Visibility is not a good measure of a well-defined relative phase

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(Received 24 August 1999; revised manuscript received 13 January 2000; published 20 July 2000)

In a recent paper Marburger and Das [J. H. Marburger III and K. K. Das, Phys. Rev. A **59**, 2213 (1999)] considered an interference visibility experiment involving two weakly interacting Bose-Einstein condensates. It was shown that condensate eigenstates of the Hermitian relative phase operator do not give interference fringes with unit visibility in a Young's double slit type of experiment. The authors concluded that ''... these states are not especially well suited to describe weakly interacting multiply occupied coherent bosonic systems.'' In this work we suggest a criterion for states with a well-defined relative phase. Subsequently we show that the relative phase operator eigenstates satisfy this criterion. This suggests that the concept of interference visibility can, and should, be generalized, since it is widely believed that interference visibility is a measure of the relative phase properties. We therefore propose a broader, but still operational, definition of interference visibility, which we call generalized visibility, and prove that the relative phase operator eigenstates indeed can show unit generalized visibility. We also derive a simple, but general, criterion for states which can display a unit generalized visibility. Somewhat surprisingly, this criterion is weaker than the criterion for a well-defined relative phase. Finally, we discuss which two-mode states can display unit (ordinary) visibility.

PACS number(s): 42.50.Dv, 03.65.Bz, 03.75.Fi

I. INTRODUCTION

The experimental observation of Bose-Einstein condensation [1–4] and subsequent refinement of the associated experimental techniques have led this field to a point where one can start to speak about an ''atom laser,'' a device emitting a beam or a pulse of atoms all belonging to a single bosonic mode [5]. Recently, several groups have demonstrated pulsed or continuous ''atom lasers'' [6–8]. In this context questions have been raised about the phase properties of the condensate and about relative phase properties between two condensates [9–14]. Observation of interference between two condensates has already been made [15].

In a recent paper Marburger and Das [16] analyzed the interference pattern that would emerge if two condensates in two plane-wave modes were prepared in an (entangled) eigenstate to the Hermitian relative phase operator derived by Luis and Sánchez-Soto [17]. One reason this state is relevant in this context is that it is simultaneously a particle number eigenstate. Bose-Einstein condensates can be trapped for long times and therefore can be measured repeatedly. In this manner the particle number of condensates can be relatively well determined. This is in contrast to electromagnetic modes where it is notoriously difficult to prepare photon number states, or near number states, with any substantial particle number. Therefore, two-mode states which are simultaneously particle number eigenstates are relevant in the context of Bose-Einstein condensation.

The main result in Marburger and Das' paper is the discovery that the interference pattern ensuing from a relative phase operator eigenstate does not have unit visibility. This is surprising, because one would expect this state to have a well-defined relative phase and hence display unit visibility interference fringes. In this paper we address the assertion expressed in the title of this paper. Marburger and Das, along with many other workers in the field, implicitly assume that visibility provide a good measure of relative phase. In this paper we argue that this is not necessarily the case. Analysis of the relative phase properties of quantum states allows, and requires, rather sophisticated experimental tools, while visibility and the measurement thereof is a classically defined concept. Therefore it is not surprising that visibility says little about the quantum phase properties of states. To say something about the latter the concept of visibility must be generalized. This is the aim of the present work.

However, our paper should not be taken as a critique of Marburger and Das' and others work. Although we argue that visibility is not a good measure of the relative phase between two bosonic modes, the interaction needed to display the relative phase properties between two modes (a generalized visibility measurement) is rather complex (in general it requires highly nonlinear mode interaction). Therefore, we believe that experimentally one will have to content oneself with visibility measurements both for condensates, as well as for photon states, for some time to come, although recently some more elaborate measurements of weakly excited photon states have been made [18]. On the other hand, the rapid progress of trapping technology, laser pulse shaping and chirping technology, and magnetic field technology makes the long term prospect of custom made nonlinear interactions seem rather good for Bose-Einstein condensates. Still, until the prospects have become reality visibility measurements are probably the best measurement of relative phase properties that can be made. Marburger and Das' analysis is therefore highly relevant. At the end of this paper we have therefore extended Marburger and Das' analysis and derived a criterion for when a two-mode state can display unit visibility.

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The outline of the paper is as follows In Sec. II we derive a criterion for a two-mode state to have a well-defined relative phase. We subsequently show that the relative phase operator eigenstates satisfy the criterion, in spite of displaying less than unity visibility. In Sec. III we review the visibility of classical fields. In Sec. IV we take a quantum-mechanical view of visibility, define what we call ''generalized visibility,'' and derive a criterion for two-mode states to display unit generalized visibility. In Sec. V we return to a quantum-mechanical analysis of the classical visibility and derive a general form of a state to display unit visibility. We show that the class is a particular subclass of the states that can display unit generalized visibility. Finally, we summarize our findings in Sec. VI.

