Condensate fluctuations in finite Bose-Einstein condensates at finite temperature

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A Langevin equation for the complex amplitude of a single-mode Bose-Einstein condensate is derived. The equation is first formulated phenomenologically, defining three transport parameters. It is then also derived microscopically. Expressions for the transport parameters in the form of Green-Kubo formulas are thereby derived and evaluated for simple trap geometries, a cubic box with cyclic boundary conditions, and an isotropic parabolic trap. The number fluctuations in the condensate, their correlation time τ_c, and the temperature-dependent collapse time of the order parameter as well as its phase-diffusion coefficient are calculated.

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I. INTRODUCTION

Bose-Einstein condensation in a weakly interacting Bose gas in three dimensions in the thermodynamic limit of an infinitely extended system is a second-order phase transition in which an order parameter, the macroscopic wave function, appears spontaneously with a fixed but arbitrary phase, turning the global U(1) gauge symmetry connected with particle-number conservation into a spontaneously broken or hidden symmetry. The rigidity of the phase of the order parameter against local perturbations and the absence of any phase diffusion gives rise to the Goldstone modes, which take the form of collisionless (zero) sound or hydrodynamic sound, respectively, depending on whether the sound frequency is in the collisionless mean-field regime or in the collision-dominated regime [1].

In finite systems, and thus also in all trapped Bose gases, sharp phase-transitions are impossible, and hidden symmetries in a rigorous sense cannot appear. However, a macroscopic wave function describing a Bose-Einstein condensate still exists [2]. Its phase cannot be stable and must undergo a diffusion process, which restores the U(1) gauge symmetry over sufficiently long-time intervals. This diffusion process is different from the Goldstone modes mentioned previously, which are oscillations around a fixed value of the phase and do not restore the symmetry. Rather, the Goldstone modes show up either as collision-dominated hydrodynamic phonons or as collisionless phonons, which have also been observed in the finite Bose-Einstein condensates. In the present paper I would like to discuss the dynamics of the complex amplitude of a Bose-Einstein condensate containing a finite number of particles, and in particular analyze the diffusion of its phase. My discussion will extend and correct in several respects the work published in Ref. [3].

The stability of the phase difference between the macroscopic wave functions of two Bose-Einstein condensates in a trap has been measured. In the experimental setup [4] the relative phase was measured using a time-domain separated-oscillatory-field condensate interferometer. Over the time interval of 100 ms scanned in the experiment, the relative phase was found to be robust. This experimental result demonstrates that the macroscopic wave functions of the condensates cannot be considered as quantum-mechanical wave functions of many-particle systems entangled with each other, whose decoherence would indeed be extremely rapid. Rather, the macroscopic wave functions are appropriately viewed as robust classical objects, their quantum-mechanical origin (just like magnets, crystals, etc.) notwithstanding. This does, of course, not preclude that there may be quantum effects, for finite condensates, which lead to corrections of the dynamics described by the underlying classical wave equation, the well-known Gross-Pitaevskii equation [5]. In a number of papers [6] the dispersion of the phase of a trapped Bose-Einstein condensate at zero temperature was considered, which is due to fluctuations $\delta\mu$ of the chemical potential μ in a finite system with fixed particle number. An extension of this mechanism to finite temperature was also proposed [7]. This effect is not an irreversible phase diffusion, but corresponds to an effect of inhomogeneous broadening, similar to the dephasing of precessing spins occurring in spin systems due to inhomogeneous broadening. As the decay of the magnetization can be reversed in spin echos, the decay of the order-parameter expectation value in Bose-Einstein condensates due to a finite variance of $\delta\mu$ is in principle reversible in ''revivals.'' Experiments in Bose-Einstein condensation are done at temperature $k_BT \gg \hbar\bar{\omega}$ and often even at $k_BT \gg \mu$, where $\bar{\omega}$ is the geometrical mean of the three main trap frequencies. A phase-diffusion process should occur in such a regime due to the interaction of the condensate with a thermal bath of collective modes and quasiparticles. An estimate of this phase diffusion is of interest for the theory of atom lasers, because the fundamental limit of the linewidth of an atom laser for a given temperature depends on it similarly to the ''Schawlow-Townes'' formula [8] for the linewidth of a laser.

In this paper a theory of dissipation and thermal fluctuations of a trapped Bose-Einstein condensate will be formulated. First a phenomenological framework for the theory in the form of a Langevin equation will be given in which dissipation appears via a phenomenological parameter and the fluctuation-dissipation relation is invoked to relate it to three maximal intensity coefficients of the fluctuations. The solution of the Langevin equation then determines the relaxation of the condensate number and the diffusion of the phase, quite similar to the dynamics of a laser amplitude above threshold. Then the Langevin equation is derived from the microscopic theory, and formulas for the phenomenological parameters are derived. These are evaluated for a boxlike trap and an isotropic harmonic trap potential as a function of temperature, particle number, and scattering length. Section IX contains a discussion of our results, and a comparison with earlier related work. The theory presented here may not apply to the critical regime, nor can we examine here to what extent it covers the regime below but close to T_c, where it may be important to take in account the dynamics of the thermal cloud of noncondensed atoms, as well as the excitations from the condensate.

II. MICROSCOPIC EQUATIONS OF MOTION

A weakly interacting Bose gas in a trap in standard notation is described by the Hamiltonian

$$\hat{H}=\int d^3x\,\hat{\psi}^\dagger\left\{-\frac{\hbar^2}{2m}\nabla^2+V(\boldsymbol{x})-\langle\mu\rangle+\frac{U_0}{2}\hat{\psi}^\dagger\hat{\psi}\right\}\hat{\psi}. \tag{2.1}$$

The total number of atoms N is fixed, i.e., the Hilbert space is the restriction of the Fock space of $\hat{\psi}$ to the subspace on which $\hat{N}=N$ is satisfied. $\langle\mu\rangle$ is the average of the chemical potential, which is a fluctuating quantity in a system where N is fixed. Below we shall denote the fluctuating part of the chemical potential by $\Delta\mu$. The presence of a Bose-Einstein condensate in equilibrium means that many ($N_0\gg 1$) particles occupy a single mode of a macroscopic classical matter wave, determined as the mode of lowest energy of the classical Hamiltonian corresponding to Eq. (2.1). The latter is obtained by replacing the field operator $\hat{\psi}(\boldsymbol{x})$ in H by the classical field $\psi(\boldsymbol{x})=\sqrt{\langle N_0\rangle}\exp(i\phi)\tilde{\psi}_0(\boldsymbol{x})$. We shall restrict our attention to sufficiently low temperatures below the critical temperature T_c, so that the interaction of the condensate with the mean field of the thermal cloud of noncondensed particles is negligible. In this way one finds that the condensate mode $\tilde{\psi}_0(\boldsymbol{x})$, which we take to be normalized to 1, satisfies the Gross-Pitaevskii equation [5]

$$-(\hbar^2/2m)\nabla^2\tilde{\psi}_0+(V(\boldsymbol{x})+U_0\langle N_0\rangle|\tilde{\psi}_0(\boldsymbol{x})|^2)\tilde{\psi}_0=\langle\mu\rangle\tilde{\psi}_0. \tag{2.2}$$

For a given $\langle N_0\rangle$ the average value of the chemical potential μ follows by imposing the normalization condition

$$\int d^3x|\tilde{\psi}_0(\boldsymbol{x})|^2=1 \tag{2.3}$$

on the solution of the Gross-Pitaevskii equation, and thereby $\langle\mu\rangle$, like $\tilde{\psi}_0(\boldsymbol{x})$, becomes a function of the mean atom number in the condensate $\langle N_0\rangle$. As an important consequence of this fact the chemical potential of the system can be expressed as a function of the average number of atoms in the condensate alone. $\langle N_0\rangle$ differs from N, the fixed total number of atoms, by the average number $\langle N'\rangle$ of noncondensed atoms, which needs to be calculated for a given $\langle N_0\rangle$. The condition $N=\langle N_0\rangle+\langle N'\rangle$ then fixes $\langle N_0\rangle$ self-consistently. In experimentally realized Bose-Einstein condensates, it is possible to measure $\langle N_0\rangle$ directly with reasonable accuracy as a function of temperature, and in practice it is therefore reasonable to regard $\langle N_0\rangle$ as an experimentally given and known function of temperature. The space-dependent mean number density of the condensate is $n_0(\boldsymbol{x})=\langle N_0\rangle|\tilde{\psi}_0(\boldsymbol{x})|^2$. We shall take the mode function $\tilde{\psi}_0(x)$ in the Gross-Pitaevskii equation to be real and positive. (This also means we are not considering condensates containing vortices.) The physical phase of the condensate is not carried by its mode function $\tilde{\psi}_0$, but by its complex amplitude denoted as α_0, where $\alpha_0=\sqrt{N_0}\exp i\phi$.

