Optimal photon cloning

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We show that optimal universal cloning of the polarization state of photons can be achieved via stimulated emission in three-level systems, both of the Λ and the V type. We establish the equivalence of our systems with coupled harmonic oscillators, which permits us to analyze the structure of the cloning transformations realized. These transformations are shown to be equivalent to the optimal cloning transformations for qubits discovered by Bužek and Hillery and Gisin and Massar. The down-conversion cloner discovered previously by some of the authors is obtained as a limiting case. We demonstrate an interesting equivalence between systems of Λ atoms and systems of pairwise entangled V atoms. Finally we discuss the physical differences between our photon cloners and the qubit cloners considered previously and prove that the bounds on the fidelity of the clones derived for qubits also apply in our situation.

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I. INTRODUCTION

An ideal quantum cloning machine is a device that produces an arbitrary number of perfect copies of a given (unknown) quantum system. Such a device would allow the exact determination of the quantum state of a system. It has been shown [1] that such a device would violate the linearity of quantum mechanics and also relativistic locality because it would make superluminal communication possible [2,3].

Nonperfect copying, though, can be realized in quantum mechanics. Since the seminal paper of Bužek and Hillery [4], quantum cloning has been extensively studied theoretically. Bruss *et al.* [5] derived bounds on the possible fidelity of quantum cloners, Gisin and Massar and Bužek and Hillery [6] discovered optimal universal cloning transformations, and finally Werner and Keyl and Werner [7] discussed optimal universal cloning in great generality.

While optimal cloning was previously discussed in terms of quantum networks, in a recent paper some of the authors have shown that optimal universal cloning can be comparatively easily realized via stimulated emission [8]. In this scheme the general qubit to be cloned is represented by the polarization state of a photon. When cloning is realized via stimulated emission, the fidelity of the clones is limited by the unavoidable presence of spontaneous emission. It was shown that the bounds on the fidelity given by the abovementioned fundamental principles can nevertheless be saturated.

In Sec. II of the present paper we present a scheme for *optimal* universal cloning based on stimulated emission in three-level systems of the Λ type. Our main analytic tool is the formal equivalence between systems of Λ atoms and coupled harmonic oscillators, which is established in Sec. II B. In Sec. II C, this equivalence is used to analyze the structure of the transformations realized in detail and to prove their optimality. More specifically, we explicitly demonstrate their equivalence to the optimal cloning transformations for qubits discovered before [4,6]. In particular, it will become clear that the atomic states play the double role of photon source and ancilla, and that the universal NOT operation is realized in the ancilla states. In the same way, we show that the down-conversion cloner presented in Ref. [8] is obtained from the present schemes as a limiting case. In Sec. III we demonstrate that optimal cloning can also be achieved with pairwise entangled V atoms, using an interesting equivalence between the two systems. In Sec. IV we discuss the physical differences that exist between our stimulated emission cloners and the qubit cloners considered previously, and we give an explicit proof that the bounds derived for qubit cloning indeed apply to our situation as well. Section V gives our conclusions.

II. CLONING VIA STIMULATED EMISSION IN Λ ATOMS

The general principles of universal cloning via stimulated emission are the following. Consider an inverted medium that can spontaneously emit photons of any polarization with the same probability. If a photon (or several) of a given polarization interacts with such a medium, it stimulates the emission of photons of the same polarization. In the final photonic state there will be a majority of photons polarized parallel to the incoming photon, while some photons will be in the orthogonal polarization due to spontaneous emission. In this way the photons in the final state can be considered as clones of the original incoming photon, where the fidelity of the clones is given by the relative frequency of photons of the correct polarization in the final state.

The inverted medium that we will use as a cloning device consists of an ensemble of Λ atoms. These are three-level systems that have two degenerate ground states $|g_1\rangle$ and $|g_2\rangle$ and an excited level $|e\rangle$. The ground states are coupled to the excited state by two modes of the electromagnetic field a_1 and a_2, respectively. These two modes define the Hilbert space of our qubit to be cloned, i.e., we want to clone general

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superposition states $(\alpha a_1^\dagger + \beta a_2^\dagger)|0,0\rangle = \alpha|1,0\rangle + \beta|0,1\rangle$. We can think of a_1 and a_2 as being orthogonal polarizations of one photon with a specific frequency, but we do not have to restrict ourselves to such a specific example, in fact we can think about other systems and other degrees of freedom, as long as they are described by the same formalism, e.g., a_1 and a_2 could also refer to the center-of-mass motion (phonons) in an ion trap. In the interaction picture, after the usual dipole and rotating wave approximations, the interaction Hamiltonian between field and atoms has the following form:

$$\mathcal{H} = \gamma\left(a_1 \sum_{k=1}^{N} |e^k\rangle\langle g_1^k| + a_2 \sum_{k=1}^{N} |e^k\rangle\langle g_2^k| \right) + \text{H.c.}$$
$$= \gamma\left(a_1 \sum_{k=1}^{N} \sigma_{+,1}^k + a_2 \sum_{k=1}^{N} \sigma_{+,2}^k \right) + \text{H.c.} \quad (1)$$

The index k refers to the kth atom. Note that in Eq. (1) the atoms couple to only one single spatial mode of the electromagnetic field. In particular this means that spontaneous emission into all other modes is neglected. Situations where this is a good approximation can now be achieved in cavity QED [10]. We also assume that the coupling constant γ is the same for all atoms, which in a cavity QED setting means that they have to be in equivalent positions relative to the cavity mode. Trapping of atoms inside a cavity has recently been achieved [11]. Finally note that our Hamiltonian has no spatial dependence, which means that the effect of the field on the motion of the atoms is neglected, their spatial wavefunction is assumed to be unchanged [12].

