Linewidth narrowing in Brillouin lasers: Theoretical analysis

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The linewidth narrowing observed in Brillouin fiber ring lasers is studied within the framework of the usual three-wave model of stimulated Brillouin scattering. We show that the phase noise of the pump laser is transferred to the emitted Stokes wave after being strongly reduced and smoothed under the combined influence of the acoustic damping and the cavity feedback. We then derive a simple analytical relation connecting the full width at half maximum of the Stokes linewidth to that of the pump laser.

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The Brillouin fiber ring laser is a highly coherent light source. Experiments have shown that the linewidth of the Stokes radiation emitted by this laser can be several orders of magnitude narrower than that of the incident beam used to pump it [1]. Although this effect is commonly used in many applications such as gyroscopes [2] or temperature sensors [3], it is not clearly understood. In fact, the questions related to the noise properties or to the temporal coherence of Brillouin lasers have not yet been attentively examined from the theoretical point of view. Brillouin lasers and conventional lasers have common noise sources. For instance, fluctuations of the pumping mechanism or of the cavity length alter the spectral purity of the emitted wave in both systems. However, contrary to lasers with population inversion, it is not spontaneous emission but spontaneous scattering that fundamentally limits the degree of monochromaticity of the Stokes radiation. The aim of this paper is to analyze the role of the different noise sources existing in Brillouin fiber ring lasers and to characterize their influence on the temporal coherence of the laser. In particular, we will show that the phase noise of the pump laser is the predominant noise source and we will study in detail the connection between the linewidth of the Brillouin laser and that of its pump laser.

Our theoretical study enters within the framework of the usual three-wave model of stimulated Brillouin scattering (SBS). This model reproduces properly most of the dynamical behaviors experimentally observed in Brillouin generators [4] or lasers [5]. It reads

$$\partial_\tau \varepsilon_p + \partial_\zeta \varepsilon_p = -gB\varepsilon_s, \tag{1a}$$

$$\partial_\tau \varepsilon_s - \partial_\zeta \varepsilon_s = gB^* \varepsilon_p, \tag{1b}$$

$$(1/\beta_A)\partial_\tau B + B = \varepsilon_p \varepsilon_s^* + f(\zeta,\tau), \tag{1c}$$

where ε_p, ε_p, and B represent, respectively, the complex amplitudes of the pump, Stokes, and acoustic waves. These equations in dimensionless form have been obtained by neglecting the weak attenuation of the fiber. The time τ is normalized to the transit time of the light inside the fiber. ζ is the space coordinate that is normalized to the fiber length. The fields ε_p and ε_s are measured in units of the maximum pump field available at the entrance end of the fiber. g is the SBS coupling constant and β_A represents the normalized damping rate of the acoustic wave. $f(\zeta,\tau)$ is a Langevin noise term describing the random thermal fluctuations of density (i.e., spontaneous scattering) occurring inside the fiber [6]. This noise is a zero-mean Gaussian process and it is δ correlated both in space and time. In the chosen normalization, f is of the order of 10^{-6} whereas the amplitude of the optical fields is of the order of unity ($|\varepsilon_p|,|\varepsilon_s|\simeq 1$). Despite its weak relative importance, spontaneous scattering plays a determining role in Brillouin generators by inducing a stochastic dynamics [4]. In Brillouin lasers, spontaneous scattering is usually considered as the effect that is responsible for the initiation of the SBS process.

As in Refs. [5] and [7], we study a Brillouin fiber ring laser in which pump recoupling is avoided by an intracavity isolator. The boundary conditions characterizing this system read

$$\varepsilon_p(\zeta=0,\tau)=\mu \quad e^{i\phi_0(\tau)}, \tag{2a}$$

$$\varepsilon_s(\zeta=1,\tau)=Re^{i\delta_s} \quad \varepsilon_s(\zeta=0,\tau). \tag{2b}$$

R is the amplitude feedback parameter and δ_s is a detuning term expressing the fact that the cavity is not necessarily resonant. In an all-fiber laser, the fluctuations of the fiber length can be reduced by stabilization techniques and the fluctuations of δ_s are then weak. Moreover, the use of couplers offers a great stability of the power feedback parameter R^2. This is not the case for Brillouin fiber lasers operating in low-finesse resonators. In these systems including an aerial arm, the relative fluctuations of R^2 can be of the order of 20% and the fluctuations of the resonator length can exceed the optical wavelength. The variations of δ_s are then greater than 2π and mode hops are observed [7]. All these fluctuations in the cavity characteristics arise from mechanical and thermal noise and their bandwidth is typically of the order of 100 Hz. These technical noise sources are responsible for slow drifts of the frequency of the Brillouin laser but they do not contribute to the intrinsic linewidth of the emitted Stokes radiation.