If $|\alpha_0|^2$ makes a small fluctuation away from its equilibrium value $\langle N_0\rangle$ the condensate mode function ψ_0 will no longer satisfy Eq. (2.2), but will change its form slightly. We shall assume that such fluctuations of $N_0=|\alpha_0|^2$ occur on a sufficiently large time scale that the new form is again determined by the Gross-Pitaevskii equation, but for the changed condensate number $|\alpha_0|^2$ and a correspondingly changed chemical potential μ_0 determined uniquely by $|\alpha_0|^2$; i.e., in Eq. (2.2) the replacements $(\tilde{\psi}_0,\langle N_0\rangle,\langle\mu\rangle)\rightarrow(\psi_0,|\alpha_0|^2,\mu_0)$ have to be made in this case:

$$-(\hbar^2/2m)\nabla^2\psi_0+(V(\boldsymbol{x})+U_0|\alpha_0|^2|\psi_0(\boldsymbol{x})|^2)\psi_0=\mu_0\psi_0. \tag{2.4}$$

We cannot expect, in general, that in any given nonequilibrium state the difference defined by $\Delta_0\mu=\mu_0-\langle\mu\rangle$ is the *total* deviation of the chemical potential from its equilibrium value, because there may obviously be states with $|\alpha_0|^2=\langle N_0\rangle$ which differ in other respects from the equilibrium state and may therefore have $\mu\neq\langle\mu\rangle$. Therefore, we use the notation μ_0 for the part of the nonequilibrium chemical potential determined by $|\alpha_0|^2$.

The presence of the highly occupied condensate mode makes the decomposition of the Heisenberg field operator

$$\hat{\psi}(\boldsymbol{x},t)=(|\alpha_0|\exp(i\phi)\psi_0(\boldsymbol{x})+\hat{\chi}(\boldsymbol{x},t))\exp(-i\langle\mu\rangle t/\hbar) \tag{2.5}$$

useful, where we follow Bogoliubov [9] and describe the condensate classically. $\hat{\chi}(\boldsymbol{x},t)$ is taken to be the field operator for the particles outside the condensate. We shall assume that the temporal changes in ϕ can be considered as slow on the time scales of the dynamics of $\hat{\chi}$. The phase ϕ and amplitude $|\alpha_0|$ are additional c-number variables in Eq. (2.5). Therefore, the taking of expectation values from now has on to include an integration over a distribution of $|\alpha_0|$, and in addition an integration over all values of ϕ. Since the total number N is fixed, $\langle\hat{\psi}\rangle=0$ must hold for all times. However, it will also be useful to consider expectation values in the Fock space of the operators $\hat{\chi}$ and $\hat{\chi}^+$ alone without averaging over ϕ. Such expectation values will be denoted as $\langle\cdots\rangle_\phi$.

Gauge invariance, strictly speaking, is lost by splitting off a c-number term from the field operator. However, this symmetry is saved by adopting the rule that the phase ϕ of the c-number term in the decomposition also changes under a gauge transformation according to $\phi\rightarrow\phi+\epsilon$. By this device we take into account the fact that the same change of phase would have occurred automatically, if we had not replaced the condensate term by a c number. The generator of gauge transformations is thus taken as

$$\hat{N}=i\frac{\partial}{\partial\phi}|_{\hat{\chi},\hat{\chi}^\dagger}+\int d^3x\hat{\chi}^\dagger\hat{\chi}, \tag{2.6}$$

from which it is clear [cf. Eq. (2.24)] that $i(\partial/\partial\phi)|_{\hat{\chi},\hat{\chi}^\dagger}$ is a representation of $\hat{N}_0$.[1] The canonical conjugate is the phase $\hat{\phi}$ with

$$\exp(i\hat{\phi})=\exp(\partial/\partial N_0)_{\hat{\chi},\hat{\chi}^\dagger}. \tag{2.7}$$

Via Eq. (2.5), the Hamiltonian furthermore splits up according to $\hat{H}=H_0+\hat{H}_1+\hat{H}_2+\hat{H}_3+\hat{H}_4$, where $\hat{H}_n$ comprises the terms of $\hat{H}$ which are of nth order in $\hat{\chi}$ and $\hat{\chi}^\dagger$. Explicitly,

$$H_0=|\alpha_0|^2\int d^3x\psi_0\left\{-\frac{\hbar^2}{2m}\nabla^2+V(\boldsymbol{x})-\mu_0+\frac{U_0}{2}|\alpha_0|^2|\psi_0|^2\right\}\psi_0+(\mu_0-\langle\mu\rangle)|\alpha_0|^2, \tag{2.8}$$

$$\hat{H}_1=|\alpha_0|\int d^3x\left\{e^{-i\phi}\hat{\chi}\left(-\frac{\hbar^2}{2m}\nabla^2+V(\boldsymbol{x})-\langle\mu\rangle+U_0|\alpha_0|^2\psi_0^2\right)\psi_0+(\text{H.c.})\right\}, \tag{2.9}$$

$$\hat{H}_2=\int d^3x\left\{\hat{\chi}^\dagger\left(-\frac{\hbar^2}{2m}\nabla^2+V(\boldsymbol{x})-\mu_0\right)\hat{\chi}+\frac{U_0}{2}|\alpha_0|^2\psi_0^2(e^{-2i\phi}\hat{\chi}^2+e^{2i\phi}\hat{\chi}^{+2}+4\hat{\chi}^\dagger\hat{\chi})+(\mu_0-\langle\mu\rangle)\hat{\chi}^\dagger\hat{\chi}\right\}, \tag{2.10}$$

$$\hat{H}_3=U_0|\alpha_0|\int d^3x\psi_0\hat{\chi}^\dagger(e^{-i\phi}\hat{\chi}+e^{i\phi}\hat{\chi}^\dagger)\hat{\chi}, \tag{2.11}$$

$$\hat{H}_4=\frac{U_0}{2}\int d^3x\hat{\chi}^\dagger\hat{\chi}^\dagger\hat{\chi}\hat{\chi}. \tag{2.12}$$

Using the Gross-Pitaevskii equation (2.4) and its derivative with respect to $|\alpha_0|^2$, in the Appendix we derive

$$H_0=\int_{\langle N_0\rangle}^{|\alpha_0|^2}dN_0(\mu_0(N_0)-\langle\mu\rangle). \tag{2.13}$$

The term $\hat{H}_1$ can be simplified using the Gross-Pitaevskii equation (2.4), and then becomes

$$\hat{H}_1=|\alpha_0|(\mu_0-\langle\mu\rangle)\int d^3x(e^{-i\phi}\hat{\chi}+e^{i\phi}\hat{\chi}^\dagger)\psi_0. \tag{2.14}$$

This expression will be seen to vanish below due to an orthogonality condition.

The first part of $\hat{H}_2$ in Eq. (2.10) is diagonalized by introducing quasiparticle operators $\hat{\alpha}_\nu,\hat{\alpha}_\nu^\dagger$ defined by the standard Bogoliubov transformation, with time-dependent $\phi(t)$:

$$\hat{\chi}(\boldsymbol{x})=e^{i\phi}\sum_\nu (u_\nu(\boldsymbol{x})\hat{\alpha}_\nu+v_\nu^*(\boldsymbol{x})\hat{\alpha}_\nu^\dagger). \tag{2.15}$$

u_ν and v_ν satisfy the usual Bogoliubov-Fetter equations

$$\begin{pmatrix}-\frac{\hbar^2}{2m}\nabla^2+U_{\text{eff}}(\boldsymbol{x})-\hbar\omega_\nu & K(\boldsymbol{x})\\ K(\boldsymbol{x}) & -\frac{\hbar^2}{2m}\nabla^2+U_{\text{eff}}(\boldsymbol{x})+\hbar\omega_\nu\end{pmatrix}\times\begin{pmatrix}u_\nu(\boldsymbol{x})\\ v_\nu(\boldsymbol{x})\end{pmatrix}=0, \tag{2.16}$$

with the abbreviations

$$\begin{aligned}U_{\text{eff}}(\boldsymbol{x})&=V(\boldsymbol{x})-\mu_0+2U_0|\alpha_0|^2\psi_0(\boldsymbol{x})^2,\\ K(\boldsymbol{x})&=|\alpha_0|^2U_0\psi_0(\boldsymbol{x})^2.\end{aligned} \tag{2.17}$$

The Hamiltonian $\hat{H}_2$ now takes the form

$$\hat{H}_2=\sum_\nu \hbar\omega_\nu(\hat{\alpha}_\nu^\dagger\hat{\alpha}_\nu+|v_\nu|^2)+(\mu_0-\langle\mu\rangle+\hbar\dot{\phi})\int d^3x\hat{\chi}^\dagger\hat{\chi}. \tag{2.18}$$

The coefficients u_ν and v_ν and the mode frequencies ω_ν also become functions of $|\alpha_0|$, and fluctuate (slowly) with that number. Their equilibrium values will be denoted by $\tilde{u}_\nu$, $\tilde{v}_\nu$, and $\tilde{\omega}_\nu$, and the corresponding operator $\hat{\chi}$ according to Eq. (2.15), as $\tilde{\hat{\chi}}$.