The Hamiltonian (1) is invariant under simultaneous unitary transformations of the vectors (a_1, a_2) and $(|g_1\rangle, |g_2\rangle)$ with the same matrix U. If one furthermore chooses an initial state of the atoms that has the same invariance, then the system behaves equivalently for all incoming photon polarizations, i.e., universal cloning is achieved. This can be seen in the following way. Consider an incident photon in a general superposition state $(\alpha a_1^\dagger + \beta a_2^\dagger)|0,0\rangle$. Together with the orthogonal one-photon state this defines a new basis in polarization space, which is connected to the original one by a unitary transformation. If the atomic states are now rewritten in the basis that is connected to the original one by the same unitary transformation, then under the above assumptions the interaction Hamiltonian and initial state of the atoms look exactly the same as in the original basis. The initial state where all atoms are excited to $|e\rangle$ has the required invariance: it is completely unaffected by the abovementioned transformations.

We can therefore, without loss of generality, restrict ourselves to the cloning of photons in mode a_1. We consider an initial state

$$|\Psi_{\text{in}}\rangle = \otimes_{k=1}^{N} |e^k\rangle \frac{(a_1^\dagger)^m}{\sqrt{m!}} |0,0\rangle, \quad (2)$$

i.e., we are starting with m photons of a given polarization, and we want to produce a certain (larger) number n of clones.

A. The simplest case

For illustrative purposes let us first consider the simplest case of one Λ atom and one photon polarized in direction 1:

$$|\Psi_{\text{in}}\rangle = |e\rangle a_1^\dagger |0,0\rangle = |e\rangle|1,0\rangle =: |\mathcal{F}_0\rangle. \quad (3)$$

To study the time development, we expand the evolution operator $e^{-i\mathcal{H}t}$ into a Taylor series and determine the action of powers of $\mathcal{H}$ on the state $|\Psi_{\text{in}}\rangle$:

$$\mathcal{H}|\Psi_{\text{in}}\rangle = \gamma(|g_1\rangle a_1^\dagger|1,0\rangle + |g_2\rangle a_2^\dagger|1,0\rangle)$$
$$= \gamma\sqrt{3}\, \frac{(\sqrt{2}|g_1\rangle|2,0\rangle + |g_2\rangle|1,1\rangle)}{\sqrt{3}} =: \gamma\sqrt{3}|\mathcal{F}_1\rangle,$$
$$\mathcal{H}^2|\Psi_{\text{in}}\rangle = \gamma^2(|e\rangle a_1\sqrt{2}|2,0\rangle + |e\rangle a_2|1,1\rangle)$$
$$= 3\gamma^2|e\rangle|1,0\rangle = 3\gamma^2|\mathcal{F}_0\rangle, \ldots \quad (4)$$

The result is

$$e^{-i\mathcal{H}t}|\Psi_{\text{in}}\rangle = \cos(\gamma\sqrt{3}t)|e\rangle|1,0\rangle$$
$$- i\sin(\gamma\sqrt{3}t)\left(\sqrt{\frac{2}{3}}|g_1\rangle|2,0\rangle + \sqrt{\frac{1}{3}}|g_2\rangle|1,1\rangle \right)$$
$$= \cos(\gamma\sqrt{3}t)|\mathcal{F}_0\rangle - i\sin(\gamma\sqrt{3}t)|\mathcal{F}_1\rangle. \quad (5)$$

$|\mathcal{F}_0\rangle$ and $|\mathcal{F}_1\rangle$ denote the states of the system atom-photons that lie in the subspace with 1 and 2 photons, respectively. $|\mathcal{F}_0\rangle$ is in the subspace where no cloning has taken place and $|\mathcal{F}_1\rangle$ in the one where one additional photon has been emitted, so that the two photons can now be viewed as clones with a certain fidelity. This way of labeling the states will turn out to be convenient below. The probability that the system acts as a cloner is $p(1) = \sin^2(\gamma\sqrt{3}t)$. The fidelity F_1 of the cloning procedure can be defined as the relative frequency of photons in the correct polarization mode in the final state $|\mathcal{F}_1\rangle$ (see Sec. IV). One finds

$$F_1 = \frac{2}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} = \frac{5}{6}, \quad (6)$$

which is exactly the optimal fidelity for a 1-to-2 cloner [4,5]. Actually, the state

$$|\mathcal{F}_1\rangle = \sqrt{\frac{2}{3}}|2,0\rangle|g_1\rangle + \sqrt{\frac{1}{3}}|1,1\rangle|g_2\rangle \quad (7)$$

