Bell theorem for the nonclassical part of the quantum teleportation process

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(Received 8 December 1999; published 3 August 2000)

The quantum teleportation process is composed of a joint measurement performed upon two uncorrelated subsystems A and B, followed by a unitary transformation (parameters of which depend on the outcome of the measurement) performed upon a third subsystem C (EPR correlated with system B). The information about the outcome of the measurement is transferred by classical means. It is shown that this measurement process, plus possible measurements on subsystem C (with the classical channel switched off), cannot be described by a local realistic theory.

PACS number(s): 03.65.Bz, 42.50.Dv

Quantum teleportation [1] is the operational protocol which enables one to transfer the quantum state of one system, say A, to another quantum system, C. The transfer can be obtained by performing a joint (''Bell-state'') measurement on A and a third system B, originally EPR entangled with C, and then unitarily transforming C according to the outcome of this measurement. Teleportation separates the complete information in A into two parts: a *classical* part carried by the outcome c of the joint measurement on A and B, and a *nonclassical* part carried by the prior entanglement between B and C.

Teleportation is strongly related to other effects, like interferometric tests of against local realism involving independent sources of particles [2], especially entanglement swapping [3]. In entanglement swapping, the particle A of the quantum teleportation protocol is originally entangled with some particle D. If, as in the case of quantum teleportation, a full Bell-state measurement is performed on A and B, and depending on the outcome, after a classical transfer of information a suitable unitary transformation is performed upon C, the particles D and C are in an entangled state. Thus entanglement swapping can be interpreted as *teleportation of entanglement* (from A to C). The final state of D and C can be used in an experiment in which Bell inequalities are violated. Since the classical information on the outcome of the Bell measurement is needed on one side only (in the present example, in the vicinity of particle C), the measurement acts on D and C can easily satisfy the necessary requirement for a Bell inequality test, namely, that of spatial separation. One may arrange the experiment in such a way that no classical information on the result of the Bell measurement upon A and B can reach D before the local measurement on D, in the Bell inequality test, is done. In such a case, the teleportation process can be treated as just a more involved scheme of the preparation of the entangled state of D and C. This suggests that there must be at least an element in the teleportation procedure which defies local and realistic interpretation.

The problem of the link or lack of link between the violations of local realism and the teleportation process was addressed by many authors [4]. In this work the following aspect of the problem will be discussed. As mentioned before, the teleportation process has its quantum and classical parts. The classical part involves communication via standard classical methods, and thus cannot be suspected of adding anything interesting to the relation of the teleportation process to the Bell theorem (except for the case of entanglement swapping, as discussed above). Even worse, the classical transfer of information from the Bell-state-measuring station (operated by Alice) to particle C makes it possible that the measurements upon C (after the full teleportation protocol) can be causally linked with the events at Alice's apparatus. Thus a Bell-type analysis is absolutely excluded. Nevertheless, as will be argued below, the quantum part of the process cannot be described by a local realistic formalism.

First one should define what is meant here by the quantum part of the process. Assume that the classical information link between Alice and Bob, which precludes a Bell-type analysis, is cut. However, both parties are still allowed to perform the usual laboratory tasks for an experiment toward verification of the actuality of the teleportation process [5]. That is, Alice herself (or, for purists, this can be done by her friend Cecil) can prepare particle A in any pure state, and subsequently she can make a Bell measurement on A and B. Bob, not knowing the result on Alice's side, nor the original state of A, instead of being totally idle, performs on particle C a measurement of a (generally randomly chosen) yes-no observable.

The formal description of the above runs as follows (as in Ref. [1] we assume all particles involved to be two-state systems). (i) The initial three particle state is

$$(\sin\beta|A_1\rangle+\cos\beta e^{-i\phi}|A_2\rangle)\sqrt{\tfrac{1}{2}}(|B_1\rangle|C_1\rangle+|B_2\rangle|C_2\rangle), \tag{1}$$

where A_i, B_i, and C_i denote the states of the three subsystems (the letter stands for the subsystem (particle), and $i=1$ and 2 is the index of two orthogonal states). The parameters ϕ and β are determined by the state preparation procedure of Cecil.