Equation (2.16) is consistent with the orthonormality conditions

$$\int d^3x(u_\nu u_\mu^*-v_\nu v_\mu^*)=\delta_{\nu\mu}, \tag{2.19}$$

$$\int d^3r(u_\nu^*v_\mu-u_\mu^*v_\nu)=0, \tag{2.20}$$

which guarantee the Bose commutation relations of the α_ν and $\alpha_\mu^\dagger$. A formal solution of Eq. (2.16) at zero energy $\hbar\omega_\nu=0$ is given by the condensate

$$u_\nu(\boldsymbol{x})=-v_\nu^*(\boldsymbol{x})=\psi_0(\boldsymbol{x}),\quad \omega_\nu=0. \tag{2.21}$$

[1]This operator with fixed $\hat{\chi},\hat{\chi}^\dagger$ has to be well distinguished from the *unrestricted* derivative operator $i(\partial/\partial\phi)$, which is a representation of the *total* particle number N and has as formal canonical-conjugate $\hat{\phi}$ with $\exp(i\hat{\phi})=\exp(\partial/\partial N)$.

This solution is obviously not normalizable in the required sense [Eqs. (2.19)] to furnish an acceptable solution for u_ν and v_ν, and must therefore be excluded from the sum over the terms containing the operators $\hat{\alpha}_\nu$ and $\hat{\alpha}_\nu^\dagger$. The existence of this formal solution implies, however, that the properly normalizable solutions u_ν and v_ν and the condensate mode ψ_0 satisfy the important orthogonality relation

$$\int d^3x \psi_0(u_\nu+v_\nu)=0. \tag{2.22}$$

It follows from Eq. (2.15) that

$$\int d^3x \psi_0(e^{-i\phi}\hat{\chi}+e^{i\phi}\hat{\chi}^+)=0, \tag{2.23}$$

which in turn implies that the reduced expression (2.14) for $\hat{H}_1$ vanishes. Using property (2.23), one can verify that the decomposition [Eq. (2.5)] of $\hat{\psi}$ implies

$$N=|\alpha_0|^2+\hat{N}', \tag{2.24}$$

with

$$\begin{aligned}\hat{N}'=&\int d^3x\hat{\chi}^\dagger(\boldsymbol{x})\hat{\chi}(\boldsymbol{x})\\ =&\sum_{\nu,\mu}\int d^3x\bigg(\hat{\alpha}_\nu^\dagger\hat{\alpha}_\mu(u_\nu^*u_\mu+v_\nu^*v_\mu)\\ &+\frac{1}{2}\hat{\alpha}_\nu\hat{\alpha}_\mu(u_\nu v_\mu+v_\nu u_\mu)\\ &+\frac{1}{2}\hat{\alpha}_\nu^\dagger\hat{\alpha}_\mu^\dagger(u_\nu^*v_\mu^*+v_\nu^*u_\mu^*)+\delta_{\nu\mu}|v_\nu|^2\bigg),\end{aligned} \tag{2.25}$$

which serves as a definition of $N_0=|\alpha_0|^2$. The mean thermal density n' in equilibrium can now be determined via

$$\begin{aligned}n'(\boldsymbol{x})=&\langle\hat{\tilde{\chi}}^\dagger(\boldsymbol{x})\hat{\tilde{\chi}}(\boldsymbol{x})\rangle\\ =&\sum_\nu\{(|\tilde{u}_\nu(\boldsymbol{x})|^2+|\tilde{v}_\nu(\boldsymbol{x})|^2)\overline{n}_\nu+|\tilde{v}_\nu(\boldsymbol{x})|^2\},\end{aligned} \tag{2.26}$$

with $\overline{n}_\nu=(\exp(\beta\hbar\tilde{\omega}_\nu)-1)^{-1}$.

The fluctuations of N_0 are similarly fixed by

$$\langle\Delta N_0^2\rangle=\langle N_0^2\rangle-\langle N_0\rangle^2=\langle\Delta\hat{N}'^2\rangle \tag{2.27}$$

$$\begin{aligned}=&\sum_\nu\sum_{\nu'}\bigg\{\overline{n}_\nu(\overline{n}_{\nu'}+1)\bigg|\int d^3x(\tilde{u}_\nu^*(x)\tilde{u}_{\nu'}(x)\\ &+\tilde{v}_{\nu'}(x)\tilde{v}_\nu^*(x))\bigg|^2+\bigg(\overline{n}_\nu\overline{n}_{\nu'}+\frac{1}{2}(\overline{n}_\nu+\overline{n}_{\nu'}+1)\bigg)\\ &\times\bigg|\int d^3x(\tilde{u}_\nu(x)\tilde{v}_{\nu'}(x)+\tilde{u}_{\nu'}(x)\tilde{v}_\nu(x))\bigg|^2\bigg\}.\end{aligned} \tag{2.28}$$

They were evaluated in Ref. [10(a)] and very generally in [10(b)], and are also needed below [see Eq. (8.20)]. For work in the mathematical physics literature on number fluctuations in the condensate of the ideal Bose gas and models of the interacting Bose gas, see Refs. [11,12], and references given there. For an alternative proposal to define and calculate the number fluctuations in a Bose condensate, see Ref. [13].

After transformation (2.15), the Hamiltonian is now in the form

$$\hat{H}=H_0+\hat{H}_2+\hat{H}_3+\hat{H}_4, \tag{2.29}$$

with H_0, $\hat{H}_2$, $\hat{H}_3$, and $\hat{H}_4$ given by Eqs. (2.13), (2.18), (2.11), and (2.12).

III. LANGEVIN EQUATION OF THE CONDENSATE AMPLITUDE

Neither the Gross-Pitaevskii equation nor the Bogoliubov-Fetter equations furnish an equation for the condensate amplitude $\alpha_0=\sqrt{N_0}\exp i\phi$. To find such an equation phenomenologically we first turn to a macroscopic quantity like the entropy $S(|\alpha_0|^2,N)$ for a fixed particle number N, but restricted to a fixed arbitrary value of $\alpha_0=\sqrt{N_0}\exp(i\phi)$, where N_0 is the instantaneous number of particles in the condensate and different from the equilibrium value $\langle N_0\rangle$ corresponding to the maximum of $S(|\alpha_0|^2,N)$. Thus $\langle N_0\rangle$ is a function of N. The fluctuations of N_0 in the closed system formed by the trapped condensate after the evaporative cooling has been switched off are governed by a canonical Boltzmann-Einstein distribution

$$P(N_0)=\Omega^{-1}\exp(S(|\alpha_0|^2,N)/k_B).$$

We shall restrict ourselves to temperatures in the condensed regime outside the critical regime, where $\langle N_0(N)\rangle$ is much larger than its root mean square $\sqrt{\langle\Delta N_0^2(N)\rangle}=\sqrt{\langle\Delta\hat{N}'^2\rangle}=(\langle\hat{N}'^2\rangle-\langle\hat{N}'\rangle^2)^{1/2}$, which is also a function of N. Then $S(|\alpha_0|^2,N)$, expanded to lowest order around its maximum, takes the form

$$S(|\alpha_0|^2,N)=S^{(eq)}(N)+\Delta S(|\alpha_0|^2,N),$$

with

$$\Delta S(|\alpha_0|^2,N)=-k_B\frac{(|\alpha_0|^2-\langle N_0\rangle)^2}{2\langle\Delta N_0^2\rangle}. \tag{3.1}$$

The entropy $S(|\alpha_0|^2,N)$ not only determines the equilibrium distribution of the condensate amplitude, but also appears in its equation of motion, both in the conservative part of the dynamics as a conserved quantity, and in the dissipative part as a potential for the irreversible part of the dynamics. Let us first consider both parts separately.