II. THE RELATIVE PHASE-SHIFT OPERATOR AND STATES WITH A WELL-DEFINED RELATIVE PHASE

In order to derive a criterion for a (two-mode) state with a well-defined relative phase, we start by considering the physical action of a relative (or differential) phase shift, i.e., letting the two condensates undergo free evolution for unequal times. The free evolution of a Bose-Einstein condensate for a time τ is described by $\hat{U}_0(\tau)=\exp(-i\hat{H}_0\tau/\hbar)$, where the Hamiltonian is $\hat{H}_0=\hbar\omega\hat{n}$, and ω and $\hat{n}$ is the angular frequency and particle-number operator of the condensate, respectively. As usual the gauge has here been chosen such that the vacuum field energy is zero. Thus the evolution operator for two condensates undergoing free evolution for times τ_1 and τ_2, respectively, becomes

$$e^{-i\omega(\tau_1\hat{n}_1+\tau_2\hat{n}_2)}=e^{-i\omega(\tau_1+\tau_2)(\hat{n}_1+\hat{n}_2)/2}e^{-i\omega(\tau_1-\tau_2)(\hat{n}_1-\hat{n}_2)/2}$$
$$=e^{-i\omega(\tau_1+\tau_2)\hat{N}/2}e^{i\phi\hat{n}_{12}}, \tag{1}$$

where $\hat{N}\equiv\hat{n}_1+\hat{n}_2$ is the total particle number operator, $\hat{n}_{12}\equiv(\hat{n}_1-\hat{n}_2)/2$ is the particle difference operator, and the differential phase shift is $\phi=\omega(\tau_2-\tau_1)$. Since the operator $\exp[-i\omega(\tau_1+\tau_2)\hat{N}/2]$ in Eq. (1) will only give all states with a particular particle number the same phase shift, and (two-mode) states with different particle number are orthogonal and cannot interfere, this operator can be neglected in the context of interference. Hence the unitary differential phase-shift operator is particle-number conserving and can be written

$$\hat{U}_{\rm PS}(\phi)=\exp(i\phi\hat{n}_{12}). \tag{2}$$

We now turn our attention to what constitutes a state with a well-defined relative phase. We take an operational approach and assign this property to any two-mode state on which (at least) two different relative phases can be encoded and read out with certainty. This definition avoids all complications with associating a well-defined relative phase with some relative phase operator, or with the properties of the state's relative phase statistical distribution. It is well known that in order to be able to encode either of two relative phases ϕ_1 or ϕ_2 so that they (at least in principle) can be read out with certainty, i.e., be projected onto orthogonal meter eigenstates, the relation

$$\langle\xi|\hat{U}^{\dagger}_{\rm PS}(\phi_1)\hat{U}_{\rm PS}(\phi_2)|\xi\rangle=\langle\xi|\hat{U}_{\rm PS}(\phi_2-\phi_1)|\xi\rangle=0 \tag{3}$$

must be fulfilled. That is, some relative phase shift $\phi=\phi_2-\phi_1$ must render the state $\hat{U}_{\rm PS}(\phi)|\xi\rangle$ orthogonal to $|\xi\rangle$. Thus, in our treatment, Eq. (3) constitutes the mathematical criterion for a state with a well-defined relative phase.

Let us write the number-basis expansion of a two-mode state as

$$|\xi\rangle=\sum_{N=0}^{\infty}\sum_{n=0}^{N}c_{N,n}|n,N-n\rangle, \tag{4}$$

where we have used the notation $|k,l\rangle\equiv|k\rangle\otimes|l\rangle$, and we write the associated bra as $\langle k,l|=\langle k|\otimes\langle l|$. In this notation, we have $\hat{n}_1|k,l\rangle=k|k,l\rangle$ and $\hat{n}_2|k,l\rangle=l|k,l\rangle$. Since $\hat{U}_{\rm PS}(\phi)|0,0\rangle=|0,0\rangle$, it is clear that we must have $\langle 0,0|\xi\rangle=0$ in order to satisfy Eq. (3).

If one wishes to formalize and quantify the ability to distinguish between two relative phases, one can define the distinguishability D (in the maximum likelihood estimation sense [19,20]) between the two relative phases ϕ_1 and ϕ_2 on $|\xi\rangle$ as [20]

$$D=\sqrt{1-4p_1p_2|\langle\xi|\hat{U}_{\rm PS}(\phi_2-\phi_1)|\xi\rangle|^2}, \tag{5}$$

where p_1 and p_2 are the *a priori* probabilities of encoding the phase shifts ϕ_1 and ϕ_2, respectively. This distinguishability limit is referred to as the Helstrom bound [21] and is well known in estimation theory. We see that for any nonzero *a priori* probabilities p_1 and p_2, unit distinguishability implies that Eq. (3) has to be fulfilled.

Let us now turn to the specific state Marburger and Das analyzed in their paper [16], namely, the eigenstate to the relative phase operator introduced by Luis and Sánchez-Soto [17]. The most general form of such an eigenstate in particle manifold N can be written [17,22]

$$|\phi_r^{(N)}\rangle=\frac{1}{\sqrt{N+1}}\sum_{n=0}^{N}\exp(in\phi_r^{(N)})|n,N-n\rangle, \tag{6}$$

where $\phi_r^{(N)}=\phi_0^{(N)}+2\pi r/(N+1)$, $r=0,1,\ldots,N$.