In Eq. (2a), μ is a dimensionless pump parameter and ϕ_0 represents the phase of the pump laser. For well-stabilized single-mode lasers usually used to pump Brillouin fiber lasers, the amplitude noise can be neglected so that only the phase is a randomly fluctuating quantity. In the phase diffusion model commonly used to describe the field emitted by a single-mode laser, μ is time independent whereas the phase $\phi_0(\tau)$ performs a random walk governed by the stochastic Langevin equation

$$\frac{d\phi_0(\tau)}{d\tau}=q(\tau), \tag{3}$$

in which $q(\tau)$ is a δ-correlated Gaussian noise of zero mean [8]. The field spectrum of the pump laser is then a Lorentzian with a full width at half-maximum (FWHM) $\Delta\nu_p$ in units of cavity free spectral range (FSR) [9].

In most of the experiments, the linewidth of the pump laser is of the order of a few tens of kHz. Apart from spontaneous scattering, the phase noise of the pump laser is the noise source presenting the wider spectrum. We are now going to analyze its influence on the noise characteristics of the emitted Stokes radiation. To that effect, we will ignore the influence of the low-frequency noise sources previously mentioned. As they only induce slow variations of R and δ_s, we can consider that these two parameters do not drift on the short time scale on which we analyze the influence of the pump phase noise. In particular, we will assume that the cavity is resonant for the Stokes wave ($\delta_s=0$). Frequency pulling effects that are discussed in Ref. [7] do not play an important role and can thus be ignored. In our analysis, we will also neglect the influence of the weak Langevin term $f(\zeta,\tau)$ describing spontaneous scattering. As with spontaneous emission in a conventional laser, this term will be responsible for the existence of a lower limit to the Stokes linewidth. We will show at the end of this paper that this limit ranges in the sub-Hertz domain. Let us finally mention that we only consider situations in which the FSR of the ring laser is comparable to the width of the Brillouin gain curve. The intensity of the backscattered Stokes wave is then time independent [10] and the corresponding operating regime is called the ''Brillouin mirror'' [5].

The first step of our theoretical analysis consists in transforming the complex amplitudes to modulus-phase form. We then obtain

$$\partial_\tau A_p+\partial_\zeta A_p=-gA_aA_s\cos\theta, \tag{4a}$$

$$\partial_\tau A_s-\partial_\zeta A_s=gA_aA_p\cos\theta, \tag{4b}$$

$$(1/\beta_A)\partial_\tau A_a+A_a=A_pA_s\cos\theta, \tag{4c}$$

$$\partial_\tau\phi_p+\partial_\zeta\phi_p=-g(A_aA_s/A_p)\sin\theta, \tag{4d}$$

$$\partial_\tau\phi_s-\partial_\zeta\phi_s=-g(A_aA_p/A_s)\sin\theta, \tag{4e}$$

$$(1/\beta_A)\partial_\tau\phi_a=-(A_pA_s/A_a)\sin\theta, \tag{4f}$$

where $\theta(\zeta,\tau)=\phi_s(\zeta,\tau)+\phi_a(\zeta,\tau)-\phi_p(\zeta,\tau)$. A_i and $\phi_i(i=p,s,a)$ represent, respectively, the amplitudes and phases of the pump, Stokes, and acoustic waves. The SBS interaction is submitted to resonance and phase matching conditions imposing that θ is necessarily a weakly fluctuating variable. As the cavity is resonant, its average value is zero. This is illustrated by numerical simulations in which Eqs. (4) and (2) are integrated in the presence of phase noise generated by Eq. (3). The numerical procedure used is based on the method of characteristics described in Ref. [11]. As shown in Figs. 1(a) and 1(b), the phase of the acoustic wave nearly follows the same temporal evolution as the phase of the pump laser. The phase fluctuations of the Stokes wave are strongly correlated to that of the pump laser but they are also much weaker. In these conditions, the variable $\theta(\zeta=0,\tau)$ oscillates slightly around 0 [Fig. 1(b)]. The trigonometric functions appearing in Eqs. (4) can thus be expanded to the lowest order in θ. The equations for the amplitudes then become independent from that governing the evolution of the phases. Since the amplitude μ of the incident pump field is time independent, we can cancel all the time derivatives appearing in Eqs. (4a), (4b), and (4c) and the field amplitudes then depend only on ζ. This is confirmed by numerical simulations that show that the phase fluctuations of the pump source do not modify significantly the amplitudes of the three waves involved in the interaction. They always remain very close to the stationary longitudinal profiles presented in Fig. 1(c). With these approximations, the equations governing the spatiotemporal evolution of the phases finally become