is exactly equivalent to the three-qubit state

$$\sqrt{\frac{2}{3}}|11\rangle|\downarrow\rangle + \sqrt{\frac{1}{3}}\left(\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \right)|\uparrow\rangle \quad (8)$$

produced by the Bužek-Hillery cloner, see Ref. [4], Eq. (3.29b). The equivalence is established, if the photonic states in Eq. (7) are identified with the corresponding *symmetrized* two-qubit states (both photons in mode 1 means both qubits in state $|1\rangle$, one photon in each mode means one qubit in state $|1\rangle$, one in state $|0\rangle$) in Eq. (8), while the atomic states $|g_1\rangle$ and $|g_2\rangle$ are identified with the states $|\downarrow\rangle$ and $|\uparrow\rangle$ of the ancillary qubit. This is another way of proving the optimality of Eq. (7). Note that in our case the *universality* follows directly from the symmetry of initial state and Hamiltonian, as explained above. In the following we show that a similar equivalence holds between our cloning scheme and the Gisin-Massar cloners in the completely general case (arbitrary numbers of photons and atoms).

B. Equivalence to coupled harmonic oscillators

We now turn to the discussion of the general case, i.e., we consider the initial state (2). We are going to show the equivalence of our system defined by Eqs. (1) and (2) to a system of coupled harmonic oscillators. First note that both the initial state (2) and the Hamiltonian (1) are invariant under permutations of the atoms, which implies that the state vector of the system will always be completely symmetric. Furthermore the Hamiltonian (1) can be rewritten as

$$\mathcal{H}=\gamma(a_1J_{+,1}+a_2J_{+,2})+\text{H.c.} \tag{9}$$

in terms of ''total angular momentum'' operators

$$J_{+,r}=\sum_{k=1}^{N}\sigma_{+,r}^{k}=\sum_{k=1}^{N}|e^k\rangle\langle g_r^k| \quad (r=1,2). \tag{10}$$

By the above considerations one is led to use a Schwinger-type representation [14] for the angular momentum operator.

$$J_{+,r}=b_rc^\dagger \quad (r=1,2), \tag{11}$$

where $c^\dagger$ is a harmonic-oscillator operator creating ''e-type'' excitations, while b_1 destroys ''g_1'' excitations. Note that $J_{+,1}$ and $J_{+,2}$ share the operator $c^\dagger$ because both ground levels g_1 and g_2 are connected to the same upper level e by the Hamiltonian (1), and correspondingly for the Hermitian conjugates. In terms of these operators, Eq. (1) acquires the form

$$\mathcal{H}_{\text{osc}}=\gamma(a_1b_1+a_2b_2)c^\dagger+\text{H.c.}, \tag{12}$$

while the initial state (2) is now given by

$$\begin{aligned}|\psi_i\rangle&=\frac{(a_1^\dagger)^m}{\sqrt{m!}}\frac{(c^\dagger)^N}{\sqrt{N!}}|0\rangle\\&=|m_{a1},0_{a2},0_{b1},0_{b2},N_c\rangle\\&\equiv|m,0,0,0,N\rangle.\end{aligned} \tag{13}$$

Actually, for reasons that will become apparent below, it is slightly more convenient for our purposes to use the following Hamiltonian instead of Eq. (12):

$$\mathcal{H}=\gamma(a_1b_2-a_2b_1)c^\dagger+\text{H.c.}, \tag{14}$$

which can be obtained from Eq. (12) by a simple unitary transformation in mode b, corresponding to a simple redefinition of the atomic states in Eq. (1). This is the Hamiltonian that is going to be used in the rest of this paper. The invariance properties of Eq. (14) are linked to those of Eq. (1) or equivalently Eq. (12) discussed above: Eq. (14) is invariant under simultaneous identical SU(2) transformations in modes a and b (because the determinant of such a transformation is equal to unity), while a phase transformation in either mode can be absorbed into γ. This ensures the universality of the cloning procedure.

We are now dealing with five harmonic oscillator modes defined by the operators c, b_1, b_2, a_1, and a_2. Action of Eqs. (14) on (13) generates Fock basis states of the general form

$$\begin{aligned}&|(m+j)_{a1},i_{a2},i_{b1},j_{b2},(N-i-j)_c\rangle\\&\quad=|m+j,i\rangle_{\text{photons}}|i,j,N-j\rangle_{\text{atoms}}.\end{aligned} \tag{15}$$

Remember that a_1 is now coupled to b_2, etc. Expressed in terms of individual atoms, $|i,j,N-i-j\rangle_{\text{atoms}}$ is the completely symmetrized state with i atoms in level g_1, j atoms in level g_2, and $N-i-j$ atoms in level e. The correctness of Eq. (11) can be checked by explicit application of left hand side and right hand side to such a general state, written in terms of the individual atoms and in terms of harmonic oscillator eigenstates respectively (see Appendix A). Note that the use of the Schwinger representation is only convenient because the initial state of the atomic system in Eq. (2) is completely symmetric under permutation of the atoms.