The conservative part of the dynamics of α_0 is connected with the dynamics of its phase ϕ. According to Eqs. (2.5) and (2.15), a change of ϕ changes the total phase of the field operator $\hat{\psi}$. For this reason the dynamics of ϕ is given by the equation of motion

$$\dot{\phi} = -\frac{1}{\hbar}\frac{\partial\langle\hat{H}\rangle}{\partial N} = -\frac{1}{\hbar}\Delta\mu, \tag{3.2}$$

where $\Delta\mu$ is the deviation of the chemical potential from its equilibrium value. Such deviations may occur as a result of any fluctuations present in the system and, as discussed already, may in particular occur as a result of fluctuations of the value of N_0 away from its average $\langle N_0\rangle$. This part of the fluctuation of μ we shall denote as $\Delta_0\mu$. Expanding again to lowest order around the equilibrium $N_0 = \langle N_0\rangle$ we can write

$$\Delta_0\mu = \frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}(|\alpha_0|^2 - \langle N_0\rangle). \tag{3.3}$$

The systematic part of the conservative part of the equation of motion of α_0 can now be written in the form

$$(i\hbar\dot{\alpha}_0)_{cons} = \Delta_0\mu\,\alpha_0. \tag{3.4}$$

It is convenient to introduce the fluctuation of the free energy by $\Delta F = -T\Delta S$. The dynamics equation (3.4) conserves $|\alpha_0|^2$ and ΔF. In equilibrium the right-hand side of this equation vanishes, because there $\Delta_0\mu = 0$, and the total phase of the condensate $\phi - \langle\mu\rangle t/\hbar$ changes only with a rate given by the *average* chemical potential $\langle\mu\rangle$ in equilibrium.

The dissipative part of the equation of motion of α_0 near thermal equilibrium is written with the help of ΔF in the general form

$$\hbar(\dot{\alpha}_0)_{diss} = -\Gamma_0\frac{\partial\Delta F(|\alpha_0|^2,N)}{\partial\alpha_0^*}, \tag{3.5}$$

which contains the positive phenomenological parameter Γ_0 and describes the relaxation of $N_0 = |\alpha_0|^2$ to its equilibrium value $\langle N_0\rangle$.

According to general principles of statistical thermodynamics [14] the relaxation process (3.5) must be accompanied by some form of noise. Adding a noise-term the total Langevin equation of α_0 can be written in the form

$$i\hbar\dot{\alpha}_0 = \Delta_0\mu\,\alpha_0 - i\Gamma_0\frac{\partial\Delta F(|\alpha_0|^2,N)}{\partial\alpha_0^*} + \xi(t)\exp(i\phi). \tag{3.6}$$

Since the condensate amplitude α_0 is a collective quantity the noise $\xi(t)$ can be assumed to be Gaussian due to the central limit theorem. In addition we shall assume $\xi(t)$ to be a white-noise force. This means that the actual correlation time τ_m of the noise ξ is assumed to be much smaller than the time scale on which the dynamics of α_0 is observed, an assumption which must be checked for its validity in any concrete microscopic description. (In the microscopic theory we describe later it is a consistent assumption because the relaxation rate γ_c of $|\alpha_0|^2$ turns out to be small compared to the time scale of motion in the trap.) Thus we assume that $\langle\xi(t)\rangle = 0$ and

$$\langle\xi^*(t)\xi(0)\rangle = \hbar k_B T(2\Gamma_0 + \Gamma')\delta(t), \tag{3.7}$$

$$\langle\xi(t)\xi(0)\rangle = \hbar k_B T(\Gamma' + i\Gamma'')\delta(t), \tag{3.8}$$

where Γ_0 reappears in Eq. (3.7) because of the fluctuation-dissipation theorem. The form of the Langevin equation (3.6) generalizes the work in Ref. [3] by taking into account a possible correlation of the phase of the condensate and of the Langevin force, which may exist in condensates with finite particle numbers due to gauge invariance, i.e., particle-number conservation. (However, it will turn out later that the coefficient Γ'' vanishes in condensates with a real condensate mode, i.e., without vortices, which can be understood generally as a consequence of time-reversal symmetry.) Gauge invariance implies that the Langevin equation for α_0, including the fluctuating term, must be invariant under the transformation $\phi \to \phi + \epsilon$. This makes it useful to write the fluctuating term as $\exp(i\phi(t))\xi(t)$, where $\xi(t)$ is a complex noise source *independent* of ϕ, which, physically, describes the scattering of particles in the condensate with those outside.[2] The coefficients Γ', and Γ'' describe a possible correlation of the phases of $F_0 = \xi\exp(i\phi)$ and α_0, i.e., the existence of a squeezing in the thermal bath of uncondensed particles, caused by the constraint of total particle-number conservation. We shall see that this effect actually does occur in finite condensates, i.e., the condensate mode imprints its (slowly) fluctuating phase on the noncondensed ''environment'' due to particle number conservation in such a way that the lowest-lying modes are nearly *totally* squeezed.

The multiplicative nature of the noise in Eq. (3.6) raises the question in which stochastic calculus this equation should be interpreted: in the sense of Ito, or Stratonovich, or in some intermediate sense? This will be specified in a moment. Within the Gauss-Markoff assumption, the form of the noise force with the same positive coefficient $\Gamma_0 \geq 0$ appearing in the dissipative part [Eq. (3.5)], and two further real coefficients Γ' and Γ'', is fixed by the requirement that the Langevin equation must be consistent with the correct equilibrium distribution [14] $P(\alpha_0,\alpha_0^*) = Z^{-1}\exp(-\Delta F(|\alpha_0|^2, N)/k_BT)$ for the condensate. Splitting into real and imaginary parts Eqs. (3.7) and (3.8) become

$$\langle\mathrm{Re}(\xi(t))\mathrm{Re}(\xi(0))\rangle = \hbar k_B T(\Gamma_0 + \Gamma')\delta(t), \tag{3.9}$$

$$\langle\mathrm{Im}(\xi(t))\mathrm{Im}(\xi(0))\rangle = \hbar k_B T\Gamma_0\delta(t), \tag{3.10}$$

$$\langle\mathrm{Re}(\xi(t))\mathrm{Im}(\xi(0))\rangle = \frac{1}{2}\hbar k_B T\Gamma''\delta(t). \tag{3.11}$$

[2]A simpler ansatz (see Ref. [3]) ignores the ϕ dependence of the Langevin force in Eq. (3.6). Then the gauge invariance of the Fokker-Planck equation which is stochastically equivalent to the Langevin equation implies $\Gamma' = \Gamma'' = 0$, i.e., the complex noise $F_0(t)$ then has random-phase fluctuations which are completely uncorrelated with and equidistributed with respect to the condensate phase ϕ. Note, however, that this achieves gauge invariance only in an ensemble sense, not for each individual stochastic physical realization which together form the ensemble. In contrast, the form of the Langevin equation considered here does enforce gauge invariance for each stochastic realization.

Equation (3.6) may now be rewritten

$$\frac{\partial N_0}{\partial t} = -2\frac{\Gamma_0}{\hbar}\left(N_0\frac{\partial \Delta F}{\partial N_0} - k_B T\right) + \frac{2}{\hbar}\sqrt{N_0}\,\mathrm{Im}(\xi(t)), \tag{3.12}$$

$$\frac{\partial \phi}{\partial t} = -\frac{1}{\hbar}\Delta_0\mu - \frac{1}{\hbar\sqrt{N_0}}\mathrm{Re}(\xi(t)), \tag{3.13}$$

and must in this form be interpreted as a stochastic differential equation in the sense of Ito.[3]

Equation (3.13) can be compared with Eq. (3.2). This comparison reveals that $\mathrm{Re}(\xi(t))$ must describe the fluctuations of the chemical potential *not* caused by deviations of $|\alpha_0|^2$ from its equilibrium value, but by other fluctuations in the system. We shall come back to this point in Sec. VI below.

The three phenomenological coefficients Γ_0, Γ', and Γ'' are dimensionless, temperature-dependent numbers, which must be determined from a microscopic theory. Only one of these coefficients Γ_0 is connected with the fluctuations of the number of condensed atoms. If fluctuations of the chemical potential due to other processes are neglected, i.e., $\mathrm{Re}\,\xi(t)=0$, the remaining two coefficients are fixed at

$$\Gamma' = -\Gamma_0, \quad \Gamma''=0. \tag{3.14}$$

This corresponds to the case of maximal squeezing of the noise in the phase direction.

