a product of two single-mode squeezed vacuum states. It is easy to see how this comes about. From Eq. (2.14), the squeezed vacuum state (for which $q=0$) is

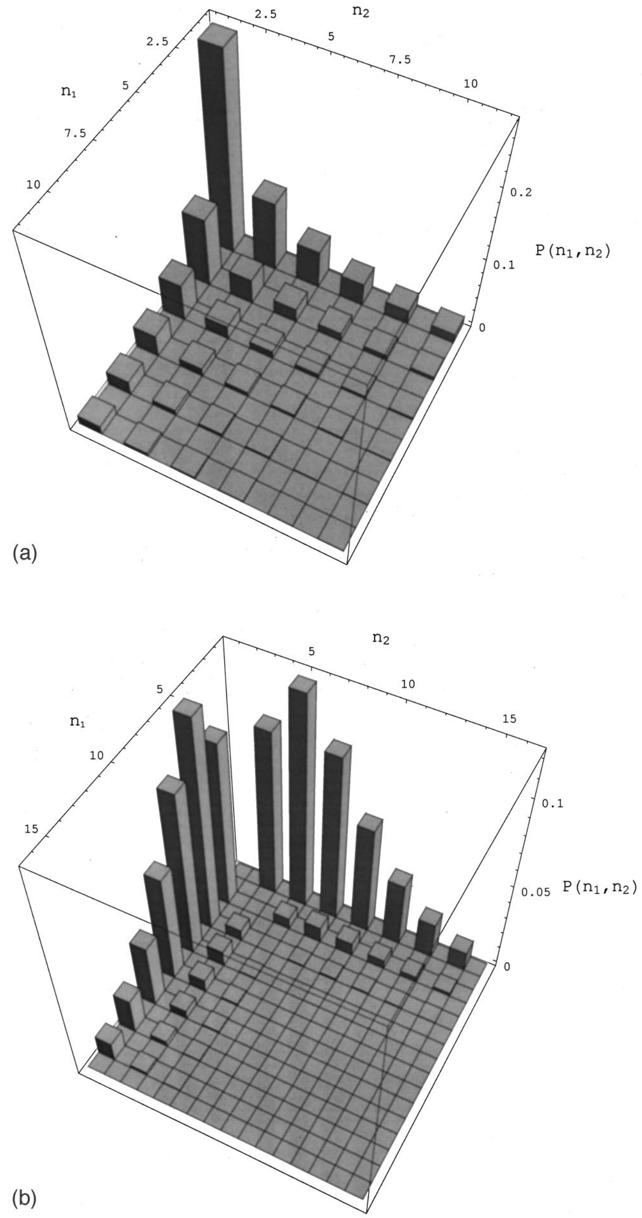

FIG. 6. Joint photon probability distribution of Eq. (4.8) for the Perelomov coherent state of Eq. (4.4) for $|\xi|=0.85$, for the case $k_1=k_2=\frac{1}{4}$, and (a) $l=0$ (hence $K=\frac{1}{2}$) and (b) $l=1$, (hence $K=\frac{3}{2}$).

$$\left|\xi,k=\frac{1}{2}\right\rangle=(1-|\xi|^2)^{1/2}\sum_{M=0}^{\infty}\xi^M|M,M\rangle \tag{4.9}$$

where we have set $n=M$. Now applying the operator U_{BS} to both sides and using Eq. (3.11), we obtain

$$U_{BS}|\xi,k=\tfrac{1}{2}\rangle=(1-|\xi|^2)^{1/2}\sum_{M=0}^{\infty}(-i\xi)^M|K=\tfrac{1}{2},M\rangle$$
$$=|-i\xi,K=\tfrac{1}{2}\rangle, \tag{4.10}$$

