Optimal quantum teleportation with an arbitrary pure state

Konrad Banaszek
Center for Quantum Information and Rochester Theory Center for Optical Science and Engineering, University of Rochester, Rochester, New York 14627
and Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, PL-00-681 Warszawa, Poland
(Received 29 February 2000; published 13 July 2000)

We present an elementary derivation of the maximum fidelity attainable in teleportation using a single copy of two d-level systems in an arbitrary pure state. This derivation provides a complete set of necessary and sufficient conditions for optimal teleportation protocols. We also discuss the information on the teleported particle that is revealed in course of the protocol using a nonmaximally entangled state.

PACS number(s): 03.67.−a, 03.65.Bz

Entanglement is a key ingredient of quantum techniques for information processing. One of the striking consequences of quantum entanglement is the existence of the procedure called quantum teleportation [1]. This procedure allows two distant parties, traditionally called Alice and Bob, to transmit faithfully the quantum state of a particle. The resources needed for this purpose is a pair of particles in a maximally entangled state shared by Alice and Bob, and the possibility to transmit classical messages from Alice to Bob. The teleportation procedure is an extremely useful tool for understanding many properties of quantum entanglement [2].

An important aspect of quantum-information theory is the characterization of the entanglement exhibited by general quantum states of bipartite systems, and the evaluation of their capability to perform various quantum-information processing tasks. In this paper, we consider the following problem. Suppose that Alice wants to teleport to Bob an unknown pure state $|\psi\rangle_1$ of a d-level particle. Alice and Bob share a single pair of d-level particles in a pure state $|t\rangle_{23}$. What is the maximum fidelity of teleportation using such a state, and what conditions have to be satisfied by a teleportation protocol to achieve this limit?

The first of these questions has been recently answered in [3] using rather intricate reasoning. The argument was based on the analysis of approximate transformations of bipartite states. This analysis, employing the concept of entanglement monotones [4], yielded in particular the singlet fraction for a partially entangled pure state. This result was subsequently combined with the earlier work of Horodecki, Horodecki, and Horodecki [5], who derived a simple algebraic link between the singlet fraction and the maximum teleportation fidelity. They also described an optimal protocol involving the so-called twirling operation followed by the standard teleportation procedure.

In this paper, we present an elementary derivation of the maximum fidelity for teleportation with an arbitrary pure state. This derivation provides also a complete set of necessary and sufficient conditions for a given protocol to be optimal. Furthermore, full characterization of optimal protocols allows us to point out an interesting issue of the information balance in teleportation. Of course, use of a nonmaximally entangled state makes the teleportation procedure imperfect. Nevertheless, we demonstrate that one can find a silver lining in such a case: namely, that the teleportation procedure reveals some information on the teleported quantum state. This information can be converted into an estimate of the quantum state of the particle initially possessed by Alice. We derive here an upper bound for the mean estimation fidelity [6], and provide an explicit recipe for constructing the quantum state estimate that saturates this bound.

In order to optimize the teleportation procedure, we shall consider a general strategy consisting of an arbitrary measurement performed on Alice's side, followed by a general transformation of Bob's particle. In the most general case, Alice's measurement is described by a certain positive operator-valued measure. Such a measure can be decomposed into rank one operators, which are represented by projections on not necessarily normalized states $|\Phi_r\rangle_{12}\langle\Phi_r|$, where the index r runs over all possible outcomes of Alice's measurement. The unnormalized state vector of the particle owned by Bob, after Alice has measured the outcome r, is given by

$$|b_r\rangle_3 = {}_{12}\langle\Phi_r|(|\psi\rangle_1 \otimes |t\rangle_{23}). \tag{1}$$

After having received from Alice the outcome of her measurement, Bob performs a general transformation of his particle, described by

$$|b_r\rangle_3\langle b_r| \rightarrow \sum_s \hat{B}_{rs}|b_r\rangle_3\langle b_r|\hat{B}_{rs}^\dagger, \tag{2}$$

where the operators $\hat{B}_{rs}$ satisfy $\Sigma_s \hat{B}_{rs}^\dagger \hat{B}_{rs} = \hat{\mathbb{1}}$ for each r. In order to simplify the notation, we shall not write explicitly the range of the parameter s, which can be different for various values of r.