Linearizing with respect to the small fluctuations $\delta N_0 \ll \langle N_0\rangle$, we find

$$\hbar\,\delta\dot{N}_0 = -\frac{2k_BT}{\langle \Delta N_0^2\rangle}\langle N_0\rangle\Gamma_0\,\delta N_0 + 2\sqrt{\langle N_0\rangle}\mathrm{Im}(\xi(t)), \tag{3.15}$$

$$\hbar\dot{\phi} = -\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\delta N_0 - \frac{1}{\sqrt{\langle N_0\rangle}}\mathrm{Re}(\xi(t)). \tag{3.16}$$

Equation (3.15) describes the relaxation of the condensate to the equilibrium at $\langle N_0\rangle = \langle|\alpha_0|^2\rangle$ on the time scale

$$\tau_c = \frac{\hbar\langle\Delta N_0^2\rangle}{2\Gamma_0\langle N_0\rangle k_BT}, \tag{3.17}$$

and the thermal fluctuations around it with the correlation function

$$\langle\delta N_0(t)\,\delta N_0(t')\rangle = \langle\Delta N_0^2\rangle e^{-|t-t'|/\tau_c}. \tag{3.18}$$

[3]Then the Fokker-Planck equation corresponding to Eqs. (3.12) and (3.13) is $\hbar\partial P/\partial t = 2\Gamma_0\partial/\partial N_0[N_0(\partial\Delta F/\partial N_0 + k_BT\partial/\partial N_0)P]$, and has the desired equilibrium distribution $\sim\exp(-\Delta F/k_BT)$. This implies that Eq. (3.6) must be interpreted in some intermediate sense, which we need not specify here further. In order to obtain its version in the sense of Ito, it is best to bring Eqs. (3.12) and (3.13) into the form of Eq. (3.6) using the Ito calculus.

The correlation time τ_c is an important time scale of the problem. The noise sources $\mathrm{Im}(\xi(t))$,$\mathrm{Re}(\xi(t))$ must have correlation times short compared to τ_c in order to be well described by white noise.

On a time scale much larger than the correlation time τ_c the fluctuations $\delta N_0(t)$ in the equation for the phase can also be considered as Gaussian white noise, with a correlation function $2\tau_c\langle\Delta N_0^2\rangle\delta(t)$. Using this long-time approximation in the equation for the phase ϕ and taking the correlation of the effective white noise $\delta N_0(t)$ with $\mathrm{Re}(\xi(t))$ properly into account, $\phi(t)$ is found to satisfy the Langevin equation of a Wiener process,

$$d\phi(t) = \sqrt{D_\phi}dw,$$

with $(dw)^2 = dt$ and a diffusion constant

$$D_\phi = \frac{\langle\Delta N_0^2\rangle}{\hbar\langle N_0\rangle}\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\left(\frac{\langle\Delta N_0^2\rangle}{k_BT}\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\frac{1}{\Gamma_0} + \frac{\Gamma''}{\Gamma_0}\right) + \frac{k_BT}{\hbar\langle N_0\rangle}(\Gamma_0+\Gamma'), \tag{3.19}$$

i.e.,

$$\langle(\phi(t)-\phi(0))^2\rangle = D_\phi|t|. \tag{3.20}$$

Equation (3.19) agrees with the result of Ref. [3] if we assume, as in Ref. [3], that there is no squeezing in the noise, i.e., $\Gamma' = \Gamma'' = 0$ and $\partial\langle\mu\rangle/\partial\langle N_0\rangle = k_BT/\langle\Delta N_0^2\rangle^{-1}$. Both assumptions will not be made in the present work, however (cf. also the corresponding discussion in Sec. IX).

The expectation value $\langle\alpha_0(t)\rangle$ then decays exponentially according to $\langle\alpha_0(t)\rangle = \sqrt{\langle N_0\rangle}e^{-\Delta\nu|t|}$, with the linewidth $\Delta\nu$ given by the Schawlow-Townes-type formula $\Delta\nu = \frac{1}{2}D_\phi$.

It is not difficult to solve Eqs. (3.15) and (3.16) for the phase fluctuations on time scales of the order of τ_c. The result for the mean square of the phase increment in time is

$$\langle(\phi(t)-\phi(0))^2\rangle = D_\phi|t| + \frac{2}{\hbar^2}\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\left(k_BT\Gamma'' + \langle\Delta N_0^2\rangle\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\right) \times \tau_c^2(e^{-|t|/\tau_c}-1). \tag{3.21}$$

This interpolates between the diffusive long-time behavior [Eq. (3.20)] for $t \gg \tau_c$ and the short-time behavior for $t \ll \tau_c$:

$$\langle(\phi(t)-\phi(0))^2\rangle = \frac{k_BT}{\hbar\langle N_0\rangle}(\Gamma_0+\Gamma')|t| + \frac{1}{\hbar^2}\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\left(\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\langle\Delta N_0^2\rangle + k_BT\Gamma''\right)t^2. \tag{3.22}$$

The first term describes phase diffusion due to thermal fluctuations of the chemical potential on time scales much shorter than τ_c. The second term describes a nondiffusive and in principle reversible phase collapse [6,7], with the collapse rate

$$\gamma_{collapse}=\frac{1}{\hbar}\sqrt{\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\left(\frac{\partial\langle\mu\rangle}{\partial\langle N_0\rangle}\langle\Delta N_0^2\rangle+k_BT\Gamma''\right)}, \tag{3.23}$$

including a contribution from the cross-correlation between both types of fluctuations.

IV. MICROSCOPIC DERIVATION OF THE LANGEVIN EQUATION

The microscopic derivation of the equation of motion for the condensate amplitude α_0 can be carried out by using Hamiltonian (2.29). As we did for phenomenological equations in Sec. III, here we wish to derive the microscopic equation of motion only to first order in the deviation $(|\alpha_0| - \sqrt{\langle N_0\rangle})$ from equilibrium.

H_0, given by Eq. (2.13), is the Hamiltonian, in the mean-field approximation, of the pure condensate. Its free equation of motion is

$$i\hbar\dot{\alpha}_0=\frac{\partial H_0}{\partial\alpha_0^*}=(\mu_0-\langle\mu\rangle)\alpha_0, \tag{4.1}$$

from which

$$\frac{d\phi(t)}{dt}=-[\mu_0(|\alpha_0(t)|^2)-\langle\mu\rangle]/\hbar \tag{4.2}$$

follows. Let us first use Eq. (4.2) in Eq. (2.18) to simplify $\hat{H}_2$, and then eliminate $\hat{H}_2$ by proceeding to the Heisenberg picture with respect to it. This changes $\hat{\chi}$ and $\hat{\chi}^+$ in $\hat{H}_3$ and $\hat{H}_4$ according to

$$\hat{\chi}\to\hat{\chi}(t)=e^{i\phi(t)}\sum_\nu\;(u_\nu\hat{\alpha}_\nu e^{-i\omega_\nu t}+v_\nu^*\hat{\alpha}_\nu^\dagger e^{i\omega_\nu t}) \tag{4.3}$$

and its adjoint.[4] The transformed time-dependent Hamiltonians will be denoted as $\hat{H}_3(t)$ and $\hat{H}_4(t)$, but $\hat{H}_4(t)$ will not be needed in the following.