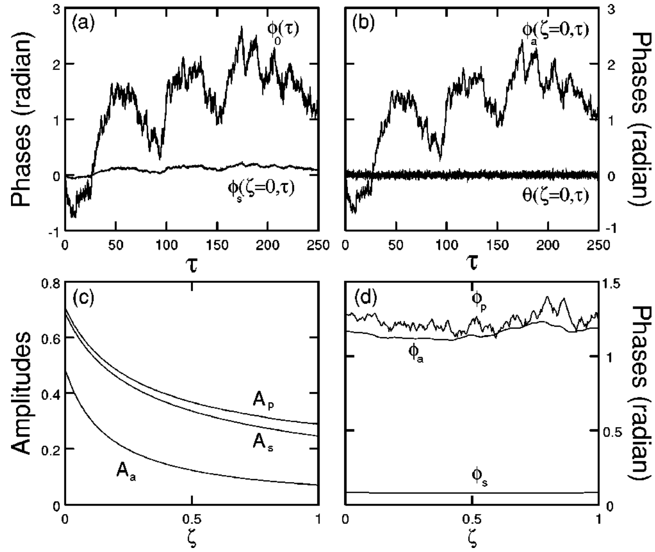

FIG. 1. Spatial profiles and temporal evolutions of the various phases and amplitudes (see the text). The parameters used in the simulations are those describing the laser of Ref. [12] ($g=6.04$, $\beta_A=10.93$, $R=0.36$, $\mu=0.7$).

$$\partial_\tau\phi_p+\partial_\zeta\phi_p=-gA_s^2(\zeta)\theta, \tag{5a}$$

$$\partial_\tau\phi_s-\partial_\zeta\phi_s=-gA_p^2(\zeta)\theta, \tag{5b}$$

$$\partial_\tau\phi_a=-\beta_A\theta. \tag{5c}$$

The terms appearing in the right-hand sides of Eqs. (5a) and (5b) are weak and comparable. However they do not play the same role in the two equations. Equation (5a) must satisfy the boundary condition $\phi_p(\zeta=0,\tau)=\phi_0(\tau)$. $\phi_0(\tau)$ acts as a source term inducing strong spatiotemporal variations of ϕ_p [see Figs. 1(a)and 1(d)]. In these conditions, the term $gA_s^2\theta$

is only perturbative and our last approximation consists in neglecting its influence. This means that the phase of the pump wave remains undisturbed in the interaction so that the solution of Eq. (5a) simply reads

$$\phi_p(\zeta,\tau)=\phi_0(\tau-\zeta). \qquad (6)$$

On the other hand, no source term appears in the boundary condition verified by the Stokes phase $[\phi_s(\zeta=1,\tau)=\phi_s(\zeta=0,\tau)]$ and the term $gA_p^2\theta$ is the source inducing the spatiotemporal variations of ϕ_s.

By taking into account the previous result [Eq. (6)] that has been verified by numerical simulations, our problem is reduced to the resolution of Eqs. (5b) and (5c) in the presence of the boundary condition characterizing the cavity resonance $[\phi_s(\zeta=1,\tau)=\phi_s(\zeta=0,\tau)]$. It can be solved analytically for any value of μ and therefore for any shape of the pump profile $A_p(\zeta)$. However, for the sake of simplicity and in order to get some insight on the physical mechanisms responsible for the effects presented in Fig. 1, we are first going to assume that the Brillouin laser does not operate very far from its threshold. In these conditions, the pump depletion effect can be neglected and the function A_p^2 can be approximated to $(-\ln R)/g$ [12]. Owing to the spatial periodicity imposed by the boundary condition $\phi_s(\zeta=1,\tau)=\phi_s(\zeta=0,\tau)$, the function $\phi_s(\zeta,\tau)$ can be decomposed as

$$\phi_s(\zeta,\tau)=\sum_{n=-\infty}^{+\infty} S_n(\tau)e^{ik_n\zeta} \qquad (7)$$