where the right-hand side is, apart from a shift in the phase of the parameter ξ, identical to Eq. (4.7) which, as we have shown, is a factorizable state of the form of Eq. (4.6). Such a disentanglement may also result from the interaction of a two-mode squeezed vacuum state with a frequency converter modeled by the Hamiltonian $H=\hbar 2\kappa J_1$, where κ depends on the second-order nonlinear susceptibility of the medium and where, as before, $J_1=(a^\dagger b+ab^\dagger)/2$. Obviously, this is essentially the same interaction as for the beam splitter, at least for certain choices of internal phases for the latter. At times t such that $2\kappa t=\pi/2$, the disentangled states appear in the output modes. Previously, Gagen and Milburn [24] studied the photon statistics of the two-mode squeezed vacuum states evolving under such an interaction. These authors made the claim that the action of the frequency converter is to produce, from a two-mode squeezed vacuum state characterized by strong correlations between the photons of the two input modes, states with anticorrelations between the photons in the two output modes. The appearance of the ''holes'' in the joint photon number probability distribution was taken as the signal for the anticorrelations, and the case for which $2\kappa t=\pi/2$ was supposed to display maximum anticorrelations. But, as we have seen, our states are in fact disentangled, and thus there are no anticorrelations or correlations between the photons in the two output modes, the effect noted by Gagen and Milburn [24] being merely the result of the direct product of two single-mode squeezed vacuum states. That such a disentanglement should occur was previously demonstrated using operator methods [25]. It is important to understand that the individual coupled basis states of the form of Eqs. (3.8b), (3.8c), etc., as generated in the experiments of Refs. [22] and [23] *do* exhibit anticorrelations, as these authors demonstrated in the laboratory, but the superposition of these states in the form of Eq. (4.7) evidently does not exhibit them.

On the other hand, if the ground state is an entangled state $(l>0)$, then the Perelomov coherent state is also an entangled state, in fact, generally an entanglement of squeezed number states of each of the modes. For example in the case for $k_1=k_2=\frac{1}{4}$ and $l=1$ with the ground state given by Eq. (3.14a) the SU(1,1)⊗SU(1,1) Perelomov coherent state is given by

$$|\xi,K=3/2\rangle=\frac{1}{\sqrt{2}}[S_1(z)|0\rangle S_2(z)|2\rangle-S_1(z)|2\rangle S_2(z)|0\rangle], \tag{4.11}$$

which is an entanglement of squeezed vacuum and squeezed two-photon states.

If we disregard mode b, the marginal photon probability distribution for mode a is given by

$$P(n_1)=\sum_{n_2} P(n_1,n_2). \tag{4.12}$$

In Fig. 7 we plot examples of this distribution for the cases $k_1=k_2=\frac{1}{4}$ for $l=0$ and 1. Oscillations in the distributions are evident in both cases, but in the former we observe the familiar ''thermal-like'' behavior expected for the single-mode squeezed vacuum state. Because of the symmetry of the states with respect to the photon numbers of the two modes, the marginal distribution for mode b is identical to that of mode a.

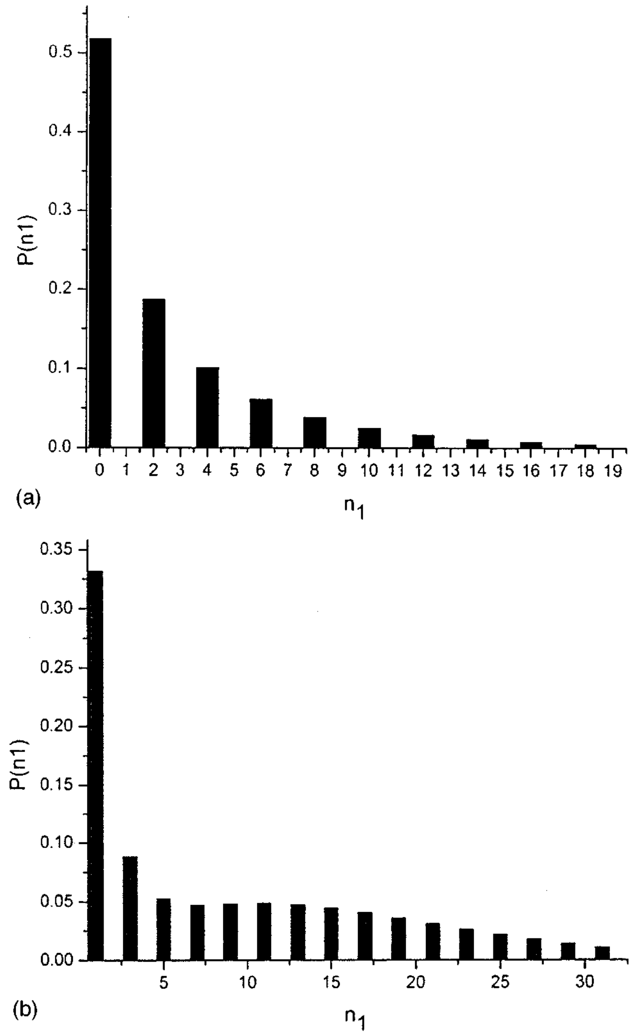

FIG. 7. For the Perelomov coherent states with $|\xi|=0.85$, the marginal photon probability distribution $P(n_1)$ vs n_1 for mode a for the case $k_1=k_2=\frac{1}{4}$, and (a) $l=0$ (hence $K=\frac{1}{2}$) and (b) $l=1$, (hence $K=\frac{3}{2}$). The marginal distribution for mode b is identical.