We shall quantify the quality of teleportation with the help of the mean fidelity. The probability that Alice obtains from her measurement the outcome r is given by the scalar product ${}_3\langle b_r|b_r\rangle_3$. The normalized state held by the Bob in this case is $|b_r\rangle_3/\sqrt{{}_3\langle b_r|b_r\rangle_3}$. After the transformation of this state described by Eq. (2), its overlap with the original state vector $|\psi\rangle$ is given by $\Sigma_s|{}_3\langle\psi|\hat{B}_{rs}|b_r\rangle_3|^2/{}_3\langle b_r|b_r\rangle_3$. Summation of this expression over r with the weights ${}_3\langle b_r|b_r\rangle_3$, and integration over all possible input states $|\psi\rangle$, yields the complete expression for the mean fidelity

$$\overline{f} = \int d\psi \sum_{rs} |({}_{12}\langle\Phi_r| \otimes {}_3\langle\psi|)\hat{B}_{rs}(|\psi\rangle_1 \otimes |t\rangle_{23})|^2, \tag{3}$$

where the integral $\int d\psi$ over the space of pure states is performed using the canonical measure invariant with respect to unitary transformations of the states $|\psi\rangle$.

Let us now select in the Hilbert spaces of the particles 2 and 3 the orthonormal bases defined by the Schmidt decomposition of the shared state $|t\rangle_{23}$:

$$|t\rangle_{23}=\sum_{k=0}^{d-1}\lambda_k|k\rangle_2\otimes|k\rangle_3, \tag{4}$$

where the nonnegative real Schmidt coefficients are put in decreasing order: $\lambda_0\geqslant\lambda_1\geqslant\cdots\geqslant\lambda_{d-1}\geqslant 0$. Using this basis, we may write each of the vectors $|\Phi_r\rangle_{12}$ as

$$|\Phi_r\rangle_{12}=\sum_{k=0}^{d-1}|\phi_r^k\rangle_1\otimes|k\rangle_2, \tag{5}$$

where the vectors $|\phi_r^k\rangle_1$ are not necessarily normalized. Applying this representation, the expression for the mean fidelity takes the form

$$\overline{f}=\int d\psi\sum_{rs}\left|\sum_{k=0}^{d-1}\lambda_k\langle\psi|\hat{B}_{rs}|k\rangle\langle\phi_r^k|\psi\rangle\right|^2, \tag{6}$$

where all scalar products are now taken in a single-particle Hilbert space. This allows us to drop the indexes labeling the particles. The fact that the operators $|\Phi_r\rangle_{12}\langle\Phi_r|$ form a decomposition of unity implies the following conditions on $|\phi_r^k\rangle$:

$$\sum_r|\phi_r^k\rangle\langle\phi_r^l|=\delta_{kl}\hat{\mathbb{1}}. \tag{7}$$

Our task is now to optimize the expression for the mean fidelity $\overline{f}$ over all possible measurements on Alice's side, and transformations performed by Bob.

We shall start by deriving an upper bound on the mean fidelity of teleportation using the state $|t\rangle_{23}$. For this purpose, let us define the vectors $|u_r^k\rangle$ such that

$$\sum_{k=0}^{d-1}\lambda_k|k\rangle\langle\phi_r^k|=\sum_{k=0}^{d-1}|u_r^k\rangle\langle k|. \tag{8}$$