Alice performs a measurement which collapses the A-B system into the four Bell states:

$$\sqrt{\tfrac{1}{2}}(|B_1\rangle|A_1\rangle-|B_2\rangle|A_2\rangle)=|\mathbf{00}\rangle, \tag{2}$$

$$\sqrt{\tfrac{1}{2}}(|B_1\rangle|A_2\rangle+|B_2\rangle|A_1\rangle)=|\mathbf{01}\rangle, \tag{3}$$

$$\sqrt{\tfrac{1}{2}}(|B_1\rangle|A_2\rangle-|B_2\rangle|A_1\rangle)=|\mathbf{10}\rangle, \tag{4}$$

$$\sqrt{\tfrac{1}{2}}(|B_1\rangle|A_1\rangle+|B_2\rangle|A_2\rangle)=|\mathbf{11}\rangle. \tag{5}$$

Note that the names of the states are binary expansions of 0,1,2, and 3. They could be the content of the classical messages of Alice, informing Bob about her results (however, this link is cut). Alice's measurement projects particle C into certain four states. Bob, cut off from Alice, in desperation performs an experiment of a dichotomic (yes-no) nature, which results in the projections into the two following orthogonal states:

$$\cos\beta'|C_1\rangle+\sin\beta'\exp(i\phi')|C_2\rangle=|0\rangle \tag{6}$$

and

$$-\sin\beta'|C_1\rangle+\cos\beta'\exp(i\phi')|C2\rangle=|1\rangle. \tag{7}$$

The probabilities of all possible eight global results (two results of Bob times four results of Alice), are

$$\begin{aligned}P(00,0)&=1/4-P(00,1)\\&=\tfrac{1}{8}[1+\cos 2\beta\cos 2\beta'\\&\quad-\sin 2\beta\sin 2\beta'\cos(\phi-\phi')],\end{aligned} \tag{8}$$

$$\begin{aligned}P(01,0)&=1/4-P(01,1)\\&=\tfrac{1}{8}[1-\cos 2\beta\cos 2\beta'\\&\quad+\sin 2\beta\sin 2\beta'\cos(\phi+\phi')],\end{aligned} \tag{9}$$

$$\begin{aligned}P(10,0)&=1/4-P(10,1)\\&=\tfrac{1}{8}[1-\cos 2\beta\cos 2\beta'\\&\quad-\sin 2\beta\sin 2\beta'\cos(\phi+\phi')],\end{aligned} \tag{10}$$

$$\begin{aligned}P(11,0)&=1/4-P(11,1)\\&=\tfrac{1}{8}[1+\cos 2\beta\cos 2\beta'\\&\quad+\sin 2\beta\sin 2\beta'\cos(\phi-\phi')].\end{aligned} \tag{11}$$

Let us assign to the four possible results of Alice's measurement, 00,01,10, and 11: four two-dimensional vectors (for some other nonconventional value assignments for experimental results, see Ref. [6])

$$\vec{A}(00)=(-1,-1),\quad \vec{A}(01)=(-1,1), \tag{12}$$
$$\vec{A}(10)=(1,-1),\quad \vec{A}(11)=(1,1).$$

The link between the vectors and binary numbers is obvious. The digit 0 has been replaced by -1. This trick makes the subsequent derivation of a Bell inequality much easier. Please note that this procedure differs from the usual one (i.e., the assignment of certain real numbers, ''eigenvalues,'' to certain projectors) by the fact that we ascribe more complicated objects to the projectors. The results of Bob's measurements 0 and 1 will be described is a similar fashion, namely, by numbers $I_B(0)=-1$ and $I_B(1)=1$.

To simplify the description of the global measurement results, one can introduce a suitably defined correlation function. One can consider such a function as the average of products of the results on each side (here vectors times numbers). That is, the result (00,0), i.e., a detection of the first Bell state, 00, by Alice, and simultaneous detection of state 0 by Bob, can be ascribed $-1(-1,-1)=(1,1)$, etc. With such definitions of the values assigned to the possible pairs of the outcomes, the correlation function

$$E(\beta,\phi;\beta',\phi')=\sum_{c=00}^{11}\sum_{i=0,1} I_B(i)\vec{A}(c)P(c,i) \tag{13}$$

acquires the form of a two-dimensional vector, and for the explicit form of the quantum prediction reads

$$\begin{aligned}&E(\beta,\phi;\beta',\phi')_{QM}\\&\quad=\sin 2\beta\sin 2\beta'(\cos\phi\cos\phi',\sin\phi\sin\phi').\end{aligned} \tag{14}$$

Let us now simplify the problem a bit. Assume that Alice prepares states of A with $\beta=45°$, and Bob fixes his apparatus at $\beta'=45°$ as well, The correlation function is then simplified to

$$E(\phi;\phi')_{QM}=(\cos\phi\cos\phi',\sin\phi\sin\phi'). \tag{15}$$

It will be shown that this correlation function cannot be modeled by local hidden variable theories.