Anticorrelations between the two modes are characterized by the normalized cross-correlation functions

$$G^{(2)}_{a,b}=\frac{\langle a^\dagger b^\dagger b a\rangle}{\langle a^\dagger a\rangle\langle b^\dagger b\rangle}. \tag{4.13}$$

Whenever this function is less than unity, the states are anticorrelated. It is evident that in the case of the Perelomov states for $l=0$ the numerator factors into the product $\langle a^\dagger a\rangle\langle b^\dagger b\rangle$, and thus the cross-correlation function takes the expected value of unity. For the case of $l=1$ we expect the states to exhibit anticorrelations, and these are seen in Fig. 8, where we plot $G^{(2)}_{a,b}$ as a function of $|\xi|$.

The BG coherent states for SU(1,1)⊗SU(1,1) are defined just as in Eq. (2.9) as eigenstates of the lowering operator $K_-=K_-^{(1)}+K_-^{(2)}$,

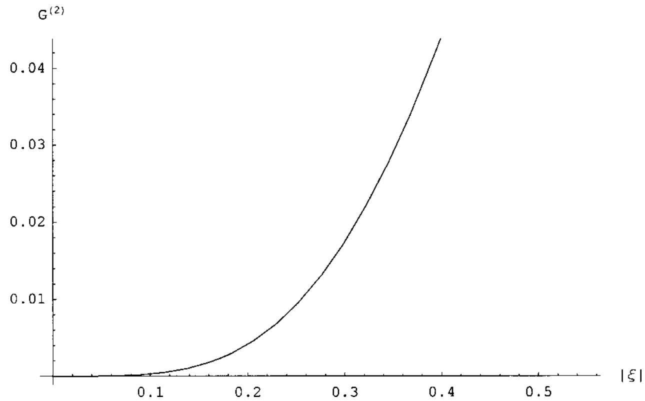

FIG. 8. The functions $G^{(2)}$ vs $|\xi|$ for the Perelomov coherent state for the case $l=1$. We note the existence of correlations between the modes.

$$K_-|\zeta,K;k_1,k_2\rangle=\tfrac{1}{2}(a^2+b^2)|\zeta,K;k_1,k_2\rangle=\zeta|\zeta,K;k_1,k_2\rangle, \tag{4.14}$$

whose solution is of the form of Eq. (2.10), which we write here as

$$|\zeta,K;k_1,k_2\rangle=N_K\sum_{M=0}^{\infty}\frac{\zeta^M}{[M!\Gamma(2K+M)]^{1/2}}|K,M;k_1,k_2\rangle, \tag{4.15}$$

where

$$N_K=[\Gamma(2K)|\zeta|^{-2K+1}I_{2K-1}(2|\zeta|)]^{-1/2}. \tag{4.16}$$

From the substitution of Eq. (3.6) into Eq. (4.15) the BG coherent state is given in terms of the basis states of the two modes as

$$|\zeta,K,k_1,k_2\rangle=\sum_{M=0}^{\infty}\sum_{m=0}^{l+M}B(l,M,m,\zeta)|k_1,m\rangle|k_2,l+M-m\rangle, \tag{4.17}$$

where

$$B(l,M,m,\zeta)=N_K\frac{\zeta^M}{[M!\Gamma(2K+M)]^{1/2}}\times C(k_1,k_2,K;m,l+M-m,M). \tag{4.18}$$

In terms of the photon number states of each mode, Eq. (4.17) may be written as

$$|\zeta,K;k_1,k_2\rangle=\sum_{M=0}^{\infty}\sum_{m=0}^{l+M}B(l,M,m,\zeta)|2(k_1+m)-1/2\rangle\times|2(k_2+l+M-m)-1/2\rangle. \tag{4.19}$$

We first consider the photon probability distributions. The joint probability of finding n_1 photons in mode a and n_2, in mode b is given by