The vectors $|u_r^k\rangle$ are uniquely defined by decomposing $\langle\phi_r^k|$ in the basis $\langle k|$ and collecting all the terms multiplying each of the $\langle k|$'s. The mean fidelity can be now represented as

$$\overline{f}=\sum_{rs}\int d\psi\left|\sum_{k=0}^{d-1}\langle\psi|\hat{B}_{rs}|u_r^k\rangle\langle k|\psi\rangle\right|^2$$
$$=\sum_{rs}\sum_{k,l=0}^{d-1}\langle u_r^k|\hat{B}_{rs}^\dagger\hat{M}_{kl}\hat{B}_{rs}|u_r^l\rangle, \tag{9}$$

where the operators $\hat{M}_{ij}$ are given by the following integrals over the space of pure states $|\psi\rangle$:

$$\hat{M}_{kl}=\int d\psi\langle\psi|k\rangle\langle l|\psi\rangle|\psi\rangle\langle\psi|=\frac{1}{d(d+1)}(\delta_{kl}\hat{\mathbb{1}}+|k\rangle\langle l|). \tag{10}$$

The second explicit form of $\hat{M}_{kl}$ is derived in the Appendix. Inserting this representation for the operators $\hat{M}_{kl}$ into Eq. (9), we can reduce the expression for $\overline{f}$ to the form

$$\overline{f}=\frac{1}{d(d+1)}\sum_r\left(\sum_{k=0}^{d-1}\langle u_r^k|u_r^k\rangle+\sum_s\left|\sum_{k=0}^{d-1}\langle k|\hat{B}_{rs}|u_r^k\rangle\right|^2\right). \tag{11}$$

The first sum over k can be transformed with the help of Eq. (8) multiplied by its Hermitian conjugate

$$\sum_{k=0}^{d-1}\langle u_r^k|u_r^k\rangle=\mathrm{Tr}\left(\sum_{k,l=0}^{d-1}|k\rangle\langle u_r^k|u_r^l\rangle\langle l|\right)$$
$$=\mathrm{Tr}\left(\sum_{k,l=0}^{d-1}\lambda_k\lambda_l|\phi_r^k\rangle\langle k|l\rangle\langle\phi_r^l|\right)$$
$$=\sum_k\lambda_k^2\langle\phi_r^k|\phi_r^k\rangle. \tag{12}$$

Furthermore, with the help of the same identity given in Eq. (8), we may convert the expression in the squared modulus in Eq. (11) to the form

$$\sum_{k=0}^{d-1}\langle k|\hat{B}_{rs}|u_r^k\rangle=\mathrm{Tr}\left(\hat{B}_{rs}\sum_{k=0}^{d-1}|u_r^k\rangle\langle k|\right)$$
$$=\mathrm{Tr}\left(\hat{B}_{rs}\sum_{k=0}^{d-1}\lambda_k|k\rangle\langle\phi_r^k|\right)$$
$$=\sum_{k=0}^{d-1}\lambda_k\langle\phi_r^k|\hat{B}_{rs}|k\rangle. \tag{13}$$

Thus we have

$$\overline{f}=\frac{1}{d(d+1)}\left(\sum_r\sum_{k=0}^{d-1}\lambda_k^2\langle\phi_r^k|\phi_r^k\rangle+\sum_{rs}\left|\sum_{k=0}^{d-1}\lambda_k\langle\phi_r^k|\hat{B}_{rs}|k\rangle\right|^2\right). \tag{14}$$

The first term in the above expression can be easily calculated using the condition defined in Eq. (7), which implies that

$$\sum_r\langle\phi_r^k|\phi_r^k\rangle=\mathrm{Tr}\left(\sum_r|\phi_r^k\rangle\langle\phi_r^k|\right)=\mathrm{Tr}\,\hat{\mathbb{1}}=d, \tag{15}$$

and consequently the double sum over r and k yields $d\sum_{k=0}^{d-1}\lambda_k^2=d$. In order to estimate the second term in Eq. (14), we will use the inequality

$$\sum_{\alpha=1}^{N}\left|\sum_{k=1}^{M}x_{k\alpha}\right|^2\leqslant\left(\sum_{k=1}^{M}\sqrt{\sum_{\alpha=1}^{N}|x_{k\alpha}|^2}\right)^2 \tag{16}$$