The equation of motion of the condensate amplitude α_0 now takes the forms,[5] with the notation $\hat{H}(t)=H_0+\hat{H}_3(t)+\hat{H}_4(t)$,

$$\hbar\frac{d\phi}{dt}=-\left(\frac{\partial\hat{H}(t)}{\partial|\alpha_0|^2}\right)_{\phi,\hat{\chi},\hat{\chi}^\dagger}, \tag{4.4}$$

$$\hbar\frac{d|\alpha_0|^2}{dt}=\left(\frac{\partial\hat{H}(t)}{\partial\phi}\right)_{|\alpha_0|,\hat{\chi},\hat{\chi}^\dagger}. \tag{4.5}$$

We obtain

$$\hbar\frac{d\phi}{dt}=-\Delta_0\mu-\frac{1}{\sqrt{\langle N_0\rangle}}\mathrm{Re}(\hat{\xi}'(t))-\frac{1}{\sqrt{\langle N_0\rangle}}\delta\,\mathrm{Re}(\hat{\xi}'(t)), \tag{4.6}$$

$$\hbar\frac{d|\alpha_0|^2}{dt}=2\sqrt{\langle N_0\rangle}\mathrm{Im}(\hat{\xi}(t))+2\sqrt{\langle N_0\rangle}\delta\,\mathrm{Im}(\hat{\xi}(t)), \tag{4.7}$$

with

$$\mathrm{Re}(\hat{\xi}'(t))=\frac{1}{2}U_0\int d^3x\left(\tilde{\psi}_0+2\langle N_0\rangle\frac{\partial\tilde{\psi}_0}{\partial\langle N_0\rangle}\right)\times\hat{\tilde{\chi}}^\dagger(t)(e^{-i\phi}\hat{\tilde{\chi}}(t)+e^{i\phi}\hat{\tilde{\chi}}^\dagger(t))\hat{\tilde{\chi}}(t), \tag{4.8}$$

$$\mathrm{Im}(\hat{\xi}(t))=\frac{1}{2i}U_0\int d^3x\tilde{\psi}_0\hat{\tilde{\chi}}^\dagger(t)(e^{-i\phi}\hat{\tilde{\chi}}(t)-e^{i\phi}\hat{\tilde{\chi}}^\dagger(t))\hat{\tilde{\chi}}(t). \tag{4.9}$$

The complex noise $\xi(t)$ in Eqs. (3.6)–(3.13) should be identified with

$$\hat{\xi}(t)=\mathrm{Re}(\hat{\xi}'(t))+i\,\mathrm{Im}(\hat{\xi}(t)). \tag{4.10}$$

It is indeed independent of ϕ, as required by the gauge invariance of the Langevin equation, as can be seen from Eqs. (4.6) and (4.7) with Eq. (4.3). We shall see in Sec. V that $\hat{\xi}'(t)$ can be replaced by a c number.

To describe fluctuations around equilibrium, in the preceding expressions we have replaced the quantities $|\alpha_0|^2$, ψ_0, $\hat{\chi}$, and $\hat{\chi}^\dagger$ by their equilibrium expressions $\sqrt{\langle N_0\rangle}$, $\tilde{\psi}_0$, $\hat{\tilde{\chi}}$ and $\hat{\tilde{\chi}}^\dagger$, and represented the difference in the nonequilibrium state by $\delta\,\mathrm{Re}(\hat{\xi}'(t))$ and $\delta\,\mathrm{Im}(\hat{\xi}(t))$ in Eqs. (4.6) and (4.7). Omitting these differences altogether amounts to neglecting the back action of the condensate on the thermal reservoir, which describes not only a modification of the fluctuating forces, which can indeed be neglected for fluctuations around a stable thermodynamic equilibrium, but also dissipation. To take the latter into account we need to calculate the averages $\delta\langle\delta\,\mathrm{Re}(\hat{\xi}'(t))\rangle_\phi, \delta\langle\delta\,\mathrm{Im}(\hat{\xi}(t))\rangle_\phi$ to lowest order in the interaction between the condensate and the thermal cloud of atoms. The form which these quantities must take is prescribed completely by the fluctuation-dissipation theorem and symmetry:

For the reversible phase dynamics the back action can only lead to a shift in the average chemical potential. Such shifts due to the interaction $\hat{H}_3$ will be small, and are neglected here. For the irreversible amplitude dynamics the fluctuation-dissipation theorem also requires the appearance of a dissipation term. If

$$S_{JJ}(t-t')=\langle\mathrm{Im}(\hat{\xi}(t))\mathrm{Im}(\hat{\xi}(t'))\rangle_\phi \tag{4.11}$$

[4]For simplicity, here we disregard the slow time dependence of the frequencies ω_ν.

[5]For the canonically conjugate pair N_0 and ϕ at fixed $\hat{\chi}$ and $\hat{\chi}^\dagger$, cf. Eqs. (2.6) and (2.7).

is the correlation function of the fluctuating force in Eq. (4.7), the back action must modify Eq. (4.7) to the form

$$\hbar \frac{d|\alpha_0|^2}{dt} = -\frac{4\langle N_0\rangle}{\hbar k_B T}\int_{-\infty}^{t} dt' S_{JJ}(t-t')\frac{\partial H_0(t')}{\partial|\alpha_0|^2} + 2\sqrt{\langle N_0\rangle}\mathrm{Im}(\hat{\xi}(t)). \tag{4.12}$$

The derivation of this equation is given in Appendix B. This stochastic differential equation still differs from the phenomenological equation (3.15) in two respects.

(i) The noise still has a finite correlation time τ_{mic}. We shall consider these correlation functions in more detail below. Taking the Markovian limit $\tau_{mic}\to 0$, with

$$S_{JJ}(t-t') = \hbar k_B T \Gamma_0 \delta(t-t'), \tag{4.13}$$

Eq. (4.12) becomes

$$\frac{d|\alpha_0|^2}{dt} = -2\frac{\Gamma_0}{\hbar}\langle N_0\rangle \frac{\partial H_0}{\partial|\alpha_0|^2} + \frac{2}{\hbar}\sqrt{\langle N_0\rangle}\mathrm{Im}(\hat{\xi}(t)). \tag{4.14}$$

(ii) The mean-field Hamiltonian $H_0(|\alpha_0|^2)$ appears in Eqs. (4.12) and (4.14) instead of the free energy $\Delta F(|\alpha_0|^2)$. This is due to the fact that the influence of the thermal excitations on the energy are not yet taken into account. Doing this under isothermal or closed-system boundary conditions we should replace the energy $H_0(|\alpha_0|^2)$ by the free energy $\Delta F(|\alpha_0|^2)$ or $-T\Delta S(|\alpha_0|^2)$, respectively.

This completes our derivation of the Langevin equation for the complex amplitude of the condensate.

V. GREEN-KUBO EXPRESSIONS FOR THE TRANSPORT COEFFICIENTS

Let us now analyze the fluctuating forces in more detail. Inserting the Bogoliubov transformation (4.3) into Eqs. (4.8) and (4.9) the fluctuating forces take the forms

$$\mathrm{Re}(\hat{\xi}'(t)) = \frac{1}{4}\sum_{\kappa\nu\mu}[((M'^{(1)}_{\kappa,\nu\mu})^* + M'^{(2)}_{\nu\mu,\kappa}) \hat{\alpha}^\dagger_\nu \hat{\alpha}^\dagger_\mu \hat{\alpha}_\kappa e^{-i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)t} + \text{H.c.}] + (\text{nonresonant terms}), \tag{5.1}$$

$$\mathrm{Im}(\hat{\xi}(t)) = \frac{1}{4i}\sum_{\kappa\nu\mu}[((M^{(1)}_{\kappa,\nu\mu})^* - M^{(2)}_{\nu\mu,\kappa}) \hat{\alpha}^\dagger_\nu \hat{\alpha}^\dagger_\mu \hat{\alpha}_\kappa e^{-i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)t} - \text{H.c.}] + (\text{nonresonant terms}). \tag{5.2}$$

Terms are called ‘‘nonresonant’’ if the frequencies of the quasiparticles cannot add up to zero. Such terms have not been written out explicitly, because below we shall restrict ourselves to the resonance or rotating wave approximation in which they do not contribute. The relevant matrix elements $M^{(1)}$ and $M^{(2)}$ are

$$M^{(1)}_{\kappa,\nu\mu} = 2U_0\int d^3x\,\tilde{\psi}_0\tilde{v}_\nu\left(\tilde{u}^*_\kappa\tilde{u}_\mu + \frac{1}{2}\tilde{v}^*_\kappa\tilde{v}_\mu\right) + (\nu\leftrightarrow\mu), \tag{5.3}$$

$$M^{(2)}_{\nu\mu,\kappa} = 2U_0\int d^3x\,\tilde{\psi}_0\tilde{u}^*_\nu\left(\tilde{v}^*_\mu\tilde{v}_\kappa + \frac{1}{2}\tilde{u}^*_\mu\tilde{u}_\kappa\right) + (\nu\leftrightarrow\mu),$$

and, very similarly,

$$M'^{(1)}_{\kappa,\nu\mu} = 2U_0\int d^3x\left(\tilde{\psi}_0 + 2\langle N_0\rangle\frac{\partial\tilde{\psi}_0}{\partial\langle N_0\rangle}\right) \times\tilde{v}_\nu\left(\tilde{u}^*_\kappa\tilde{u}_\mu + \frac{1}{2}\tilde{v}^*_\kappa\tilde{v}_\mu\right) + (\nu\leftrightarrow\mu), \tag{5.4}$$

$$M'^{(2)}_{\nu\mu,\kappa} = 2U_0\int d^3x\left(\tilde{\psi}_0 + 2\langle N_0\rangle\frac{\partial\tilde{\psi}_0}{\partial\langle N_0\rangle}\right) \times\tilde{u}^*_\nu\left(\tilde{v}^*_\mu\tilde{v}_\kappa + \frac{1}{2}\tilde{u}^*_\mu\tilde{u}_\kappa\right) + (\nu\leftrightarrow\mu).$$

The matrix-elements $M'^{(1)}$ and $M'^{(2)}$ coincide with $M^{(1)}$ and $M^{(2)}$ if the dependence of $\tilde{\psi}_0$ on $\langle N_0\rangle$ is negligible or vanishes, as, e.g., in homogeneous systems.