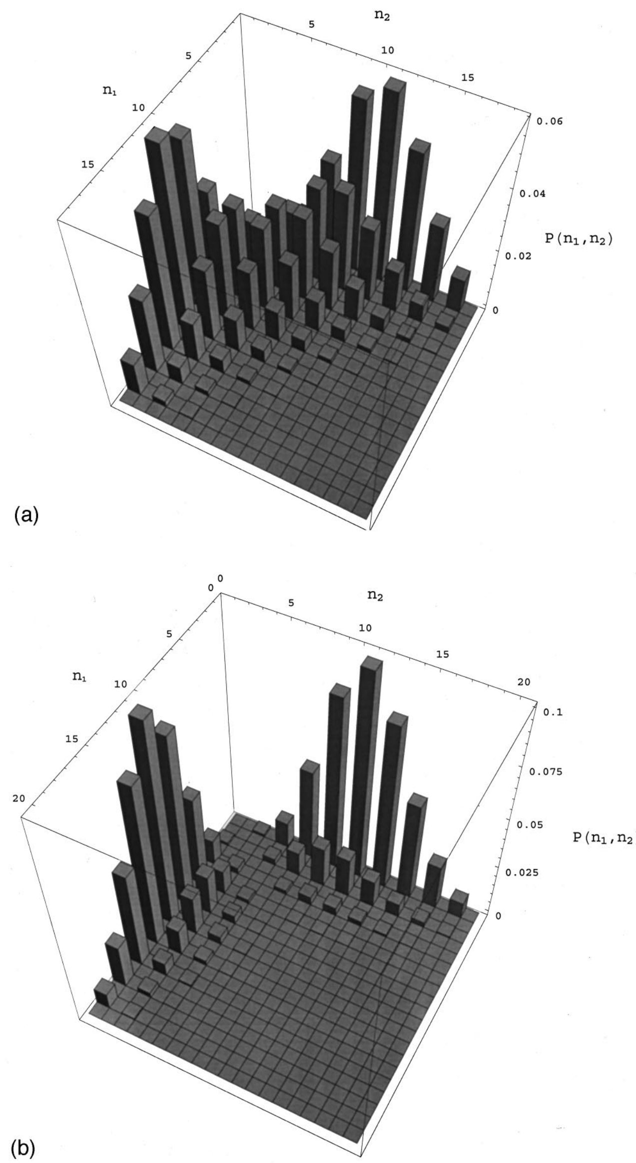

FIG. 9. Joint photon number distribution for the BG coherent state with $k_1=k_2=\frac{1}{4}$ and $\zeta=5.5$ for (a) $l=0$ and (b) $l=1$.

$$P(n_1,n_2)=|\langle n_1,n_2|\zeta,K;k_1,k_2\rangle|^2=\left|\sum_{M=0}^{\infty}\sum_{m=0}^{l+M}B(l,M,m,\zeta)\delta_{n_1,2(m+k_1)-1/2}\times\delta_{n_2,2(l+M-m-k_2)-1/2}\right|^2. \tag{4.20}$$

We display this function in Fig. 9 for various values of $|\zeta|$ for the cases $l=0$ and 1. In Fig. 10 we display the corresponding marginal distributions $P(n_1)$. In the present cases, in contrast to the Perelomov-type coherent state of Eq. (4.2) for $K=k_1+k_2(l=0)$, the BG coherent state of Eq. (4.9) is an entangled state. To show that this is the case, we calculate the trace of the square of the reduced density operator of the a mode, which works out to be