valid for arbitrary complex numbers $x_{k\alpha}$. This is simply the triangle inequality for M complex N-dimensional vectors $\mathbf{x}_k=(x_{k1},\ldots,x_{kN})$ with the standard quadratic norm $||\mathbf{x}_k||^2=\sum_{\alpha=1}^{N}|x_{k\alpha}|^2$. When all the vectors $\mathbf{x}_k\neq\mathbf{0}$, the equality sign in Eq. (16) holds if and only if there exist M strictly positive numbers $a_1,\ldots,a_M$ such that $a_l\mathbf{x}_k=a_k\mathbf{x}_l$ for every pair k,l.

With the help of the triangle inequality, we can easily find an upper bound for the sum over rs in Eq. (14):

$$\sum_{rs}\left|\sum_{k=0}^{d-1}\lambda_k\langle\phi_r^k|\hat{B}_{rs}|k\rangle\right|^2\leqslant\left(\sum_{k=0}^{d-1}\lambda_k\sqrt{\sum_{rs}|\langle\phi_r^k|\hat{B}_{rs}|k\rangle|^2}\right)^2. \tag{17}$$

The sum over rs under square root in the above expression can be estimated by

$$\begin{aligned}\sum_{rs}|\langle\phi_r^k|\hat{B}_{rs}|k\rangle|^2&=\sum_r\langle\phi_r^k|\phi_r^k\rangle\sum_s\langle k|\hat{B}_{rs}^\dagger\frac{|\phi_r^k\rangle\langle\phi_r^k|}{\langle\phi_r^k|\phi_r^k\rangle}\hat{B}_{rs}|k\rangle\\&\leqslant\sum_r\langle\phi_r^k|\phi_r^k\rangle\sum_s\langle k|\hat{B}_{rs}^\dagger\hat{\mathbb{1}}\hat{B}_{rs}|k\rangle\\&=\sum_r\langle\phi_r^k|\phi_r^k\rangle=d.\end{aligned} \tag{18}$$

In deriving Eq. (18), we have implicitly assumed that $|\phi_r^k\rangle\neq0$, but of course the above inequalities hold also in the case when $|\phi_r^k\rangle$ is zero. Thus we finally obtain the following upper bound on the mean fidelity:

$$\overline{f}\leqslant\frac{1}{d+1}\left[1+\left(\sum_{k=0}^{d-1}\lambda_k\right)^2\right]. \tag{19}$$

We will now analyze necessary and sufficient conditions for a given teleportation protocol to be an optimal one. This is the case if the inequality signs in Eqs. (17) and (18) are replaced by equalities. Let us denote by m the maximum index for which λ_m is nonzero, i.e., $\lambda_{m+1}=\cdots=\lambda_{d-1}=0$. Of course, it is sufficient to characterize the vectors $|\phi_r^k\rangle$ for $k\leqslant m$, and the action of the operators $\hat{B}_{rs}$ on the subspace spanned by the vectors $|0\rangle,\ldots,|m\rangle$.

The inequality sign in Eq. (17) becomes equality if and only if there exist $m+1$ nonnegative numbers $a_0,\ldots,a_m$ such that

$$a_l\lambda_k\langle\phi_r^k|\hat{B}_{rs}|k\rangle=a_k\lambda_l\langle\phi_r^l|\hat{B}_{rs}|l\rangle \tag{20}$$

for any pair $k,l\leqslant m$. Furthermore, equality in Eq. (18) takes place if and only if

$$|\langle\phi_r^k|\hat{B}_{rs}|k\rangle|^2=\langle\phi_r^k|\phi_r^k\rangle\langle k|\hat{B}_{rs}^\dagger\hat{B}_{rs}|k\rangle. \tag{21}$$