Since the eigenstates are orthogonal, they will fulfill Eq. (3) above for $N\neq 0$ and $\phi=2\pi k/(N+1)$, $k=1,2,\ldots,N$. Thus, in spite of displaying nonunit visibility, as was shown in Ref. [16], these states have a well-defined relative phase. In Fig. 1 the scalar product $|\langle\phi^{(N)}|\hat{U}_{\rm PS}(\phi)|\phi^{(N)}\rangle|^2$ is plotted as a function of ϕ. The function is identical to the relative phase distribution function defined in [22]. Since the smallest differential phase shift that can be resolved with certainty is $2\pi/(N+1)$, these states fulfill the Heisenberg limit in phase measurements.

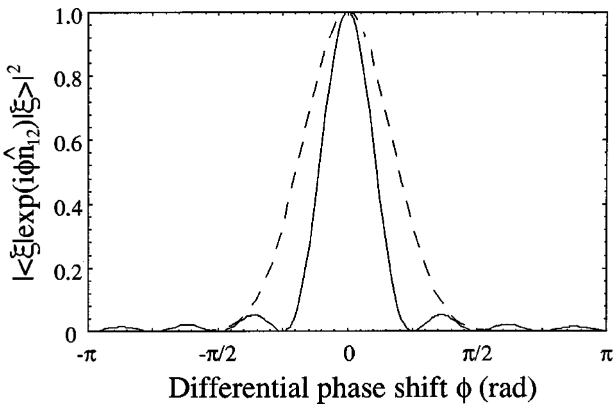

FIG. 1. The scalar product squared $|\langle \xi|\hat{U}_{\rm PS}(\phi)|\xi\rangle|^2$ as a function of the differential phase shift ϕ. The solid curve represents the scalar product for a relative phase eigenstate and the dashed curve holds for a two-mode symmetric binomial state, both with $N=7$.

III. A SHORT REVIEW OF CLASSICAL VISIBILITY

In this section we will make a short review of the properties a classical field and an interferometer must have in order to display unit visibility. While Marburger and Das considered a Young's double slit type of experiment, we shall consider a Mach-Zehnder interferometer to avoid the unnecessary complication of taking the spatial evolution of the wave function between the double slits and the screen into account. A classical version of the experiment we consider is depicted in Fig. 2(a). Two fields interfere in a beam splitter. The relative phase between the two fields can be varied by the means of a phase shifter. After the beam splitter the outgoing field intensities are monitored by two detectors. The visibility is defined in terms of the ensemble averaged modulation of the measured intensities as a function of the relative phase shift. Classically, the visibility gives an indication of the relative phase properties of the two fields. Specifically, if the two field's respective phases are random, the visibility is zero.

To make a quantitative analysis of classical visibility we shall consider the interference of two harmonic waves (sufficiently generally) described by

$$E_1(t)=E_1\cos(\omega t+\theta) \quad \text{and} \quad E_2(t)=E_2\cos(\omega t), \tag{7}$$

where E_1 and E_2 are (real) electric field amplitudes. The phase shifter transforms field $E_1(t)$ to

$$E_1(t)=E_1\cos(\omega t+\theta+\phi). \tag{8}$$

The beam splitter is described by the unitary transformation

$$\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{9}$$

The transformation assumes, without loss of generality, that reference planes have been chosen so that all coefficients are real. If the beam splitter transmittance $T=\cos(\alpha)$ is neither zero nor unity, the (time-averaged) intensity I detected by either the two detectors is a sinousoidally varying function of the phase shift ϕ. The visibility for each detector is defined

$$V=\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}, \tag{10}$$

where $I_{\max}$ and $I_{\min}$ are the maximum and minimum intensities measured by the detector as a function of the phase shift ϕ. Note that, in general, the visibility is not equal for the two detectors. To get unit visibility one must arrange so that $I_{\min}=0$ while $I_{\max}\neq 0$. Also note that the intensity detected in each detector is a second order correlation function. For the special case when $|\cos(\alpha)|=|\sin(\alpha)|=1/\sqrt{2}$, the detector signals provide a measure of the complex degree of coherence between the two fields [23].

Using Eqs. (7), (8), and (9), it is straightforward to show that $I_{\min}=0$ for either detector implies a phase shift ϕ fulfilling $\theta+\phi=0$ or $\theta+\phi=\pi$. In the first case it is possible to get the intensity falling onto detector D_1 to be zero if $\tan(\alpha)=-E_1/E_2$. To make the intensity falling onto detector D_2 be zero, one must require $\tan(\alpha)=E_2/E_1$. If the phase ϕ is instead set to $\pi-\theta$, then detector D_1 ''sees'' zero intensity for $\tan(\alpha)=E_1/E_2$ and D_2 ''sees'' zero intensity for $\tan(\alpha)=-E_2/E_1$. In both cases we can only get the visibility to equal unity in both beam splitter output modes if $E_1=\pm E_2$ and $|\cos(\alpha)|=|\sin(\alpha)|=1/\sqrt{2}$.