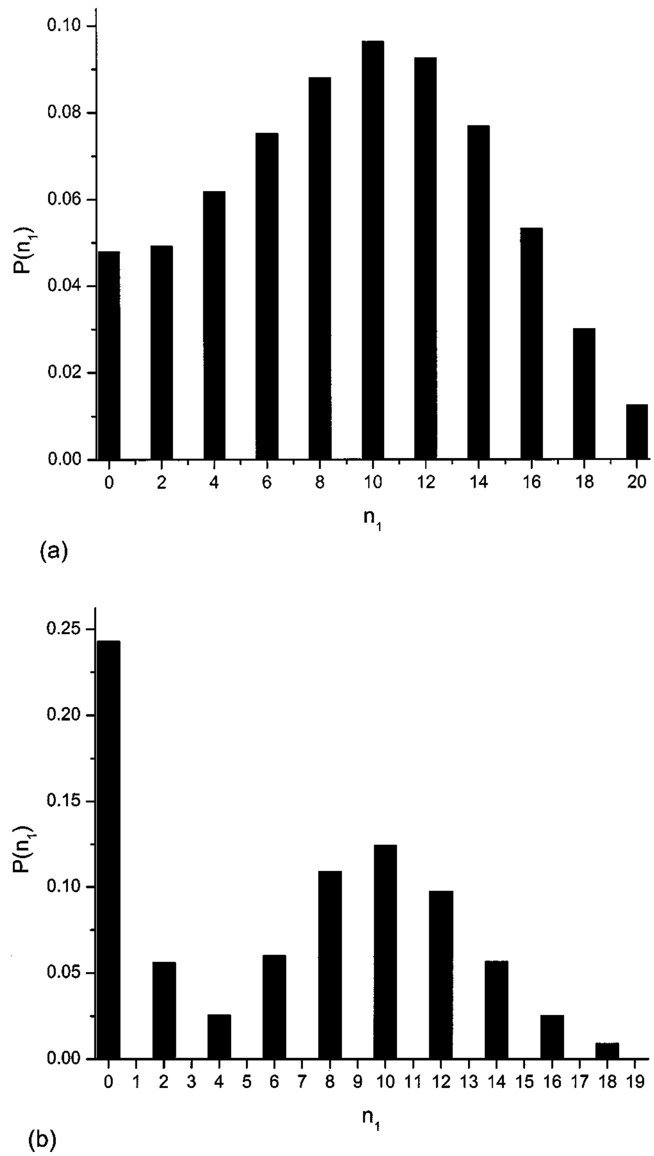

FIG. 10. For the BG coherent states for $|\zeta|=5.5$, the marginal photon probability distribution $P(n_1)$ vs n_1 for mode a for the case $k_1=k_2=\frac{1}{4}$, and (a) $l=0$, (hence $K=\frac{1}{2}$) and (b) $l=1$, (hence $K=\frac{3}{2}$). The marginal distribution for mode b is identical.

$$\mathrm{Tr}_a(\rho_a^2)=\sum_{s=0}^{\infty}\sum_{\nu=0}^{\infty}\left|\sum_{r=0}^{\infty}B(l,s+r-l,r,\zeta)\times B^*(l,\nu+r-l,r,\zeta)\right|^2. \tag{4.21}$$

In Fig. 11 we plot this quantity as a function of $|\zeta|$, where it is evident that the trace of ρ_a^2 is less than unity, and thus we have an entangled state.

As in the case of the Perelomov states, a 50/50 beam splitter, acting as previously described in terms of the operator J_1, can be used to generate the BG coherent states for the case with $k_1=k_2=\frac{1}{4}$. Only this time the incident state on the beam splitter must be a pair-coherent state with degeneracy parameter $q=0$ (or $k=\frac{1}{2}$), where, from Eq. (2.18),

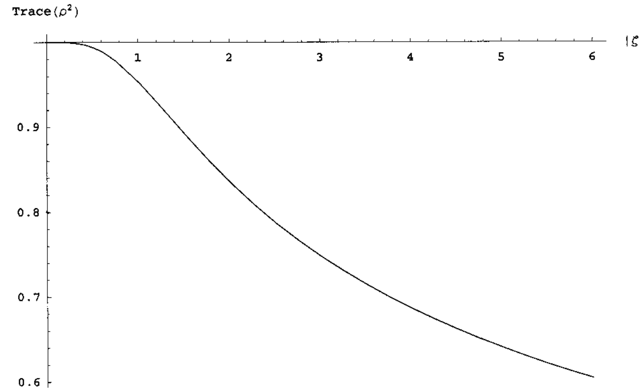

FIG. 11. $Tr(\rho_a^2)$ vs $|\zeta|$ for the BG coherent state for the case with $l=0$. We note that the state is increasingly entangled for increasing $|\zeta|$.

$$|\zeta,k=\tfrac{1}{2}\rangle=[I_0(2|\zeta|)]^{-1/2}\sum_{M=0}^{\infty}\frac{\zeta^M}{M!}|M,M\rangle, \tag{4.22}$$

where we have set $n=M$. Applying the operator to both sides and using Eq. (3.11), we obtain

$$U_{BS}|\zeta,k=\tfrac{1}{2}\rangle=[I_0(2|\zeta|)]^{-1/2}\sum_{M=0}^{\infty}\frac{(-i\zeta)^M}{M!}|K=\tfrac{1}{2},M\rangle =|-i\zeta,K=\tfrac{1}{2}\rangle, \tag{4.23}$$

where the last equality follows from Eq. (4.12), with the understanding that $k_1=k_2=\frac{1}{4}$.

In Fig. 12 we plot the corresponding cross-correlation function $G_{a,b}^{(2)}$ of Eq. (4.13) for the BG coherent states as a function of $|\zeta|$ for the case $l=0$. It is apparent that our BG state exhibits anticorrelations over a wide range of the parameter $|\zeta|$.