$M^{(1)}_{\kappa,\nu\mu}$, and similarly $M'^{(1)}_{\kappa,\nu\mu}$, describes a scattering process in which one atom is scattered out of the condensate by the absorption of the two quasiparticles ν and μ from—and the emission of the new quasiparticle κ into—the thermal bath. Likewise $M^{(2)}_{\nu\mu,\kappa}$, and similarly $M'^{(2)}_{\nu\mu,\kappa}$, describes a scattering process where an incoming thermal quasiparticle κ is absorbed, again an atom is kicked out from the condensate, and two quasiparticles ν and μ are emitted into the thermal bath. The scattering amplitudes for both processes are linearly superimposed due to the phase coherence of the condensate, which exists on the time scale of the relaxation process induced by the scattering process even if it is destroyed on a much longer time scale.

We can now calculate the correlation functions of the fluctuating forces. Their averages over the bath of quasiparticles vanish, $\langle\mathrm{Re}(\hat{\xi}(t))\rangle = 0 = \langle\mathrm{Im}(\hat{\xi}(t))\rangle$. Their second-order correlation functions are obtained as

$$\langle\mathrm{Re}(\hat{\xi}'(t))\mathrm{Re}(\hat{\xi}'(t'))\rangle_\phi = \frac{1}{8}\sum_{\kappa,\nu,\mu}|(M'^{(1)}_{\kappa,\nu\mu})^* + M'^{(2)}_{\nu\mu,\kappa}|^2 \times\{\overline{n}_\kappa(\overline{n}_\nu+1)(\overline{n}_\mu+1)e^{i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)(t-t')} + \overline{n}_\nu\overline{n}_\mu(\overline{n}_\kappa+1)e^{-i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)(t-t')}\}, \tag{5.5}$$

$$\langle \mathrm{Im}(\hat{\xi}(t))\mathrm{Im}(\hat{\xi}(t'))\rangle_\phi = \frac{1}{8}\sum_{\kappa,\nu,\mu} |(M^{(1)}_{\kappa,\nu\mu})^* - M^{(2)}_{\nu\mu,\kappa}|^2 \times \{\overline{n}_\kappa(\overline{n}_\nu+1)(\overline{n}_\mu+1)e^{i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)(t-t')} + \overline{n}_\nu\overline{n}_\mu(\overline{n}_\kappa+1)e^{-i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)(t-t')}\}, \quad (5.6)$$

$$\langle \mathrm{Re}(\hat{\xi}'(t))\mathrm{Im}(\hat{\xi}(t'))\rangle_\phi = \frac{1}{8i}\sum_{\kappa,\nu,\mu}\{((M^{(1)}_{\kappa,\nu\mu})^* - M^{(2)}_{\nu\mu,\kappa})(M'^{(1)}_{\kappa,\nu\mu} + (M'^{(2)}_{\nu\mu,\kappa})^*) \times(\overline{n}_\nu+1)(\overline{n}_\mu+1)\overline{n}_\kappa e^{i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)(t-t')} - (M^{(1)}_{\kappa,\nu\mu} - (M^{(2)}_{\nu\mu,\kappa})^*)((M'^{(1)}_{\kappa,\nu\mu})^* + M'^{(2)}_{\nu\mu,\kappa}) \times(\overline{n}_\kappa+1)\overline{n}_\nu\overline{n}_\mu e^{-i(\tilde{\omega}_\kappa-\tilde{\omega}_\nu-\tilde{\omega}_\mu)(t-t')}\}. \quad (5.7)$$

These correlation functions can be replaced by δ functions, provided that the frequency sums contain a flat quasicontinuum of nearly resonant terms in a neighborhood of the resonance $\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu = 0$ which is broad compared to the damping rates we calculate here. This assumption will be satisfied in sufficiently large condensates. The strengths of the δ functions can then be extracted from expressions (5.5), (5.6), and (5.7) by taking the time averages $\int_{-\infty}^{\infty} d(t-t')\langle \hat{\xi}(t)\hat{\xi}(t')\rangle_\phi$ and $\int_{-\infty}^{\infty} d(t-t')\langle \hat{\xi}^+(t)\hat{\xi}(t')\rangle_\phi$.

$\mathrm{Re}(\hat{\xi}'(t))$, and $\mathrm{Im}(\hat{\xi}(t))$ here are given as expressions involving *operators*. Provided the Markovian approximation is satisfied, the average of their commutators over the quasiparticle bath are again given by δ functions in time. Explicitly, for the coefficients of the δ functions we obtain

$$\int_{-\infty}^{\infty} dt \langle [\mathrm{Re}(\hat{\xi}'(t)), \mathrm{Re}(\hat{\xi}'(0))]\rangle_\phi = 0 = \int_{-\infty}^{\infty} dt\langle[\mathrm{Im}(\hat{\xi}(t)),\mathrm{Im}(\hat{\xi}(0))]\rangle_\phi,$$
$$\int_{-\infty}^{\infty} dt\langle[\mathrm{Re}(\hat{\xi}'(t)),\mathrm{Im}(\hat{\xi}(0))]\rangle_\phi = \frac{\pi}{8i}\sum_{\kappa,\nu,\mu}[((M^{(1)}_{\kappa,\nu\mu})^* - M^{(2)}_{\nu\mu,\kappa})(M'^{(1)}_{\kappa,\nu\mu} + (M'^{(2)}_{\nu\mu,\kappa})^*) - \mathrm{c.c.}]\{(\overline{n}_\nu+1)(\overline{n}_\mu+1)\overline{n}_\kappa - (\overline{n}_\kappa+1)\overline{n}_\nu\overline{n}_\mu\} \times \delta(\tilde{\omega}_\kappa - \tilde{\omega}_\nu - \tilde{\omega}_\mu) = 0. \quad (5.8)$$

It can easily be verified that the bracket $\{\cdots\}$ in the last line of Eq. (5.8) vanishes if it is multiplied by a δ function expressing energy conservation. As a result the fluctuating force $\hat{\xi}$ in the Markovian limit can indeed be treated as a c number, and will henceforth again be denoted by ξ. This also serves as a nice consistency-check that it is indeed possible to treat the condensate classically, even after taking its interaction with the quasiparticles into account.

Let us now proceed to derive formulas for the three transport parameters Γ_0, Γ', and Γ''. From

$$2\Gamma_0 + \Gamma' = \frac{1}{\hbar k_B T}\int_{-\infty}^{+\infty} dt \langle \xi^*(t)\xi(0)\rangle_\phi, \qquad \Gamma' + i\Gamma'' = \frac{1}{\hbar k_B T}\int_{-\infty}^{+\infty} dt\langle \xi(t)\xi(0)\rangle_\phi, \quad (5.9)$$

implied by Eqs. (3.7) and (3.8), we obtain the representations

$$\Gamma_0 = \frac{1}{\hbar k_B T}\int_{\infty}^{\infty} d(t-t')\langle \mathrm{Im}(\xi(t))\mathrm{Im}(\xi(t'))\rangle_\phi, \quad (5.10)$$

$$\Gamma_0 + \Gamma' = \frac{1}{\hbar k_B T}\int_{\infty}^{\infty} d(t-t')\langle \mathrm{Re}(\xi'(t))\mathrm{Re}(\xi'(t'))\rangle_\phi, \quad (5.11)$$

$$\Gamma'' = \frac{2}{\hbar k_B T}\int_{\infty}^{\infty} d(t-t')\langle \mathrm{Re}(\xi'(t))\mathrm{Im}(\xi(t'))\rangle_\phi, \quad (5.12)$$

which have the form of Green-Kubo relations for the transport coefficients. Using the explicit forms (5.1) and (5.2) of the fluctuating forces, the thermal averages can be taken, and the time-integrals in Eqs. (5.11), (5.10), and (5.12) can be carried out, which leads to the formulas