Studying the Hamiltonian in the form (14) instead of Eq. (1) is helpful in several respects. The number of atoms N that is explicit in the Hamiltonian (1) now appears only as a part of the initial conditions of our system, which makes it easy to treat the general case of N atoms in one go. We will do this in the next subsection.

Furthermore, the connection to cloning by parametric down conversion (PDC) as proposed in Ref. [8] is now obvious. The Hamiltonian (14) can also be seen as a Hamiltonian for down conversion with a quantized pump mode described by the operator c, while a_r and b_r are the signal and idler modes respectively, where r labels the polarization degree of freedom. There is only one difference between Eq. (14) and the Hamiltonian used in Ref. [8] [see Eq. (6) of that reference]: in Ref. [8] the operator c of Eq. (14) is replaced by a c number. In the context of down conversion, this corresponds to the limit of a classical pump field. Thus the PDC scheme, which was shown to achieve optimal universal cloning in Ref. [8], is obtained as a limiting case from the schemes discussed here.

In passing we note that the above dynamical equivalence generalizes to atoms with more than two ground states $|g_n\rangle$ that are coupled each to a different degree of freedom of photons a_n. By similar arguments a system of N identical atoms with r ground states $\{|g_1\rangle,\dots,|g_r\rangle\}$ governed by a Hamiltonian

$$\mathcal{H}^r=\gamma\sum_{k=1}^{N}\sum_{n=1}^{r}|e^k\rangle\langle g_n^k|a_n+\text{H.c.} \tag{16}$$

is equivalent to a system of $r+1$ coupled harmonic oscillators with lowering operators c and $b_1,\ldots,b_r$ governed by the interaction Hamiltonian

$$\mathcal{H}^r_{\rm osc}=\gamma\sum_{n=1}^{r} c b_n^\dagger a_n^\dagger+\text{H.c.} \qquad (17)$$

C. Cloning of *m* photons with *N* Λ atoms: Proof of optimality

We are now going to show that the system defined by Eqs. (13) and (14) indeed realizes optimal cloning for arbitrary N and m. The idea of the proof is the following. After evolution in time the system that started with a certain photon number m will be in a superposition of states with different total photon numbers, where total means counting photons in mode a_1 and a_2, i.e., both ''good'' and ''bad'' copies. We will show that the general form of the state vector after a time interval t is

$$|\Psi(t)\rangle=e^{-i\mathcal{H}t}|\Psi_{\rm in}\rangle=\sum_{l=0}^{N} f_l(t)|\mathcal{F}_l\rangle, \qquad (18)$$

where l denotes the number of *additional* photons that have been emitted and

$$|\mathcal{F}_l\rangle:=\binom{m+l+1}{l}^{1/2}\sum_{i=0}^{l}(-1)^i\sqrt{\binom{m+l-i}{m}}\times|(m+l-i)_{a1},i_{a2},i_{b1},(l-i)_{b2},(N-l)_c\rangle. \qquad (19)$$

Note that the number of photons can never become smaller than m since all the atoms start out in the excited state. $|\mathcal{F}_l\rangle$ is a normalized state of the system with $m+l$ photons in total. To see that $|\mathcal{F}_l\rangle$ is properly normalized note that $\Sigma_{i=0}^{l}\binom{m+i}{m}=\binom{m+l+1}{l}$.

The states $|\mathcal{F}_l\rangle$ are formally identical to the states obtained in Ref. [9], which have been shown to realize optimal universal cloning and the optimal universal NOT simultaneously. The ideal universal NOT is an operation that produces the orthogonal complement of an arbitrary qubit. As with perfect cloning, it is prohibited by quantum mechanics. The transformation in [9] links universal cloning and universal NOT (anticloning): the ancilla qubits of the cloning transformation are the anticlones. In our case, the clones are the photons in the a modes and the anticlones are the atoms in the b modes (atomic ground states g_1 and g_2). From the Hamiltonian (14) and (19) it is clear that for every ''good'' emitted photon-clone (in mode a_1) there is an excitation in mode b_2 which corresponds to an anticlone (atomic ground state $|g_2\rangle$). The only difference to the states in Ref. [9] is the presence of the fifth harmonic oscillator mode c, describing the ''e-type'' excitations, which counts the total number of clones that have been produced (equal to the number of atoms having gone to one of the ground states) and does not affect any of the conclusions.

A distinguishing feature of our cloner is that the output state (19) is a superposition of states with different total numbers of clones. Cloning with a certain fixed number of produced copies can be realized by measuring the number of atoms in the excited state $|e\rangle$ (corresponding to mode c) and post-selection.

To see that the $|\mathcal{F}_l\rangle$ are indeed the output of an optimal cloner, let us calculate the fidelity of the cloning, given by the mean relative frequency of photons in the correct mode (a_1 in our case). In the state $|m+l-i,i\rangle_{\rm photons}$ the relative frequency of correct photons is $(m+l-i)/(m+l)$. Therefore

$$F_l=\binom{m+l+1}{l}^{-1}\sum_{i=0}^{l}\binom{m+l-i}{m}\frac{m+l-i}{m+l}=\frac{m(m+2)+l(m+1)}{(m+l)(m+2)} \qquad (20)$$

which corresponds to the fidelity of an optimal universal $m\to m+l$ cloner [5]. Note again that the universality in our case follows from the symmetry of the Hamiltonian and the initial atomic state.