Let us note that for a given k, the scalar products $\langle\phi_r^k|\hat{B}_{rs}|k\rangle$ cannot be identically equal to zero. Otherwise, Eq. (21) implies a contradiction: $0=\Sigma_{rs}|\langle\phi_r^k|\hat{B}_{rs}|k\rangle|^2=\Sigma_{rs}\langle\phi_r^k|\phi_r^k\rangle\langle k|\hat{B}_{rs}^\dagger\hat{B}_{rs}|k\rangle=\Sigma_r\langle\phi_r^k|\phi_r^k\rangle=d$. Consequently, a_k must be strictly positive for $k\leqslant m$, as discussed after Eq. (16). By taking the squared modulus of Eq. (20), making use of Eq. (21), and performing the summation over s and r, we find $a_l\lambda_k=a_k\lambda_l$ for each pair k,l. Thus we obtain

$$\langle\phi_r^k|\hat{B}_{rs}|k\rangle=\langle\phi_r^l|\hat{B}_{rs}|l\rangle\quad\text{and}\quad\langle\phi_r^k|\phi_r^k\rangle=\langle\phi_r^l|\phi_r^l\rangle \tag{22}$$

for any $k,l\leqslant m$. Furthermore, Eq. (21) implies that

$$\hat{B}_{rs}|k\rangle=\mu_{rsk}|\phi_r^k\rangle, \tag{23}$$

where μ_{rsk} are certain complex numbers. By taking the scalar product of this identity with $\langle\phi_r^k|$, and making use of Eq. (22) we see that the coefficients μ_{rsk} are independent of k: $\mu_{rsk}=\mu_{rs}$. Then by taking the scalar product of Eq. (23) with the Hermitian-conjugated identity $\langle l|\hat{B}_{rs}^\dagger=\langle\phi_r^l|\mu_{rs}^*$, and performing the summation over s we obtain

$$\langle\phi_r^k|\phi_r^l\rangle\sum_s|\mu_{rs}|^2=\sum_s\langle k|\hat{B}_{rs}^\dagger\hat{B}_{rs}|l\rangle=\delta_{kl}. \tag{24}$$

As $\Sigma_s|\mu_{rs}|^2=0$ would imply that for given r all operators $\hat{B}_{rs}=0$, the above identity means that all the vectors $|\phi_r^k\rangle$ are mutually orthogonal for $k\leqslant m$. Consequently, the action of the operator $\hat{B}_{rs}$ in the subspace spanned by $|0\rangle,\ldots,|m\rangle$ is equivalent, up to a multiplicative constant, to the action of $\Sigma_{k=0}^m|\phi_r^k\rangle\langle k|$. Since for a given r the vectors $|\phi_r^k\rangle$ are mutually orthogonal and have equal norm, the action of each of the operators $\hat{B}_{rs}$ is proportional to the same unitary transformation on the relevant subspace. Thus, in order to reach the upper bound for fidelity, it is sufficient for Bob to perform a unitary transformation described by the following operation on the subspace spanned by $|0\rangle,\ldots,|m\rangle$:

$$\hat{B}_r=\frac{1}{\sqrt{\langle\phi_r^0|\phi_r^0\rangle}}\sum_{k=0}^m|\phi_r^k\rangle\langle k|. \tag{25}$$

Necessary and sufficient conditions for Alice's measurement to be optimal are given by Eq. (7) and the requirement that for $k\leqslant m$ all vectors $|\phi_r^k\rangle$ have equal norm and are mutually orthogonal. It is straightforward to check that these conditions are fulfilled by the standard teleportation protocol [1] described by $|\phi_{r=p+qd}^k\rangle=e^{2\pi ikp/d}|(k+q)\bmod d\rangle/\sqrt{d}$, where $p,q=0,\ldots,d-1$ and the index r runs from 0 to d^2-1. Consequently, the standard teleportation protocol with appropriately adjusted bases saturates the upper bound on the mean fidelity derived in Eq. (19).