To get a situation similar to a unit visibility in a Young's double slit experiment (which simultaneously probes all relative phases, usually over a several 2π interval) it is necessary to obtain unit visibility in both outputs, that is, a complete transfer of the intensity from one beam splitter output mode to the other as ϕ is varied over some π interval. Therefore, this is what we will require in the following and when we henceforth refer to unit visibility we shall always assume that it holds for both output modes. In summary, from the classical viewpoint, in order to be able to get unit visibility in the sense just defined, the field amplitudes must be equal and the beam splitter needs to have equal transmission and reflection. To simplify the notation in the following we shall refer to the visibility of an ideal measurement of the type just described, with $|\cos(\alpha)|=|\sin(\alpha)|=1/\sqrt{2}$, simply as the visibility of the two-mode state.

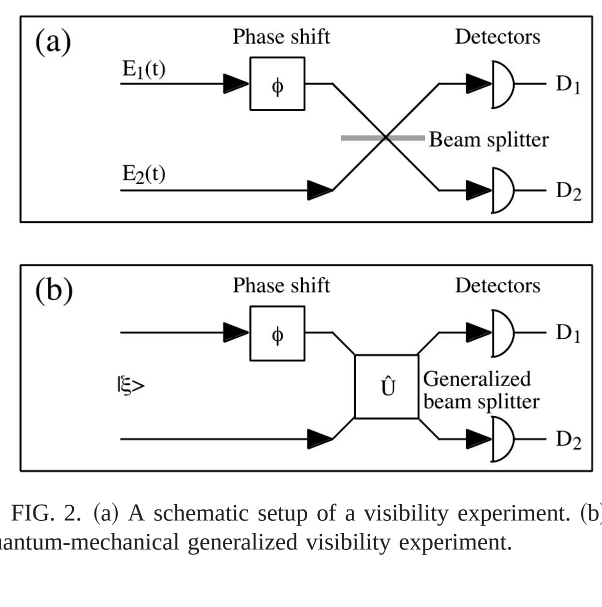

FIG. 2. (a) A schematic setup of a visibility experiment. (b) A quantum-mechanical generalized visibility experiment.

IV. VISIBILITY FROM A QUANTUM-MECHANICAL POINT OF VIEW

Now let us examine the quantum mechanical situation depicted in Fig. 2(b). One mode of the two-mode state $|\xi\rangle$ is phase shifted by a relative amount ϕ to the other. Subsequently the state is transformed by a generalized beam splitter, described by the unitary transformation $\hat{U}$ (whose essential properties will be defined below) and finally the output modes of the generalized beam splitter are detected by two particle counting detectors. The generalized visibility is, in analogy with the classical definition of visibility, defined in terms of the ensemble averaged modulation of the detected particle number in the respective particle detectors. The main difference between the quantum and the classical setups is that the generalized beam splitter is typically not a linear beam splitter but a nonlinear beam splitter that has to match the impinging two-mode state. Different two-mode states require different generalized beam splitters. If, by a proper choice of the generalized beam splitter, it is possible to find two relative phase shift settings such that for one setting all the particles of state $|\xi\rangle$ impinge on detector D_1, and for the other setting all the particles impinge on detector D_2, then we *define* the state $|\xi\rangle$ to have unit generalized visibility. Comparing this operational definition with our operational definition of a state with a well-defined relative phase, one has reason to suspect that the two definitions are interrelated.

Let us next cast generalized visibility in a mathematical framework. We begin by defining the quantum-mechanical visibility of the setup in Fig. 2(b) in analogy with the classical visibility

$$V=\frac{\langle\hat{n}\rangle_{\max}-\langle\hat{n}\rangle_{\min}}{\langle\hat{n}\rangle_{\max}+\langle\hat{n}\rangle_{\min}}, \tag{11}$$

where $\hat{n}$ is the number operator associated with the pertinent detector. In order to get unit generalized visibility in one of the output modes, we must have $\langle\hat{n}\rangle_{\min}=0$ while $\langle\hat{n}\rangle_{\max}\neq 0$. This implies that for some suitable differential phase shift ϕ the vacuum state $|0\rangle$ must impinge on the detector, since this is the only single-mode state with $\langle\hat{n}\rangle=0$.

Now consider a two-mode quantum state $|\xi\rangle$, which in general is partially entangled. It is always possible to find a unitary and particle number conserving transformation $\hat{U}$ such that

$$\hat{U}|\xi\rangle=|\psi\rangle\otimes|0\rangle, \tag{12}$$

i.e., the state $|\xi\rangle$ is transformed to a factorizable state with no excitation in one of the modes (i.e., all particles are found in the other mode). In a classical visibility measurement, $\hat{U}$ is assumed to describe the action of a 50/50 beam splitter, but for a generalized visibility measurement we must only require that $\hat{U}$ is unitary and particle number conserving. The latter requirement is reasonable as we want interference, rather than particle loss, to determine the generalized interference. A consequence of the particle number conservation is that

$$\hat{U}|0,0\rangle=\exp(i\zeta)|0,0\rangle, \tag{13}$$

where ζ is a real number. In order to get unit generalized visibility in the measurement depicted in Fig. 2(b) it is necessary that if the state $|\xi\rangle$ is differentially phase shifted by an appropriate amount ϕ, $\hat{U}$ must transform this new state to a state of the form $|0\rangle\otimes|\phi\rangle$. Hence,

$$\hat{U}\hat{U}_{\mathrm{PS}}(\phi)|\xi\rangle=|0\rangle\otimes|\phi\rangle. \tag{14}$$

Equations (12) and (14) together with the requirement that $|\xi\rangle\neq|0,0\rangle$ are, by the operational definition, sufficient and necessary conditions to get unit generalized visibility in the output modes of a generalized beam splitter.