We have already discussed methods that might be used to generate the Perelomov states. All that is required is a pair of degenerate parametric down-converters acting on suitably prepared entangled or unentangled ''ground'' states. Furthermore, in the case when the ''ground'' state is in fact that the

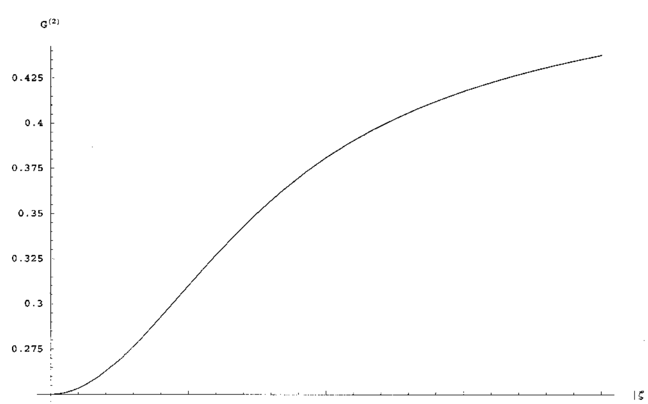

FIG. 12. The functions $G^{(2)}$ vs $|\zeta|$ for the BG coherent state for the case $l=0$. We note the existence of correlations between the modes.

two-mode vacuum state, the corresponding Perelomov state consists merely of a product of two single-mode squeezed vacuum states. Such a state results from the passage of a two-mode squeezed vacuum state through a 50/50 beam splitter, with certain choices of internal phases.

On the other hand, if a pair-coherent state with parameter $q=0$ is incident on a 50/50 beam splitter, the output state in this case will be a BG coherent state of the group SU(1,1)⊗SU(1,1). Unlike the case of the incident two-mode squeezed vacuum state, in the present case the output modes are entangled. Incident pair coherent states in cases for which $q>0$ do not seem to yield coherent states of SU(1,1)⊗SU(1,1) in the output beams.

Finally we describe a possible mechanism for generating our states, not using passive optical devices such as beam splitters, and perhaps able to generate a wider class of states than is possible with such devices. As mentioned in Sec. I, the competition between the interactions described by the Hamiltonians of Eqs. (1.2) and (1.3) in the steady state produces eigenstates of the operator $a^2+b^2=2K_-$, one example of which is the BG state. A detailed analysis following along the lines in Ref. [26] shows that, upon the adiabatic elimination of the atomic states, the dynamical evolution of the density operator ρ of the field is described by the master equation

$$\frac{\partial\rho}{\partial t}=-\frac{i}{\hbar}[H_{\rm eff},\rho]-2\kappa(K_+K_-\rho-2K_-\rho K_++\rho K_+K_-), \tag{4.24}$$

where the effective Hamiltonian is

$$H_{\rm eff}=2\hbar(GK_++G^*K_-), \tag{4.25}$$

and where κ is related to a third-order susceptibility for two-channel two-photon absorption. Elsewhere we shall numerically study the time evolution of various initial states, but here we are interested mainly in the steady-state ($t\rightarrow\infty$) solutions. To this end we introduce the operator

$$B=K_-+i\frac{G}{\kappa}, \tag{4.26}$$

in terms of which the master equation of Eq. (4.24) may be rewritten as

$$\frac{\partial\rho}{\partial t}=-2\kappa(B^\dagger B\rho-2B\rho B^\dagger+\rho B^\dagger B). \tag{4.27}$$

Evidently, the steady-state solutions, for which $\partial\rho/\partial t=0$, are those density operators satisfying the eigenvalue problem

$$B\rho=0=\rho B^\dagger. \tag{4.28}$$

The general solution is a mixed state, but for certain initial conditions the solution may be a pure state in which case we may write $\rho=|\psi\rangle\langle\psi|$, where $|\psi\rangle$ satisfies the eigenvalue problem $B|\psi\rangle=0$, or, equivalently,

$$K_-|\psi\rangle=-i\frac{G}{\kappa}|\psi\rangle. \tag{4.29}$$

Now if the initial state of the system is a ''ground'' state from the SU(1,1)⊗SU(1,1) basis, $|K,M=0;k_1,k_2\rangle$, then, because the interactions contained in the master equation of Eq. (4.24) create or destroy photons two at a time competitively in the two modes, only those states can be generated that are contained within the basis $\{|K,M;k_1,k_2\rangle\}$. In the steady-state limit, the pure state solution is the SU(1,1)⊗SU(1,1) BG state $|\psi\rangle=|\zeta,K;k_1,k_2\rangle$ given by Eq. (4.12) with $\zeta=-iG/\kappa$. Interestingly, in the limit of *short* time, for which we may neglect the dissipative term of the master equation, the initial state $|K,M=0;k_1,k_2\rangle$ evolves to the SU(1,1)⊗SU(1,1) Perelomov coherent state $|\xi,K;k_1,k_2\rangle$ with $\xi=-\tanh(\kappa|G|t)\exp[i(\phi_G+\pi/2)]$, where ϕ_G is the phase of G.