with $k_n=2\pi n$ and $S_n^\star(\tau)=S_{-n}(\tau)$. The problem is then solved by Fourier analysis and by using the orthonormality condition

$$\int_0^1 e^{ik_n\zeta}e^{-ik_m\zeta}d\zeta=\delta_{nm},$$

it can be shown that

$$\tilde{S}_0(\nu)=\frac{-\ln R}{\beta_A-\ln R+i2\pi\nu}\quad\frac{e^{-i\pi\nu}\sin\pi\nu}{\pi\nu}\quad\tilde{\phi}_0(\nu), \qquad (8)$$

where $\tilde{\phi}_0(\nu)$ and $\tilde{S}_0(\nu)$, respectively, represent the Fourier transforms of $\phi_0(\tau)$ and $S_0(\tau)$. Equation (8) shows that the incident phase noise is filtered by the association of two linear systems [14]. The first one is a low-pass filter that reduces the amplitude of the pump fluctuations by a factor $K=(\beta_A-\ln R)/(-\ln R)$ over a bandwidth equal to $(\beta_A-\ln R)/2\pi$ in units of cavity FSR. The second one is a system that smoothes the fluctuations by averaging them over a time interval $\Delta\tau=1$. As shown in Fig. 1(d), the function $\phi_s(\zeta=0,\tau)$ can be well approximated to $S_0(\tau)$ and although it has been established by neglecting the pump depletion, Eq. (8) describes qualitatively well the behavior reported in Fig. 1(a). The fluctuations of the Stokes phase are indeed much weaker than the phase fluctuations of the pump laser and the high frequencies found in $\phi_0(\tau)$ do not appear in $\phi_s(0,\tau)$. The optical coherence time, defined as the time it takes to the phase to diffuse over one radian on the average [13], is much greater for the Brillouin laser than for the pump laser.

If pump depletion is taken into account, $\tilde{S}_0(\nu)$ and $\tilde{\phi}_0(\nu)$ are no longer connected by a term that can be factorized into two parts. This term can be calculated by taking into account the expression of $A_p^2(\zeta)$ given in Ref. [12] but contrary to Eq. (8), the averaging effect does not clearly appear in its complicated analytical expression. However this term shows that the amplitude of the pump fluctuations is reduced by the factor K not only near the threshold but whatever the value of μ. As we are now going to see, this is the most important consideration for the evaluation of the Stokes linewidth.

The quantities of interest for the determination of the linewidths are the variances $\sigma_p^2(\tau)=\langle[\phi_0(\tau)-\phi_0(0)]^2\rangle$ and $\sigma_s^2(\tau)=\langle[\phi_s(0,\tau)-\phi_s(0,0)]^2\rangle$ [9]. The phase noise being not a stationary process, the brackets denote an ensemble average performed at the time τ. In our phase diffusing model, $\sigma_p^2(\tau)$ is a linear function of time $[\sigma_p^2(\tau)=2\pi\Delta\nu_p\tau]$ that measures the evolution of the dispersion of the values taken by the pump phase. As the phase fluctuations of the Stokes wave are much weaker than that of the pump wave, their dispersion is also much weaker. In fact, σ_p^2 and σ_s^2 are simply connected through the relation $\sigma_p^2=K^2\sigma_s^2$. Whatever the incident pump power, the FWHM of the Stokes spectrum is therefore given by

$$\Delta\nu_s=\frac{\Delta\nu_p}{K^2}. \qquad (9)$$

By returning to the physical variables [12], one can easily show that $K=1+\gamma_A/\Gamma_c$, where γ_A and Γ_c, respectively, represent the damping rate of the acoustic wave and the cavity loss rate. γ_A is equal to $\pi\Delta\nu_B$, where $\Delta\nu_B$ represents the FWHM of the Brillouin gain curve. Γ_c is equal to $-c\ \ln R/nL$ where c/n is the light velocity in the fiber of length L. Equation (9) represents the main result of this paper and the previous expression of the coefficient K clearly shows that the narrowing effect observed in Brillouin lasers is due to the combined influence of the acoustic damping and of the cavity feedback.