To prove that the system is indeed always in a superposition of the states $|\mathcal{F}_l\rangle$ as in Eq. (19) we use induction. The initial state of the system is $|\Psi_{\rm in}\rangle=|\mathcal{F}_0\rangle$. Now we will show that if $|\Phi\rangle$ is a superposition of states $|\mathcal{F}_l\rangle$ then $\mathcal{H}|\Phi\rangle$ is so, too. Then, since $|\Psi(t)\rangle=e^{-i\mathcal{H}t}|\Psi_{\rm in}\rangle=\Sigma_p(-i\mathcal{H}t)^p/p!|\Psi_{\rm in}\rangle$ this implies that $|\Psi(t)\rangle$ will be a superposition of $|\mathcal{F}_l\rangle$. Explicit calculation shows that

$$\mathcal{H}|\mathcal{F}_i\rangle=\gamma\left(\sqrt{(l+1)(N-l)(m+l+2)}|\mathcal{F}_{l+1}\rangle+\sqrt{l(N-l+1)(m+l+1)}|\mathcal{F}_{l-1}\rangle\right),\quad 1\leq l<N,$$

$$\mathcal{H}|\mathcal{F}_0\rangle=\gamma\sqrt{N(m+2)}|\mathcal{F}_1\rangle, \qquad (21)$$

$$\mathcal{H}|\mathcal{F}_N\rangle=\gamma\sqrt{N(m+N+1)}|\mathcal{F}_{N-1}\rangle,$$

which completes the proof.

Note that the form of the coefficients $f_l(t)$ did not play any role in our proof. Actually, the f_l are in general hard to determine exactly. Solutions have been found in limiting cases. For the limit of a classical pump field (c replaced by a c number), the solution can be found by standard methods and is given in Ref. [8]. The solution in the case of large incoming photon numbers ($m\gg N$) is presented in Appendix B.

Let us pause here for a moment and summarize what we have found. Our system consisting of an ensemble of Λ atoms in the excited state is indeed equivalent to a superposition of optimal cloning machines in the manner of Bužek and Hillery or Gisin and Massar, producing various numbers of clones. The atoms play the double role of photon source and of ancilla, the atomic ground states can be identified with the ancilla states in the qubit cloners. As for the corresponding qubit cloners, those ancillary atoms can also be seen as the output of a universal NOT gate. On the other hand, the atoms that end up in the excited state provide information about the number of clones that has actually been produced. This can be used to realize cloning with a fixed number of output clones by postselection.

III. THE EQUIVALENCE BETWEEN PAIRS OF V ATOMS AND Λ ATOMS

In this section we present an alternative (but similar) way of realizing optimal universal cloning that uses entangled pairs of V atoms instead of Λ atoms. We prove optimality by showing that the system can be exactly mapped onto the system with Lambda atoms that we discussed above.

The two degenerate upper levels of each V atom, $|e_1\rangle$ and $|e_2\rangle$, are coupled to the ground state $|g\rangle$ via the two orthogonal modes a_1 and a_2, respectively. The Hamiltonian describing the interaction of atom and field is

$$\begin{aligned}\mathcal{H}_V&=\gamma\left(a_1^\dagger\sum_{k=1}^{N}|g^k\rangle\langle e_1^k|+a_2^\dagger\sum_{k=1}^{N}|g^k\rangle\langle e_2^k|\right)+\text{H.c.}\\&=\gamma\left(a_1^\dagger\sum_{k=1}^{N}\sigma_{-,1}^k+a_2^\dagger\sum_{k=1}^{N}\sigma_{-,2}^k\right)+\text{H.c.}\end{aligned}\tag{22}$$

It arises from similar assumptions as in Eq. (1). In contrast to before we now choose an entangled state of the atoms as the initial state. This is motivated by the fact that the initial atomic state has to be a *singlet* under polarization transformations in order for our cloning device to be again universal.

Let us first examine the simplest case of two entangled V atoms A and B, and one incoming photon. The initial state of the system is

$$|\Psi_{\text{in}}\rangle=\frac{1}{\sqrt{2}}(|e_1^A e_2^B\rangle-|e_2^A e_1^B\rangle)\otimes|1,0\rangle.\tag{23}$$

Developing the time evolution operator $e^{-i\mathcal{H}t}$ into a power series, one easily finds:

$$\begin{aligned}e^{-i\mathcal{H}t}|\Psi_{\text{in}}\rangle&=\cos(\gamma\sqrt{3}t)\frac{|e_1^A e_2^B\rangle-|e_2^A e_1^B\rangle}{\sqrt{2}}|1,0\rangle-i\sin(\gamma\sqrt{3}t)\\&\quad\times\left(\sqrt{\frac{2}{3}}\frac{|g^A e_2^B\rangle-|e_2^A g^B\rangle}{\sqrt{2}}|2,0\rangle\right.\\&\quad\left.+\sqrt{\frac{1}{3}}\frac{|e_1^A g^B\rangle-|g^A e_1^B\rangle}{\sqrt{2}}|1,1\rangle\right).\end{aligned}\tag{24}$$