V. CONCLUSIONS

In this paper, we have constructed two sets of coherent states for the direct product group associated with two modes of the quantized field. The construction of the basis states of the group comes about in much the same way as in the coupling of two angular-momentum systems through the use of the Clebsch-Gordon coefficients for SU(2), only here we use the corresponding SU(1,1) Clebsch-Gordon coefficients. We have discussed coherent states of the Perelomov and Barut-Girardello types, and presented schemes for generating such states by the use of beam spitters and competitive processes involving two channels. This work could be extended in at least two directions: (i) to the SU(1,1)⊗SU(1,1) counterparts to the more general SU(1,1) states such as the intelligent state [27]; and (ii) to the coupling of more than two SU(1,1) representations, such as the three mode states for SU(1,1)⊗SU(1,1)⊗SU(1,1), etc. In the latter case, it would be necessary to employ the corresponding Racah coefficients for SU(1,1). It is not clear at this time if the coupling of three SU(1,1) representations is of any physical relevance, so we do not pursue such states here.

Finally, we point out that in all the above, we have had optical fields in mind. However, it was previously shown that various one- and two-mode SU(1,1) states may also be realizable in the vibrational motion of trapped ions [28]. Thus there is the strong possibility that, with modifications of the procedures proposed in Refs. [28], perhaps incorporating the notion that the two-dimensional motion of a trapped ion can act as a beam splitter [29], our states may be generated in the motion of a trapped ion. This possibility will be explored elsewhere.

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APPENDIX

In this appendix, we derive the SU(1,1) Clebsch-Gordon (CG) coefficients for the coupling of two discrete unitary representations of SU(1,1) of Bargmann indices k_1 and k_2. Numerous derivations of these coefficients have been given in the literature, usually in the notation analogous to that used in the case of angular momentum, SU(2) [20]. As stated in Sec. II, this notation has the disadvantage that it does not separate the integer and fractional parts of the spectrum of the operator K_0, in contrast to the notation used in this paper. Furthermore, we have found a difficulty with the results previously derived when it comes to the coupling of the nonstandard cases for which the Bargmann indices take on the values $\frac{1}{4}$ or $\frac{3}{4}$.

We begin by writing the basis of the SU(1,1)⊗SU(1,1) states as

$$|K,M;k_1,k_2\rangle = \sum_{m_1,m_2} C(k_1,k_2,K;m_1,m_2,M)|k_1,m_1\rangle|k_2,m_2\rangle, \tag{A1}$$

where the numbers $C(k_1,k_2,K;m_1,m_2,M)$ are the SU(1,1) CG coefficients. We first consider the ground state of the coupled representation, where $M=0$:

$$|K,0;k_1,k_2\rangle = \sum_{m_1,m_2} C_{m_1,m_2}|k_1,m_1\rangle|k_2,m_2\rangle, \tag{A2}$$

where for the moment and for convenience we have set $C(k_1,k_2,K;m_1,m_2,0)=C_{m_1,m_2}$. Acting on the state of Eq. (A2) with the operator K_- of Eq. (3.3c) gives us, since we have a ground state,

$$K_-|K,0;k_1,k_2\rangle = 0 = \sum_{m_1,m_2} C_{m_1,m_2}(K_-^{(1)}+K_-^{(2)})|k_1,m_1\rangle|k_2,m_2\rangle, \tag{A3}$$

or, equivalently,

$$0=\sum_{m_1,m_2} C_{m_1,m_2}\{[m_1(m_1+2k_1-1)]^{1/2}|k_1,m_1-1\rangle|k_2,m_2\rangle + [m_2(m_2+2k_2-1)]^{1/2}|k_1,m_1\rangle|k_2,m_2-1\rangle\}. \tag{A4}$$

We may rewrite this last expression as

$$\sum_{m_1,m_2}\{C_{m_1+1,m_2}[(m_1+1)(m_1+2k_1)]^{1/2} + C_{m_1,m_2+1}[(m_2+1)(m_2+2k_2)]^{1/2}\}|k_1,m_1\rangle|k_2,m_2\rangle = 0, \tag{A5}$$

from which follows the recursion relation

$$C_{m_1+1,m_2} = -C_{m_1,m_2+1}\left[\frac{(m_2+1)(m_2+2k_2)}{(m_1+1)(m_1+2k_1)}\right]^{1/2}. \tag{A6}$$