To see what requirements these equations imply for the state $|\xi\rangle$, let us compute the scalar product between the final states (12) and (14):

$$(\langle\psi|\otimes\langle 0|)(|0\rangle\otimes|\phi\rangle)=\langle\psi|0\rangle\langle 0|\phi\rangle. \tag{15}$$

Using the left hand sides of Eqs. (12) and (14), the scalar product can be rewritten as

$$\begin{aligned}\langle\psi|0\rangle\langle 0|\phi\rangle&=\langle\xi|0,0\rangle\langle 0,0|\xi\rangle=\langle\xi|\hat{U}^{\dagger}\hat{U}\hat{U}_{\mathrm{PS}}(\phi)|\xi\rangle\\&=\langle\xi|\hat{U}_{\mathrm{PS}}(\phi)|\xi\rangle,\end{aligned} \tag{16}$$

where we have used the fact that vacuum is unaffected by a phase shift and Eq. (13) to arrive at the second of the four expressions in Eq. (16). A rearrangement now gives us our final result, delineating a necessary and sufficient condition to achieve unit generalized visibility as

$$\langle\xi|\hat{U}_{\mathrm{PS}}(\phi)|\xi\rangle-|\langle 0,0|\xi\rangle|^2=0. \tag{17}$$

This is our central result. That the condition is sufficient follows from that any state fulfilling Eq. (17) can also fulfill both Eqs. (12) and (14), which is necessary and sufficient to get unit generalized visibility. That the condition is necessary follows from the fact that any state that does not fulfill the condition (17) can fulfill only one of Eqs. (12) and (14), not both. We see that in order to get unit generalized visibility there must exist a differential phase shift ϕ such that all the excited particle number manifolds of the state $\hat{U}_{\mathrm{PS}}(\phi)|\xi\rangle$ are rendered orthogonal to the excited particle number manifolds of the initial state $|\xi\rangle$.

Comparing Eqs. (3) and (17), one sees that the former condition is stronger than the latter in that every state fulfilling Eq. (3) will also fulfill Eq. (17), but a state satisfying Eq. (17) does not necessarily satisfy Eq. (3). Why then are these conditions different? The physical reason is that since visibility (both the classical and the generalized) is an ensemble averaged quantity the fact that $\hat{U}_{\mathrm{PS}}(\phi)$ does nothing to the state $|0,0\rangle$ is not important to the generalized visibility. Null particle counts will contribute neither to the minima nor to the maxima (as ϕ is varied) of the particle counter interference pattern. As long as the state $|\xi\rangle$ contains excitation in higher manifolds, and each of these excited manifold states are simultaneously rotated to an orthogonal state for some differential phase shift ϕ, an interference pattern with unit generalized visibility can be observed. On the contrary, in order to predict a phase shift with certainty (i.e., for each and every individual detected state) the state must not contain any component of the vacuum state, since every time the state $\hat{U}_{\mathrm{PS}}(\phi_1)|\xi\rangle$ or $\hat{U}_{\mathrm{PS}}(\phi_2)|\xi\rangle$ collapses into the vacuum state it leads to an inconclusive result of which phase shift ϕ_1 or ϕ_2 was used. Hence, the relation (3) is sharper than the requirement of unit generalized visibility (17) in that every state fulfilling Eq. (3) can also display unit generalized visibility, while a perfect generalized visibility does not ensure that the relative phase of the state is precisely defined.

Visibility and coherence are intimately connected. Let us therefore briefly discuss the connection between generalized visibility and coherence. In terms of coherence theory, the detectors in Fig. 2(b) measure a superposition of even order correlation functions. The explicit choice of $\hat{U}$ will determine the particular superposition. (The fact that particle detectors are quadratic in the incident fields assures that no odd order coherence functions are measured.) If the two-mode state is in a particle number eigenstate with a total of N particles, only the even ordered correlation functions up to order $2N$ are measured. This is due to the fact that the interaction Hamiltonian realizing any particle number conserving, two-mode, unitary transformation $\hat{U}$ can be synthesized by a normally ordered polynomial of order $2N$ in the creation and annihilation operators of the fields [24]. As discussed by Mandel and Wolf, the coherent properties of a N particle state is not simply given by the $2N$th order correlation function [25]. Yet, the criterion for when a two-mode state has a well-defined relative phase is surprisingly simple (3). A classical visibility measurement is a special case of a generalized visibility measurement where $\hat{U}$ has a particular form (expressed in creation and annihilation operators it contains only the linear term of each mode) so that no correlation functions higher than of the second order are measured.