As expected, Eq. (19) shows that use of a nonmaximally entangled state makes the teleportation imperfect. However, suppose that Alice would like to use the result of her measurement to estimate the quantum state that has been teleported. Thus, for each outcome r of her measurement, she would like to assign a state $|\psi_r^{\text{est}}\rangle$, which is her guess for the teleported state. This state can be represented as a result of a unitary transformation $\hat{U}_r$ performed on a reference state $|0\rangle$: $|\psi_r^{\text{est}}\rangle=\hat{U}_r|0\rangle$. Given the input state $|\psi\rangle$, the probability that Alice's measurement yields the outcome r equals $\Sigma_{k=0}^{d-1}\lambda_k^2|\langle\phi_r^k|\psi\rangle|^2$. The fidelity of the corresponding estimate is then $|\langle\psi|\psi_r^{\text{est}}\rangle|^2=|\langle\psi|\hat{U}_r|0\rangle|^2$. Thus, the mean fidelity of Alice's estimate is given by

$$\overline{f}_{\text{est}}=\sum_r\int d\psi|\langle\psi|\hat{U}_r|0\rangle|^2\sum_{k=0}^{d-1}\lambda_k^2|\langle\phi_r^k|\psi\rangle|^2. \tag{26}$$

Using the invariance of the measure $d\psi$ with respect to unitary transformations, we may change the integration according to $|\psi\rangle\to\hat{U}_r|\psi\rangle$. This yields

$$\overline{f}_{\rm est} = \sum_r \sum_{k=0}^{d-1} \lambda_k^2 |\langle \psi|0\rangle|^2 |\langle \phi_r^k|\hat{U}_r|\psi\rangle|^2$$
$$= \sum_r \sum_{k=0}^{d-1} \lambda_i^2 \langle \phi_r^k|\hat{U}_r \hat{M}_{00} \hat{U}_r^\dagger|\phi_r^k\rangle, \qquad (27)$$

where $\hat{M}_{00}$ is defined in Eq. (10). By inserting its explicit form, we obtain that

$$\overline{f}_{\rm est} = \frac{1}{d(d+1)} \left(\sum_{k=0}^{d-1} \lambda_k^2 \sum_r \langle \phi_r^k|\phi_r^k\rangle + \sum_{k=0}^{d-1} \lambda_k^2 \sum_r |\langle \phi_r^k|\hat{U}_r|0\rangle|^2 \right). \qquad (28)$$

The first double sum over k and r gives d, which follows from Eq. (15). The second sum can be estimated using the fact that for a given r, all the vectors $|\phi_r^k\rangle$ are orthogonal and have equal norm for $k \leqslant m$. Thus, the second sum over k is maximized if the operator $\hat{U}_r$ maps the vector $|0\rangle$ onto the subspace spanned by the vectors $|\phi_r^k\rangle$ corresponding to the maximum λ_k. As λ_k are ordered decreasingly, we obtain the following upper bound on the estimation fidelity:

$$\overline{f}_{\rm est} \leqslant \frac{1}{d+1}\left(1 + \frac{\lambda_0^2}{d}\sum_r \langle \phi_r^0|\phi_r^0\rangle\right) = \frac{1+\lambda_0^2}{d+1}. \qquad (29)$$

It is straightforward to see that the optimal estimation strategy is given by $|\psi_r^{\rm est}\rangle = |\phi_r^0\rangle/\sqrt{\langle \phi_r^0|\phi_r^0\rangle}$. Of course, if several of λ_k have the same maximum value, then Alice can take as a guess any linear combination of the corresponding vectors $|\psi_r^k\rangle$.

It is interesting to compare two extreme cases: if the state $|t\rangle_{23}$ is maximally entangled, the maximum estimation fidelity is $1/d$, which corresponds to making completely random guesses by Alice. This is clear, as perfect teleportation with a maximally entangled state cannot reveal any information on the teleported state. On the other hand, if the state $|t\rangle_{23}$ is completely disentangled, the maximum estimation fidelity is $2/(d+1)$, which corresponds to the optimal state estimation of a d-level system from a single copy [7]. In this case, the optimal teleportation strategy reduces to the optimal state estimation procedure, with Bob generating on his side an imperfect copy according to the classical message obtained from Alice.