With the substitution

$$\begin{aligned}&\frac{|e_1^A e_2^B\rangle-|e_2^A e_1^B\rangle}{\sqrt{2}}\to|\tilde{e}\rangle,\\&\frac{|g^A e_2^B\rangle-|e_2^A g^B\rangle}{\sqrt{2}}\to|\tilde{g}_1\rangle,\\&\frac{|e_1^A g^B\rangle-|g^A e_1^B\rangle}{\sqrt{2}}\to|\tilde{g}_2\rangle.\end{aligned}\tag{25}$$

the state (24) has exactly the same form as the corresponding state (5) for one Lambda-atom, which implies that it also implements optimal universal $1\to2$ cloning.

Actually, the correspondence goes much further. Consider an initial atomic state consisting of N pairs of V atoms, where each pair is in a singlet state:

$$|\psi_i\rangle=\otimes_{k=1}^{N}|\tilde{e}^k\rangle\tag{26}$$

with $|\tilde{e}\rangle$ as defined in Eq. (25).

It is easy to see that the action of the Hamiltonian (22) on each pair only generates one of the three antisymmetric atomic states in Eq. (25). Because of the invariance of the Hamiltonian under permutations, and in particular under the exchange of two atoms belonging to the same pair, transitions between states with different symmetry properties are impossible. In fact, with the identification (25) the Hamiltonian (22) has exactly the same form as the Hamiltonian for Λ atoms (1). The analysis made for Λ atoms in Sec. II now goes through unchanged and we obtain the same cloning properties of a system of pairwise entangled V atoms as we had before for Λ atoms, i.e., we have found another way of realizing optimal universal cloning. Although this scheme would without doubt be more difficult to realize experimentally, we believe that the underlying equivalence between the two systems is interesting and may be useful in other contexts as well.

IV. CLONING OF PHOTONS VERSUS CLONING OF QUBITS

In this section we are going to discuss the physical differences that exist in spite of the formal equivalence proven above between our photon cloners based on stimulated emission and the qubit cloners as usually considered [4,6]. In particular, we will show that the claim that optimal cloning is realized by our devices is justified in spite of these differences.

In most of the previous work cloning was discussed in terms of quantum networks. In general, the situation considered in these papers is the following: one has a certain number of qubits that are localized in different positions, which makes them perfectly distinguishable. At the beginning, some of those qubits are the systems to be cloned, the others play the role of ancillas. After the cloning procedure, which consists of several joint operations on the qubits that can be expressed in terms of quantum gates, some of the qubits are the clones, the rest are ancillas, which for a specific form of the optimal cloning transformation can also be seen as outputs of the universal NOT operation. As a consequence of localization, it is possible to address individual clones.

In our stimulated emission cloners, the situation is different. All input systems (photons) are in the same spatial mode (called mode a in this paper), and, even more importantly, all clones are produced into that mode. Note that this is completely unavoidable if stimulated emission is to be used. One can say that this is the price one has to pay for the great conceptual simplicity of the cloning procedure itself.

However, having all clones in the same spatial mode is not necessarily an important disadvantage. For example, if perfect cloning of that kind were possible, one could still determine the polarization of the original photon to arbitrary precision by performing measurements on the clones. This would still make superluminal communication possible [13]. If one wants to distribute the clones to different locations, this can for example be achieved using an array of beam splitters. However, this does not lead to a situation where one can be sure to have exactly one photon in each mode. If one wants to have at most one photon in each mode, the array has to have many more output modes than there are photons.

Another distinguishing feature of our cloners compared to the usual qubit cloners is the fact that the same procedure is used to produce different numbers of clones. While in the qubit case the network to be used depends on the number of desired clones, in our case the final state is a superposition of states with different numbers of clones. Of course, the average number of clones produced depends on the number of atoms present in the system and the interaction time. As discussed in Sec. II cloning with a fixed number of output clones can be achieved by post selection based on a measurement of the number of excited atoms in the final state.

The formal equivalence between the qubit cloners and our one-mode cloners can arise because the output state produced by the optimal qubit cloners is completely symmetric under the exchange of clones [4,6]. Because of the bosonic nature of the photons there is a one-to-one-mapping between completely symmetric qubit states and photonic states. For a completely symmetric qubit state the two concepts of relative frequency of qubits in the ''correct'' basis state and of single-particle fidelity are equivalent. This can be seen in the following way. Let $|\psi\rangle$ denote the state that is to be copied. Then the usual definition of the (single-particle) cloning fidelity is

$$F=\langle\psi|\rho_{\mathrm{red}}|\psi\rangle, \tag{27}$$

where ρ_{red} is the reduced density matrix of one of the clones, say the first one, i.e.,

$$\rho_{\mathrm{red}}=\mathrm{Tr}_{2,3,\ldots,N}[\rho]. \tag{28}$$

Then F can also be expressed as

$$F=\mathrm{Tr}[\rho|\psi\rangle\langle\psi|_1\otimes I_2\otimes\cdots\otimes I_N]. \tag{29}$$