The analytical result given by Eq. (9) has been verified by the numerical integration of Eqs. (4) and (2) over an ensemble of 50 000 realizations of the random process $\phi_0(\tau)$. Such a statistical treatment permits us to determine the variances σ_p^2 and σ_s^2 at different times. The data thus obtained are then fitted to straight lines (see Fig. 2) and the ratio between the two slopes gives the numerical value of the coefficient K^2. For parameter values characterizing a Brillouin laser operating in a low-finesse resonator [7,11,12], the value of K^2 given by the analytical relation is 137 and that deduced from numerical simulations presented in Fig. 2 is 144. The slight relative difference between the two values arises from the fact that the function $\phi_s(\zeta,\tau)$ has been approximated to $S_0(\tau)$ and that the influence of the terms $S_n(\tau)$ $(n\neq 0)$ has been neglected.

If the linewidth of the pump laser is of the order of several tens of kHz, a Brillouin laser operating in a low-finesse cav-

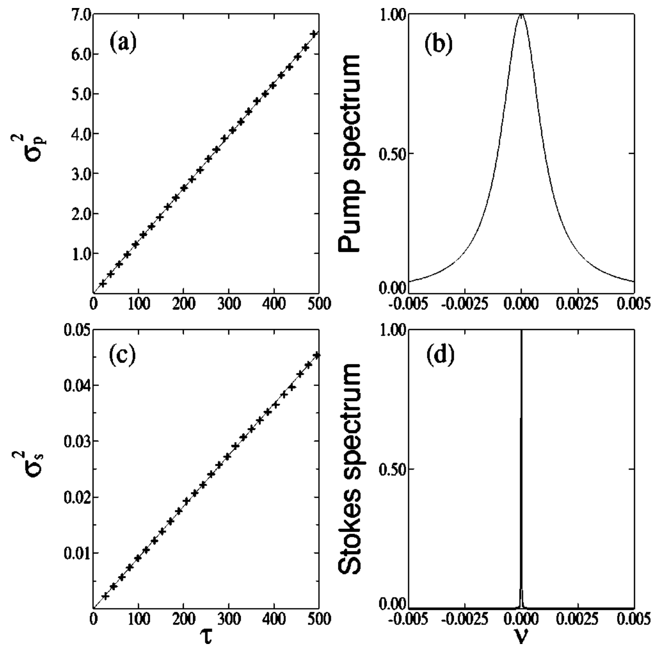

FIG. 2. Same parameters as in Fig. 1. Temporal evolution of the variance of the phase noise (diffusion process) and corresponding normalized Lorentzian spectrum of the (a),(b) incident pump field; (c),(d) Stokes field.

ity will emit a Stokes radiation with a linewidth of several hundreds of Hz. In an all-fiber ring resonator, a finesse greater than 100 can be easily achieved and the ratio γ_A/Γ_c is then typically of the order of 100. This means that the Stokes linewidth is at least 10^4 times narrower than the linewidth of the pump laser. This order of magnitude is effectively the one that has been experimentally measured in that kind of laser [1]. In these conditions, the linewidth of the Stokes radiation becomes of the order of a few Hz and the approximation that consists in neglecting spontaneous scattering must be questioned. Before analyzing the importance of this effect, let us first recall that spontaneous emission can be seen as a process that randomly perturbs the amplitude and the phase of the optical field emitted by a conventional laser [15]. By analogy with that description, spontaneous scattering can be considered as an effect perturbating the amplitude and the phase of the acoustic wave. In particular, it will lead to a diffusion of the phase of the acoustic wave. The influence of this diffusion process will become noticeable when it will be comparable to the diffusion process induced by the pump laser. As the phase of the acoustic wave nearly follows the same evolution than the phase of the pump laser [see Figs. 1(a) and 1(b)], spontaneous scattering cannot be neglected if $q(\tau)$ is of the same order of magnitude as the Langevin term $f(\zeta,\tau)$. The pump spectrum of Fig. 2(b) has been obtained for a value of $q(\tau)$ of the order of $10^{-2}(\langle q^2\rangle \simeq 10^{-4})$ and its dimensionless FWHM is 0.002. This corresponds to a pump linewidth of 40 kHz for a Brillouin laser characterized by a FSR of 20 MHz. As $f(\zeta,\tau)$ is of the order of 10^{-6}, spontaneous scattering is then effectively negligible. This is no longer the case for pump linewidths as narrow as 10 Hz and Stokes linewidths that fall in the sub-Hertz domain. Obviously, the previous analysis just allows us to give an idea of the order of magnitude of the effect. Additional studies must be undertaken to precisely characterize the influence of spontaneous scattering on the linewidth of the Brillouin laser.

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