It is evident that only those states are coupled for which $m_1+m_2=\text{const}$. Setting $m_1+m_2=l$, $l=0,1,2\ldots,\infty$, the recursion can be solved to yield

$$C_{q,l-q} = (-1)^q\left[\binom{l}{q}\frac{\Gamma(2k_1)\Gamma(2k_2+l)}{\Gamma(2k_1+q)\Gamma(2k_2+l-q)}\right]^{1/2} C_{0,l}, \tag{A7}$$

where $C_{0,l}$ is determined from normalization to be

$$C_{0,l} = \left[\sum_{r=0}^{l}\binom{l}{r}\frac{\Gamma(2k_1)\Gamma(2k_2+l)}{\Gamma(2k_1+r)\Gamma(2k_2+l-q)}\right]^{-1/2}. \tag{A8}$$

Thus our ground state may now be written as

$$|K,0;k_1,k_2\rangle = \sum_{q=0}^{l} C(k_1,k_2,K;q,l-q,0)|k_1,q\rangle|k_2,l-q\rangle, \tag{A9}$$

where we have set $C(k_1,k_2,K;q,l-q,0)=C_{q,l-q}$. Applying the operator $K_0=K_0^{(1)}+K_0^{(2)}$ to this last equation it is easy to show that the allowed values of K, and hence the allowed representations of SU(1,1)⊗SU(1,1), are given by $K=k_1+k_2+l$, $l=0,1,2,\ldots,\infty$.

To obtain the states for $M>0$, we now apply to Eq. (A9) the raising operator K_+ of Eq. (3.3b) M times. However, making use of Eq. (2.4), we first rewrite Eq. (A9) as

$$|K,0,k_1,k_2\rangle = \sum_{q=0}^{l} C(k_1,k_2,K;q,l-q,0) \times\left[\frac{\Gamma(2k_1)\Gamma(2k_2)}{q!(l-q)!\Gamma(2k_1+q)\Gamma(2k_2+l-q)}\right]^{1/2} \times (K_+^{(1)})^q(K_+^{(2)})^{l-q}|k_1,0\rangle|k_2,0\rangle. \tag{A10}$$

Now writing

$$K_+^M = (K_+^{(1)}+K_+^{(2)})^M = \sum_{p=0}^{M}\binom{M}{p}(K_+^{(1)})^p(K_+^{(2)})^{M-p}, \tag{A11}$$

applying it to Eq. (A10) and making multiple uses of Eq. (2.4), we obtain

$$|K,M;k_1,k_2\rangle=\left[\frac{\Gamma(2K)}{M!\Gamma(2K+M)}\right]^{1/2}\sum_{q=0}^{l}\sum_{p=0}^{M}C(k_1,k_2,K;q,l-q,0)\binom{M}{p}\times\left[\frac{(p+q)!(l+M-p-q)!\Gamma(2k_1+p+q)\Gamma(2k_2+l+M-p-q)}{q!(l-q)!\Gamma(2k_1+q)\Gamma(2k_2+l-q)}\right]^{1/2}|k_1,p+q\rangle|k_2,l+M-p-q\rangle. \tag{A12}$$

We note that there is a degeneracy with respect to p and q. We thus need to collect the coefficients for with constant $p+q$. To this end we rewrite Eq. (A12) as

$$|K,M;k_1,k_2\rangle=\sum_{m=0}^{l+M}C(k_1,k_2,K;m,l+M-m,M)|k_1,m\rangle|k_2,l+M-m\rangle, \tag{A13}$$

where the $C(k_1,k_2,K;m,l+M-m,M)$ are the SU(1,1) Clebsch-Gordon coefficients, and are given by

$$C(k_1,k_2K;m,l+M-m,M)=\sum_{q=0}^{l}\sum_{p=0}^{M}\delta_{m,p+q}(-1)^q\frac{1}{q!(l-q)!\Gamma(2k_1+q)\Gamma(2k_2+l-q)}\binom{M}{p}\times\left\{\frac{l!\Gamma(2K)\Gamma(2k_1)\Gamma(2k_2+l)(p+q)!(l+M-p-q)!\Gamma(2k_1+p+q)\Gamma(2k_2+l+M-p-q)}{M!\Gamma(2K+M)}\right\}^{1/2}\times\left[\sum_{r=0}^{l}\binom{l}{r}\frac{\Gamma(2k_1)\Gamma(2k_2+l)}{\Gamma(2k_1+r)\Gamma(2k_2+l-r)}\right]^{-1/2}, \tag{A14}$$

where, of course, it is understood that $K=k_1+k_2+l$.

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