Imagine that a hidden variable λ specifies the future results of the experiments of Alice and Bob. The product of such predictions reads

$$I_B(\phi',\lambda)\vec{A}(\phi,\lambda), \tag{16}$$

where $I_B(\phi',\lambda)=\pm 1$ is the local hidden variable (LHV) prediction for the result of the measurement by Bob (for the given value of the hidden parameter λ, and the local observable defined by ϕ') and the vector $\vec{A}(\phi,\lambda)$, which is the LHV prediction for the Alice's result, depends on λ and ϕ, and takes one of the four values of Eqs. (12). The local hidden variable prediction for the correlation function is an average of Eq. (16) over a certain (properly normalized) distribution $\rho(\lambda)$, namely,

$$E(\phi;\phi')_{LHV}=\int d\lambda\,\rho(\lambda)I_B(\phi',\lambda)\vec{A}(\phi,\lambda). \tag{17}$$

Now let us assume that Alice can set the values of phase ϕ, which prepares the state of particle A, at 0° or 90°, whereas Bob can play with ϕ' at $-45°$ and $+45°$.

To show that $E(\phi;\phi')_{QM}$ cannot be modeled by $E(\phi;\phi')_{LHV}$, the geometric approach of Ref. [7] will be used. It is based on the following simple observation. Assume that one knows the components of a certain vector $\mathbf{q}$ (the *known* vector) belonging to some vector space, whereas for a second vector $\mathbf{h}$ (the *test* vector) one is only able to establish that its scalar product with q satisfies the inequality

$\langle \mathbf{h}|\mathbf{q}\rangle < ||\mathbf{q}||^2$. The immediate implication is that these two vectors cannot be equal: $\mathbf{q}\neq \mathbf{h}$.

To form a vector for such an argument, one can take the values of the quantum correlation function at the $2\times 2=4$ pairs of the possible settings of the macroscopic parameters controlled by Alice and Bob (ϕ,ϕ'). In this way a "supervector" if $V^{\mathbf{QM}}$ is built. The first component of the supervector, for the settings $(0°,-45°)$, reads

$$\vec{V}_1^{QM} = E(0;-45)_{QM} = (\sqrt{1/2},0), \tag{18}$$

the second, for (45,90), reads

$$\vec{V}_2^{QM} = E(45;90)_{QM} = (\sqrt{1/2},0), \tag{19}$$

the third, for $(90,-45)$, reads

$$\vec{V}_3^{QM} = E(90;-45)_{QM} = (0,-\sqrt{1/2}), \tag{20}$$

and the fourth, for (90,45), reads

$$\vec{V}_4^{QM} = E(90;45)_{QM} = (0,\sqrt{1/2}). \tag{21}$$

The square of the norm of such a supervector $\|V^{\mathbf{QM}}\|^2$ can be defined as the sum of the squares of the norms of all the components, where the square of the norm of a component is in turn the sum of the squares of its two components. Therefore, one has

$$\|V^{\mathbf{QM}}\|^2 = \sum_{i=1}^{4} |\vec{V}_i^{QM}|^2 = 2. \tag{22}$$

Let us estimate the scalar product of the quantum supervector with analogous supervector $V^{\mathbf{HV}}$, which has the structure characteristic of (deterministic) local hidden variables. The aforementioned scalar product is defined in a way compatible with the norm (i.e. it is a sum of the products of the respective components, and the product of two components is again the sum of the products of the respective elements of the components):

$$(V^{\mathbf{QM}},V^{\mathbf{LHV}}) = \sum_{i=1}^{4} \vec{V}_i^{QM}\cdot \vec{V}_i^{LHV}, \tag{23}$$

with $\vec{V}_i^{LHV}$ equal to the value of $E(\phi,\phi')_{LHV}$ for appropriate pairs of settings. As is usual in proofs of the Bell theorem, it is better first to consider the hidden variable prediction for a single specified λ, and only later to average this over the distribution of the hidden variables.

Thus, what we should do [7] is to estimate the scalar product of a supervector constructed out of hidden-variable predictions for the specified λ with the quantum supervector (defined above). The hidden variable supervector for a specific λ, which will be denoted by $H(\lambda)$, has the following components:

$$H(\lambda)_1 = I_B(-45,\lambda)(I_A(0,\lambda)_1, I_A(0,\lambda)_2), \tag{24}$$

$$H(\lambda)_2 = I_B(45,\lambda)(I_A(0,\lambda)_1, I_A(0,\lambda)_2), \tag{25}$$

$$H(\lambda)_3 = I_B(-45,\lambda)(I_A(90,\lambda)_1, I_A(90,\lambda)_2), \tag{26}$$

$$H(\lambda)_4 = I_B(1,45,\lambda)(I_A(90,\lambda)_1, I_A(90,\lambda)_2). \tag{27}$$