On the other hand, the relative frequency of qubits in the state $|\psi\rangle$ can be written as

$$\frac{1}{N}\mathrm{Tr}[\rho(|\psi\rangle\langle\psi|_1\otimes I_2\otimes\cdots\otimes I_N+I_1\otimes|\psi\rangle\langle\psi|_2\otimes\cdots\otimes I_N+\cdots$$
$$+I_1\otimes\cdots\otimes|\psi\rangle\langle\psi|_N)]. \tag{30}$$

If ρ is invariant under exchange of any two clones, it is obvious that Eq. (30) is equal to Eq. (29), i.e., for symmetric cloners the two concepts are completely equivalent. This justifies our definition of fidelity via the relative frequency in the case of photon cloning (see Sec. II).

Let us finally address the issue of optimally in the context of stimulated emission cloners. In this paper we have shown the formal equivalence of our scheme and the optimal schemes for qubit cloning. As a consequence, the fidelity of the clones saturates the bounds derived for the cloning of qubits. However, it is not entirely obvious that the bounds derived for the situation of distinct well-localized qubits also apply to our situation. Could one maybe achieve even higher fidelity in our one-mode case? The following argument shows that the bounds indeed apply in our situation as well, i.e., that photon cloning is not allowed to be better than qubit cloning.

Let us assume that we had a single-mode cloning machine that clones photons with a better fidelity than given by the bounds for qubits. Consequently, the relative frequency of ''correct'' photons has to exceed the bound for at least one value of the final total photon number M. This is obvious if M has been fixed by postselection. Otherwise the fidelity has to be defined as the average of the relative frequencies over all final total photon numbers. This average can only exceed the bound for qubits if the bound is violated for at least one particular value M of the final photon number.

As a consequence, we have a universal map from the N-photon Hilbert space to the M-photon Hilbert space that achieves a relative frequency of correct photons in the final state that is higher than the qubit bound. But the existence of such a map is equivalent to the existence of a universal map from the totally symmetric N-qubit space to the totally symmetric M-qubit space with a single-particle fidelity equal to the relative frequency. The existence of the latter map is excluded by the theorems on cloning of qubits [7]. This justifies our claim that the schemes presented in the previous sections realize *optimal* cloning of photons.

V. CONCLUSIONS AND OUTLOOK

In this paper we have shown that optimal universal cloning can be realized via stimulated emission in three-level systems. The permutation symmetry of the interaction allowed to map our system onto bosonic modes independent of the number of atoms used. Furthermore, we have found an equivalence between single Λ atoms and entangled pairs of V systems, which might be fruitful in other contexts as well.

The connection between stimulated emission and optimal cloning is remarkable. Our results show that a task previously discussed in terms of rather complicated quantum networks can be realized in an elegant way using basic quantum systems and interactions. While it was clear from the beginning that perfect cloning is prohibited by fundamental principles, it is interesting to see how this impossibility arises in a concrete physical system. In our case, the physical process limiting the fidelity of the clones is spontaneous emission. It is fascinating that in this way spontaneous emission ensures that there cannot be any superluminal communication.

It might be interesting to investigate possible experimental realizations of our proposal, e.g., using a combination of cavity QED and Bose-Einstein condensates. This would potentially allow the creation of macroscopic numbers of clones.

Quantum cloners are often discussed in the context of eavesdropping in quantum cryptography. Currently all cryptography schemes rely on photons. Therefore devices based on the principles presented here could be useful to a future eavesdropper.

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APPENDIX A: SCHWINGER REPRESENTATION

As noted above, the action of the Hamiltonian (1) on the initial state (2) only generates completely symmetric states of the atomic system. These states have the general form

$$\binom{N}{i,j}^{-1/2}\sum_{\alpha}|g_1^{\alpha_1},g_1^{\alpha_2},\ldots,g_1^{\alpha_i},\, g_2^{\alpha_{i+1}},\ldots,g_2^{\alpha_{i+j}},e^{\alpha_{i+j+1}},\ldots,e^{\alpha_N}\rangle =:|i,j,N-i-j\rangle_{\text{atoms}}, \tag{A1}$$

where the sum is over all arrangements α of the N-i-j levels $|e\rangle$, the i levels $|g_1\rangle$, and the j levels $|g_2\rangle$ on the N atoms, and $\binom{N}{i,j}=N!/i!j!(N-i-j)!$ is the multinominal coefficient giving the number of such arrangements.