Let us next show that unit generalized visibility does not require any symmetry of the state $|\xi\rangle$ with the respect of permutation of modes. As demonstrated above this is necessary in a classical experiment. To show this we construct a simple example, e.g., the state

$$|\delta\rangle=\sqrt{\frac{3}{10}}(|0,N\rangle+|1,N-1\rangle)+\frac{2}{\sqrt{10}}|3,N-3\rangle, \quad (18)$$

where $N\geqslant 3$. This state has no symmetry with respect of permutation of modes, and its average excitation in the second mode is much larger than its excitation in the first mode, if N is large. Yet, for $\phi=\pm\arcsin(\sqrt{15}/4)\approx\pm 0.42\pi$ rad Eq. (3) is satisfied. Hence, the state has a well-defined relative phase and can therefore display unit generalized visibility. In Fig. 3 the scalar product $|\langle\delta|\hat{U}_{\mathrm{PS}}(\phi)|\delta\rangle|^2$ is plotted as a function of ϕ. The function has zeros at $\phi=\pm\arcsin(\sqrt{15}/4)$. Using the state (18), we can take $(|0,0\rangle+|\delta\rangle)/\sqrt{2}$ as an example of a state that does not have a well-defined relative phase but still can attain unit generalized visibility.

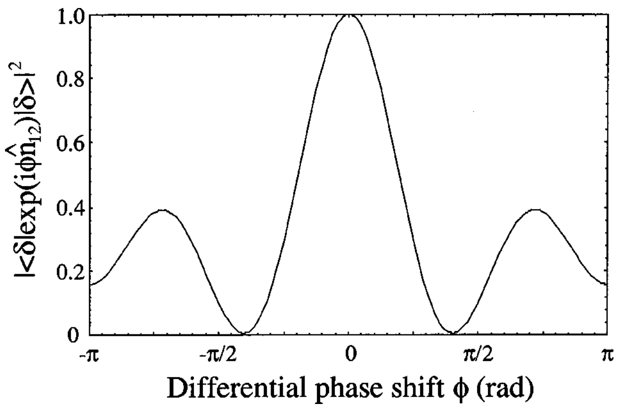

FIG. 3. The scalar product squared $|\langle\xi|\hat{U}_{\mathrm{PS}}(\phi)|\xi\rangle|^2$ as a function of the differential phase shift ϕ for the state given by Eq. (18). The curve is independent of N as long as $N\geqslant 3$. For any such N the function is symmetric with respect to the argument.

Let us now turn to the specific question Marburger and Das raised in their paper [16]: What visibility can one observe from a (two-mode) eigenstate to the relative phase operator introduced by Luis and Sánchez-Soto [17]? Marburger and Das showed that for $N\geqslant 2$ the state displays less than unit visibility. However, since the state fulfills Eq. (3), the state can display unit generalized visibility. To give a specific example of a unitary transformation which gives any state of the type (6) a unit generalized visibility, consider the unitary transformation

$$\hat{U}=\sum_{N=0}^{\infty}\sum_{r=0}^{N}|r,N-r\rangle\langle\phi_r^{(N)}|. \quad (19)$$

In every particle manifold the state $|\phi_r^{(N)}\rangle$ is transformed into the number difference state $|r,N-r\rangle$. In Fig. 4 the expectation values $\langle\hat{n}_1\rangle(\phi)$ and $\langle\hat{n}_2\rangle(\phi)$ of detectors D_1 and D_2 are plotted as a function of the differential phase shift ϕ. The two curves show unit generalized visibility. In order to observe these particular curves experimentally, the generalized beam splitter described by $\hat{U}$ corresponds to a rather nonlinear Hamiltonian, as pointed out in Ref. [22]. Note that this is

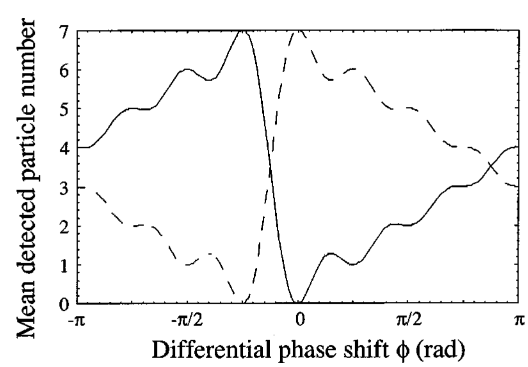

FIG. 4. The particle number expectation value for detector D_1 (solid) and D_2 (dashed) as a function of the differential phase shift ϕ for the state $|\phi_0^{(7)}\rangle$. The particular implementation of a generalized beam splitter given by Eq. (19) has been used.

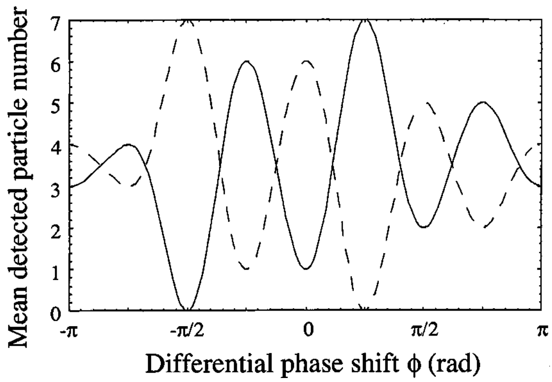

FIG. 5. The particle number expectation value for detector D_1 (solid) and D_2 (dashed) as a function of the differential phase shift ϕ for the state $|\phi_0^{(7)}\rangle$. The unitary transformation of the generalized beam splitter is given by Eq. (20).