In conclusion, we have derived an upper bound for fidelity of teleportation using an arbitrary pure bipartite system, and characterized optimal teleportation protocols. We have also presented an optimal strategy for estimating the quantum state given result of the measurement performed in course of teleportation.

The author thanks Professor J. H. Eberly for valuable comments on the manuscript, and P. Horodecki, D. Jonathan, and V. Vedral for helpful discussions. This research was partially supported by ARO-administered MURI Grant No. DAAG-19-99-1-0125, NSF Grant No. PHY-9415583, and KBN Grant No. 2 P03B 089 16.

APPENDIX: Evaluation of integrals in Eq. (10)

Because of symmetry, it is sufficient to consider two cases: $i=0, j=0$, and $i=0, j=1$. We will use the following parametrization of the state vector $|\psi\rangle$ in the basis $|k\rangle$:

$$|\psi\rangle = \begin{pmatrix} e^{i\xi}\cos\theta \\ \sin\theta\cos\varphi \\ z_3 \sin\theta\sin\varphi \\ \vdots \\ z_d \sin\theta\sin\varphi \end{pmatrix}, \qquad (A1)$$

where $0 \leqslant \xi \leqslant 2\pi$, $0 \leqslant \theta$, $\varphi \leqslant \pi/2$, and $z_3, \dots, z_d$ are complex numbers satisfying $|z_3|^2 + \cdots + |z_d|^2 = 1$. This parameterization is a straightforward generalization of the method used in [8]. Following [8], the invariant volume element in this parametrization is given by

$$d\psi = \frac{(d-1)!}{4\pi^{d-1}} (\sin\theta)^{2d-3} (\sin\varphi)^{2d-5} \times d(\sin\theta) d(\sin\varphi) d\xi dS_{2d-5}, \qquad (A2)$$

where dS_{2d-5} is the volume element of the unit sphere S_{2d-5}. For the case $i=0, j=0$ all the off-diagonal elements vanish, and we need to calculate only two elements: $\langle 0|\hat{M}_{00}|0\rangle = \int d\psi \cos^4\theta = 2/[d(d+1)]$, and $\langle 1|\hat{M}_{00}|1\rangle = \int d\psi \sin^2\theta \cos^2\theta \cos^2\varphi = 1/[d(d+1)]$. Due to symmetry, we have $\langle k|\hat{M}_{00}|k\rangle = 1/[d(d+1)]$ for all $k \neq 0$. For the operator $\hat{M}_{01}$, the only nonvanishing element is $\langle 0|\hat{M}_{01}|1\rangle = \int d\psi \sin^2\theta \cos^2\theta \cos^2\varphi = 1/[d(d+1)]$.

[1] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] M.B. Plenio and V. Vedral, Contemp. Phys. **39**, 431 (1998).

[3] G. Vidal, D. Jonathan, and M. A. Nielsen, Phys. Rev. A **62**, 012304 (2000).

[4] M.A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999); D. Jonathan and M.B. Plenio, *ibid.* **83**, 1455 (1999); G. Vidal, *ibid.* **83**, 1046 (1999).

[5] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).

[6] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995); R. Derka, V. Bužek, and A.K. Ekert, *ibid.* **80**, 1571 (1998); G. Vidal, J.I. Lattorre, P. Pascual, and R. Tarrach, Phys. Rev. A **60**, 126 (1999).

[7] D. Bruss and C. Macchiavello, Phys. Lett. A **253**, 149 (1999); A. Acín, J.I. Latorre, and P. Pascual, Phys. Rev. A **61**, 022113 (2000).

[8] R. Schack, G.M. D'Ariano, and C.M. Caves, Phys. Rev. E **50**, 972 (1994).