For the scalar product $V^{QM},H(\lambda)$, since $I_B(\phi',\lambda)=\pm 1$ and $I_A(\phi,\lambda)_i=\pm 1$, one obtains

$$\begin{aligned} -2\sqrt{1/2} &\leqslant (V^{\mathbf{QM}},H(\lambda)) \\ &= \sqrt{1/2}\{I_A(0,\lambda)_1[I_B(-45,\lambda)+I_B(45,\lambda)] \\ &\quad + I_A(90,\lambda)_2[I_B(45,\lambda)-I_B(-45,\lambda)]\} \\ &\leqslant 2\sqrt{1/2}. \end{aligned} \tag{28}$$

Thus, if one now averages this inequality over the distribution of the hidden variables $\rho(\lambda)$, the following relation emerges:

$$-\sqrt{2} \leqslant (V^{\mathbf{LHV}},V^{\mathbf{QM}}) \leqslant \sqrt{2} < \|V^{\mathbf{QM}}\|^2 = 2. \tag{29}$$

This implies simply that $V^{\mathbf{LHV}}\neq V^{\mathbf{QM}}$, that is, *no local hidden variable correlation function can reproduce the quantum prediction* (we have a Bell theorem for the process). Note that the appropriate Bell inequality is given here by the first two inequalities in Eq. (29).

This method can still be expanded to cover many more settings of the variables; here only the simplest case was presented. It is an interesting fact, that needs further investigation, that the Bell inequality presented here is violated by the same factor $\sqrt{2}$ as the CHSH inequality for the usual Bell theorem involving a pair of particles in a maximally entangled state. This may imply that the quantum component of the teleportation process cannot be described in a local and realistic way, as long as the initial states of B and C admit no such models.

The present result also explains why the current local hidden variable model explaining the low detection efficiency teleportation [8] cannot be extended into the high efficiency case. Simply, had this been possible, such a model would constitute a LHV model of the process considered here, which by Eq. (29), is impossible. For the same reason, considerations with toy models, like those in Ref. [9], cannot be extended in such a way that they can fully reproduce the quantum teleportation. Nevertheless, the conclusions reached in Ref. [10], that one can model the teleportation process with specific local hidden variables and a classical communication channel, requiring the transfer of 2.19 bits on average, are not in disagreement with the present result.

Inequality (29) can also serve as a Bell-type inequality for the experiment of Boschi *et al* [11]. In this experiment only two systems (photons) were used. System A was replaced by the polarization degree of freedom of one of the photons of the EPR entangled pair. The EPR entanglement itself was realized by a path entanglement of the two photons. In this way a measurement discriminating between the four correlated states of polarization and momentum direction of a photon, which are formally equivalent to Eqs. (2)–(5), can be performed with standard quantum interferometric techniques. Thus all observables involved in the present scheme found their representation in the experiment. However, due to the measurement settings chosen, one cannot directly apply inequality (29). Nevertheless the very high visibility of the two-particle fringes obtained is well above the threshold (71%) indicated by Eq. (29). This indirectly rules out a LHV model for the experiment (if one accepts the usual fair sampling assumption).

In the teleportation experiment involving all three particles (with A emitted independently of the emission of the EPR pair B and C) [12], due to fundamental technical limitations one currently cannot distinguish between all four states [Eqs. (2)–(5)]. Thus the inequality cannot be applied. However, the extension of the experiment to the teleportation of entanglement, i.e. entanglement swapping, [13,14], results in entangling previously independent photons, on which in turn a Bell-type experiment can be performed. Such an experiment is possible on a subensemble of events for which only one of the states of Eqs. (2)–(5) is measured, i.e., the process does not need a full Bell-state measurement to be undescribable by LHV theories. Unfortunately, the visibility in Ref. [13] was around 65%, i.e., within a zone for which one can build explicit LHV models [15]. Thus a high visibility realization of entanglement swapping would constitute an important fact in the empirical knowledge on the nature of quantum teleportation.

The presented results cannot be applied directly to a teleportation experiment involving continuous variables of Ref. [16]. However, perhaps the result of Ref. [17], concerning the Bell theorem for the original EPR state, may, after some extensions, lead to the same conclusion.

The author acknowledges support from the University of Gdańsk, Grants No. BW-5400-5-0062-5 (1995) and No. BW/5400-5-0264-9 (1999). The author thanks Anton Zeilinger, Harald Weinfurter, and the Horodecki family for discussions.

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