not the only possible implementation of $\hat{U}$ giving unit generalized visibility for this state. In Fig. 5 the same quantities are plotted for the relevant unitary transformation

$$\hat{U}=|1,6\rangle\langle\phi_0^{(7)}|+|7,0\rangle\langle\phi_1^{(7)}|+|2,5\rangle\langle\phi_2^{(7)}|+|5,2\rangle\langle\phi_3^{(7)}| +|3,4\rangle\langle\phi_4^{(7)}|+|4,3\rangle\langle\phi_5^{(7)}|+|0,7\rangle\langle\phi_6^{(7)}|+|6,1\rangle\langle\phi_7^{(7)}|. \tag{20}$$

The resulting curves look quite different from those in Fig. 4 but the generalized visibility is still unity. If, however, a 50/50 beam splitter is used, the result derived by Marburger and Das holds, namely, the visibility is only unity for the case $N=1$ and then decreases monotonically with increasing N to approach the limiting value $V=\pi/4$ for large N.

As should be clear by now, the generalized visibility depends critically on the choice of generalized beam splitter. Most generalized beam splitters will not give a unit generalized visibility even to states fulfilling Eq. (3). As was discussed in Ref. [20], optimal resolution of the relative phase requires careful matching between the impinging states and the generalized beam splitter.

Before ending this section let us say something about the generalized visibility of mixed states. Mixed states can also have unit generalized visibility provided that all the eigenstates of the density operator fulfill Eq. (17) for the same differential phase shift ϕ. To give one example, the state $\hat{\rho}=P|\phi_0^{(2)}\rangle\langle\phi_0^{(2)}|+(1-P)|\phi_0^{(5)}\rangle\langle\phi_0^{(5)}|$ has unit generalized visibility for any value $0<P<1$, since, e.g., $\langle\phi_0^{(2)}|\hat{U}_{\mathrm{PS}}(2\pi/3)|\phi_0^{(2)}\rangle=\langle\phi_0^{(5)}|\hat{U}_{\mathrm{PS}}(2\pi/3)|\phi_0^{(5)}\rangle=0$.

V. MEASURING GENERALIZED VISIBILITY WITH A BEAM SPLITTER

Since it is unlikely that arbitrary generalized beam splitters can be experimentally realized in the near future one will probably have to stick with ''ordinary'' beam splitters, or Young's double-slit type of experiments, for some time to come. In light of this Marburger and Das' analysis is highly relevant, but could be extended to answer the question: What two-mode particle number eigenstates can yield unit visibility when they are mixed by an ordinary beam splitter? In Refs. [14,20] it is shown that the ''N-coherent states'' (in the language of Marburger and Das) have this property. In fact, these are the only two-mode particle number eigenstates which will exhibit unit visibility when $\hat{U}$ represents a 50/50 beam splitter. This can easily be shown by noting that if $|\xi\rangle$ is a particle number eigenstate containing N particles or quanta, then state $|\psi\rangle$ in Eq. (12) must be the single mode state $|N\rangle$. Using equality (12), we get

$$|\xi\rangle=\hat{U}_{\mathrm{BS}}^{\dagger}|N,0\rangle, \tag{21}$$

where $\hat{U}_{\mathrm{BS}}$ is the 50/50 beam splitter unitary transformation. The states are found to be what we have called the symmetric binomial two-mode states [14,20]

$$|\xi\rangle=\sqrt{\frac{N!}{2^N}}\sum_{n=0}^{N}\frac{|n,N-n\rangle}{\sqrt{n!(N-n)!}}. \tag{22}$$

These are the states Castin and Dalibard call ''phase states'' [14] and Marburger and Das call ''N-coherent states.'' A comprehensive treatment of these states in conjunction with interferometry has been made by Campos, Saleh, and Teich [26]. Similar single mode states have been treated by Radcliffe, who referred to those as ''coherent spin states'' [27], and by Stoler, Saleh, and Teich [28], who referred to them as ''binomial states.'' The scalar product $|\langle\xi|\hat{U}_{\mathrm{PS}}(\phi)|\xi\rangle|^2$ for these states is also displayed in Fig. 1.

Marburger and Das conclude their paper by stating that '' . . . [the relative phase operator eigenstates] are not especially well suited to describe weakly interacting multiply occupied coherent bosonic systems. The coherent-state-like 'N-coherent' states appear to be the natural generalization of coherent states for this purpose.'' If we interpret ''weakly interacting'' as linearly interacting, such as in an ordinary linear beam splitter, then it is obvious from the analysis in the preceeding paragraph why it is indeed so. A linear beam splitter is the appropriate generalized beam splitter for symmetric binomial two-mode states while the appropriate generalized beam splitter for the two-mode relative phase states is a nonlinear beam splitter [22].

Going beyond particle number eigenstates one can ask the question: What two-mode states will display unit visibility if a 50/50 beam splitter is used? To answer this question we use the observation made in Sec. II that states having different total particle number will not interfere. Therefore, using the fact that $\hat{U}_{\mathrm{BS}}$ is particle number conserving and the superposition principle, we conclude that (a pure) such state must have the general form

$$|\xi\rangle=\sum_{n=0}^{\infty}c_n\sqrt{\frac{n!}{2^n}}\sum_{k=0}^{n}\frac{|k,n-k\rangle}{\sqrt{k!(n-k)!}}, \tag{23}$$

where c_n are the respective particle number manifold's probability amplitudes. That the condition is necessary follows from the fact that every state which can be expressed $\hat{U}_{\mathrm{BS}}^{\dagger}|\psi\rangle\otimes|0\rangle$ will have the general form specified above.