Now study the action of a typical term in the Hamiltonian (1) on the system whose state we will write as $|i,j,N-i-j\rangle_{\text{atoms}}\otimes|m+i,j\rangle_{\text{photons}}$:

$$\begin{aligned}
&\left(\sum_{k=1}^{N}|g_1^k\rangle\langle e^k|\right)a_1^\dagger|i,j,N-i-j\rangle_{\text{atoms}}\otimes|m+i,j\rangle_{\text{photons}}\\
&=\sum_{k=1}^{N}|g_1^k\rangle\langle e^k|\sqrt{\frac{i!j!(N-i-j)!}{N!}}\sum_{\alpha}|g_1^{\alpha_1},\ldots,g_1^{\alpha_i},g_2^{\alpha_{i+1}},\ldots,e^{\alpha_N}\rangle\otimes a_1^\dagger|m+i,j\rangle_{\text{field}}\\
&=(i+1)\sqrt{\frac{i!j!(N-i-j)!}{N!}}\sum_{\alpha}|g_1^{\alpha_1},\ldots,g_1^{\alpha_i},g_1^{\alpha_{i+1}},g_2^{\alpha_{i+2}},\ldots,e^{\alpha_N}\rangle\otimes a_1^\dagger|m+i,j\rangle_{\text{field}}\\
&=\sqrt{i+1}\sqrt{N-i-j}\sqrt{\frac{(i+1)!j!(N-i-j-1)!}{N!}}\sum_{\alpha}|g_1^{\alpha_1},\ldots,g_1^{\alpha_i},g_1^{\alpha_{i+1}},g_2^{\alpha_{i+2}},\ldots,e^{\alpha_N}\rangle\otimes a_1^\dagger|m+i,j\rangle_{\text{field}}\\
&=\sqrt{i+1}\sqrt{N-i-j}|i+1,j,N-i-j-1\rangle_{\text{atoms}}\otimes a_1^\dagger|m+i,j\rangle_{\text{photons}}.
\end{aligned}\tag{A2}$$

Here the factor $(i+1)$ arises from the number of different configurations that a given arrangement α can be reached by. This shows that this term acts exactly as a term $a_1^\dagger b_1^\dagger c$. Similar calculations can be made for the other terms in the Hamiltonian. Together, they justify the Schwinger representation (11).

APPENDIX B: LIMIT OF LARGE PHOTON NUMBER

Here we determine the coefficients $f_l(t)$ of Eq. (19) in the limit of large m ($m\gg N$, many incoming photons, small number of atoms). For that case, the recursion (21) becomes

$$\begin{aligned}
\mathcal{H}|\mathcal{F}_l\rangle&=\gamma\sqrt{m}\big(\sqrt{(l+1)(N-l)}|\mathcal{F}_{l+1}\rangle\\
&\quad+\sqrt{l(N-l+1)}|\mathcal{F}_{l-1}\rangle\big),\quad 1\leqslant l<N,\\
\mathcal{H}|\mathcal{F}_0\rangle&=\gamma\sqrt{m}\sqrt{N}|\mathcal{F}_1\rangle,\\
\mathcal{H}|\mathcal{F}_N\rangle&=\gamma\sqrt{m}\sqrt{N}|\mathcal{F}_{N-1}\rangle.
\end{aligned}\tag{B1}$$

It is possible to diagonalize the ''transfer'' matrix A acting on the vector $(f_0,\ldots,f_N)$ that corresponds to the action of $\mathcal{H}$ on $|\Psi\rangle=\Sigma_{l=0}^N f_l|\mathcal{F}_l\rangle$: $A_{l,l+1}=\gamma\sqrt{m}\sqrt{(l+1)(N-l)}=A_{l+1,l}$. This allows to exponentiate A and to determine the final state of the system after a time t:

$$|\Psi(t)\rangle=\sum_{l=0}^{N}(-i)^l\sqrt{\binom{N}{l}}\cos^{N-l}(\gamma\sqrt{m}t)\sin^l(\gamma\sqrt{m}t)|\mathcal{F}_l\rangle. \tag{B2}$$

Differentiating Eq. (B2) and using Eq. (B1) one can show that this state fulfills Schrödinger's equation with the correct initial condition.

In this big-m-limit the probability to observe the system as an $m\rightarrow m+l$ cloner (i.e., the probability that l additional photons are emitted) is

$$p(l)=\binom{N}{l}\cos^{2(N-l)}(\gamma\sqrt{m}t)\sin^{2l}(\gamma\sqrt{m}t). \tag{B3}$$

This is a binomial distribution with a probability $\sin^2(\gamma\sqrt{m}t)$ for each atom to emit a photon. Setting $N=1$ or comparison with Eq. (5) shows that this is identical to the probability for the case of only one atom in the case of large m. This means that in this limit each atom interacts independently with the electromagnetic field, because the effect of the other atoms on the field is negligible. In the short-time limit $p(l)=O(t^{2l})$. Furthermore the expected average number of "clones" $N_c=\Sigma_{l=0}^N lp(l)=N\sin^2(\gamma\sqrt{m}t)$ oscillates with an m-dependent frequency.

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[12] This leads to the question what the spatial wave function could be. The most fascinating possibility is probably to imagine a BEC inside a cavity.
[13] It may be interesting to note that in the paper that started the whole discussion about quantum cloning, Herbert [3] considered cloning via stimulated emission and therefore necessarily into a single spatial mode.
[14] See, e.g., J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, New York, 1994), Sec. 3.8.