That the condition is sufficient follows from Marburger and Das' analysis [any state of the form (22) can display unit visibility] and the superposition principle. Hence, it is possible to prepare any unit visibility two-mode Bose-Einstein condensate by linearly and symmetrically splitting an appropriate single-mode Bose-Einstein condensate. Conversely, any single-mode Bose-Einstein condensate split symmetrically by a linear beam splitter can display unit visibility. The latter conclusion is not surprising since, if one considers the overall unitary transformation of, e.g., a Mach-Zehnder interferometer with two beam splitters with transmittivity T_1 and T_2, and a relative phase shift ϕ in between, the transformation is equivalent to the unitary transformation of a single beam splitter with transmittivity $T_1+T_2-2T_1T_2+2\sqrt{T_1T_2(1-T_1)(1-T_2)}\cos(\phi)$. We see that if $T_1=T_2=T$ the total interferometer equivalent transmittivity becomes $2T(1-T)[1+\cos(\phi)]$. Therefore, if $T=1/2$, changing the phase shift from $\phi=0$ to $\phi=\pi$ will divert all the incident particles from total transmission (e.g., to detector D_1) to total reflection (e.g., to detector D_2), giving unit visibility.

However, only a small fraction of all two-mode states that display a unit generalized visibility are of the form (23). To give a simple counterexample consider the Schrödinger cat state

$$\frac{1}{\sqrt{2}}(|N,0\rangle+|0,N\rangle), \tag{24}$$

where $N>1$. The state is of interest in interferometric applications since, for a given mean energy, it is the state that will be transformed into an orthogonal state (and hence display unit generalized visibility) by the smallest relative phase shift [29,30]. It is also an energy eigenstate, so as to display unit visibility the state must be projected by the beam splitter to the state $|N\rangle\otimes|0\rangle$ for some setting of the relative phase shift ϕ. However, as shown in Eq. (22) above, a symmetric two-mode binomial state is the only state that will be transformed to this final state by a 50/50 beam splitter. Hence, the Schrödinger cat state (24) will not display unit visibility although it fulfills Eq. (17) and even Eq. (3).

Some approximate generalized visibility measurements have already been performed on the state (24), above. In, e.g., an experiment by Rarity and Tapster [31] such a state with $N=2$ was transformed to the state $|1,1\rangle$ by the use of an ordinary beam splitter. The state $|1,1\rangle$ was subsequently detected by coincidence photodetection. If the state is differentially phase-shifted by an amount $\phi=\pi/2$, the state becomes orthogonal to the original state, and therefore a beam splitter (being lossless and unitary) will project the state onto a state orthogonal to $|1,1\rangle$. In this manner a unit-visibility fourth-order correlation curve was obtained, proving that the state (24) can be transformed into an orthogonal state by a differential phase shift of $\pi/2$. However, as already mentioned, a linear beam splitter cannot project the state (24) onto the state $|0,2\rangle$ (nor onto $|2,0\rangle$) due to the symmetry of the state. Therefore an experiment of the type Rarity and Tapster performed does not constitute a true measurement of generalized visibility although their experiment relies on the fact that the state has a well-defined relative phase.

VI. CONCLUSIONS

We have discussed measurements of the relative phase of two-mode bosonic states, especially in the context of Bose-Einstein condensation where particle number eigenstates are of particular significance. The germinal hypothesis in our paper was that visibility, in general, is not a good measure of how well defined the relative phase between the two modes is. We suggested a criterion (which we believe is widely accepted) defining a state with a well-defined relative phase. The relative phase operator eigenstates satisfy this criterion in spite of displaying less than unity visibility. This led us to suggest that a generalization of a visibility measure is called for, and we subsequently derived a criterion for when the generalized visibility can equal unity. We showed that all states with a well-defined relative phase can display unit generalized visibility, whereas the converse is not true. We also showed that the unit-visibility states (23) form only a small subset of the two-mode states that fulfill the condition for unit generalized visibility (17). Therefore we conclude that visibility measurements made by a beam splitter, or equivalently, with a Young's double slit apparatus, do not, in general, give a good measure of the relative phase properties of a two-mode state. Moreover, as we have shown above, unless the state is void of any excitation amplitude of the vacuum state, not even a generalized visibility measurement will provide a good measure of the relative phase. The reason is that the vacuum state does not have a well-defined relative phase and therefore any state containing some component of the vacuum state will have some relative-phase ambiguity. (In the theory of Luis and Sánchez-Soto [17] the two-mode vacuum state is an eigenstate of the relative phase operator with a fixed associated eigenvalue. Although the vacuum state's relative phase hence is well defined in this theory, it cannot be made to change. Therefore the theory asserts that the vacuum state is inappropriate to use to display any relative phase properties.)

ACKNOWLEDGMENTS

This work was supported by grants from the Swedish National Board for Industrial and Technical Development through the QNANO program, the Swedish Technical Science Research Council, and the Swedish Natural Science Research Council.

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