$$\Gamma_0 = \frac{\pi}{2\hbar k_B T}\sum_{\kappa,\nu,\mu} |(M^{(1)}_{\kappa,\nu\mu})^* - M^{(2)}_{\nu\mu,\kappa}|^2 \overline{n}_\nu\overline{n}_\mu(\overline{n}_\kappa+1) \times \delta(\tilde{\omega}_\kappa - \tilde{\omega}_\mu - \tilde{\omega}_\nu), \quad (5.13)$$

$$\Gamma_0 + \Gamma' = \frac{\pi}{2\hbar k_B T}\sum_{\kappa,\nu,\mu} |(M'^{(1)}_{\kappa,\nu\mu})^* + M'^{(2)}_{\nu\mu,\kappa}|^2 \overline{n}_\nu\overline{n}_\mu \times(\overline{n}_\kappa+1)\,\delta(\tilde{\omega}_\kappa - \tilde{\omega}_\mu - \tilde{\omega}_\nu), \quad (5.14)$$

$$\Gamma'' = \frac{-i\pi}{2\hbar k_B T}\sum_{\kappa,\nu,\mu}\{((M^{(1)}_{\kappa,\nu\mu})^* - M^{(2)}_{\nu\mu,\kappa})(M'^{(1)}_{\kappa,\nu\mu} + (M'^{(2)}_{\nu\mu,\kappa})^*) - (M^{(1)}_{\kappa,\nu\mu} - (M^{(2)}_{\nu\mu,\kappa})^*)((M'^{(1)}_{\kappa,\nu\mu})^* + M'^{(2)}_{\nu\mu,\kappa})\}\overline{n}_\nu\overline{n}_\mu(\overline{n}_\kappa+1)\,\delta(\tilde{\omega}_\kappa - \tilde{\omega}_\mu - \tilde{\omega}_\nu). \quad (5.15)$$

These expressions constitute our general results for the three transport parameters. They have to be evaluated separately for each individual trap geometry.

VI. RELATION TO THE FLUCTUATION AND DISSIPATION OF THE EXCITATIONS

As pointed out after Eq. (3.13), by phenomenological arguments, the noise term $\mathrm{Re}(\xi(t))$ is not connected with the fluctuations of the number of particles in the condensate, but must be due to other fluctuations, which are then necessarily thermal fluctuations of the amplitudes of the excited states. In our microscopic results this can be seen from the fact that the fluctuating force $\mathrm{Re}(\hat{\xi}'(t))$ according to Eq. (4.8) contains precisely the same operator which also appears in $\hat{H}_3(t)$, and couples the atoms in the thermal cloud to the condensate.

In the special case where the difference between the coupling matrix elements $M'^{(1,2)}$ and $M^{(1,2)}$ is negligible (which is exactly satisfied in boxlike traps, cf. Sec. VII) the intensity $\Gamma_0+\Gamma'$ of the noise source can be expressed entirely as a property of the excitations, as we shall now demonstrate.[6] For the amplitudes $\hat{\alpha}_\nu(t)$ and $\hat{\alpha}_\nu^+(t)$, a quantum-Langevin equation can be derived microscopically along the same lines employed here for the condensate amplitude. We have done this elsewhere [15] (also see Ref. [3]) with the result, in the Markovian limit,

$$\frac{d\hat{\alpha}_\nu(t)}{dt}=-i\omega_\nu\hat{\alpha}_\nu(t)-\gamma_\nu\hat{\alpha}_\nu(t)+\hat{\xi}_\nu(t), \tag{6.1}$$

with Gaussian fluctuating force operators with vanishing average and

$$\begin{aligned}\langle\hat{\xi}_\nu^+(t)\hat{\xi}_\mu(t')\rangle&=2\gamma_\nu\bar{n}_\nu\delta(t-t')\delta_{\nu\mu},\\ \langle[\hat{\xi}_\nu(t),\hat{\xi}_\mu^+(t')]\rangle&=2\gamma_\nu\delta(t-t')\delta_{\nu\mu},\end{aligned} \tag{6.2}$$

where the damping rates γ_ν are given by

$$\begin{aligned}\gamma_\nu=\frac{\pi\langle N_0\rangle}{\hbar^2}\sum_{\kappa,\mu}\Big\{&|(M^{(1)}_{\kappa,\nu\mu})^*+M^{(2)}_{\nu\mu,\kappa}|^2(\bar{n}_\mu-\bar{n}_\kappa)\\ &\times\delta(\omega_\kappa-\omega_\mu-\omega_\nu)+|M^{(1)}_{\nu,\kappa\mu}+(M^{(2)}_{\kappa\mu,\nu})^*|^2\\ &\times\left(\bar{n}_\kappa+\frac{1}{2}\right)\delta(\omega_\kappa+\omega_\mu-\omega_\nu)\Big\}.\end{aligned} \tag{6.3}$$

The first term describes the Landau damping of the mode ν by scattering a quasiparticle from mode μ to mode κ, and is equivalent to a result derived in Ref. [16] by the golden rule. The second term in Eq. (6.3) describes Beliaev damping, where the mode ν decays into two modes κ and μ. It survives even for $T\to0$, where $\bar{n}_\kappa\to0$ for all modes.

Let us now establish the connection between $\Gamma_0+\Gamma'$ and the damping rates γ_ν, as given by Eq. (6.3). We shall show that the simple sum rule

$$\Gamma_0+\Gamma'=\frac{\hbar}{3\langle N_0\rangle k_BT}\sum_\nu \bar{n}_\nu(\bar{n}_\nu+1)\gamma_\nu \tag{6.4}$$

holds. To see this, we need to consider

$$\begin{aligned}\sum_\nu \bar{n}_\nu(\bar{n}_\nu+1)\gamma_\nu=\frac{\pi\langle N_0\rangle}{\hbar^2}\sum_{\kappa\mu\nu}&|(M^{(1)}_{\kappa,\nu\mu})^*+M^{(2)}_{\nu\mu,\kappa}|^2\\ &\times\delta(\omega_\kappa-\omega_\nu-\omega_\mu)\Big\{(\bar{n}_\mu-\bar{n}_\kappa)\bar{n}_\nu(\bar{n}_\nu+1)\\ &+\frac{1}{2}(\bar{n}_\mu+\bar{n}_\nu+1)\bar{n}_\kappa(\bar{n}_\kappa+1)\Big\}.\end{aligned} \tag{6.5}$$

The second term in the curly bracket arises from the second term in Eq. (6.3) by first exchanging the notations for the summation indices ν and κ and then symmetrizing in ν and μ, because the matrix elements are already symmetric in these indices. The remainder of the proof then consists simply of noting that for $\omega_\kappa=\omega_\nu+\omega_\mu$ the identities

$$\begin{aligned}(\bar{n}_\mu-\bar{n}_\kappa)\bar{n}_\nu(\bar{n}_\nu+1)&=\bar{n}_\mu\bar{n}_\nu(\bar{n}_\kappa+1),\\ (\bar{n}_\mu+\bar{n}_\nu+1)\bar{n}_\kappa(\bar{n}_\kappa+1)&=\bar{n}_\mu\bar{n}_\nu(\bar{n}_\kappa+1)\end{aligned} \tag{6.6}$$

hold. Using this in Eq. (6.5), and then comparing with Eq. (5.14), establishes the sum rule. We can also note that processes due to Landau scattering contribute to the sum rule with precisely twice the strength of those due to Beliaev scattering.

Thus we see that, in general, the noise amplitudes proportional to the combination of matrix elements $(M^{(1)}_{\kappa,\nu\mu})^*-M^{(2)}_{\nu\mu,\kappa}$ drive the *number* fluctuations in the condensate, while those proportional to $(M'^{(1)}_{\kappa,\nu\mu})^*+M'^{(2)}_{\nu\mu,\kappa}$ are due to fluctuations of the occupation numbers in the excited states, couple in the Hamiltonian to the particle number in the condensate, and therefore drive the *phase* fluctuations in the condensate.

VII. EVALUATION OF THE TRANSPORT PARAMETERS FOR A BOXLIKE TRAP

For simplicity we consider now a trap consisting of a cube of length L with cyclic boundary conditions. In the following equilibrium values of all parameters are implied, but in this section we shall omit the tilde and write μ for $\langle\mu\rangle$ to simplify our notation. The normalized u and v coefficients in this case are

$$u_\nu=\frac{E_\nu+p_\nu^2/2m}{\sqrt{2E_\nu p_\nu^2/m}}\frac{1}{\sqrt{V}}e^{i\vec{p}_\nu\cdot\vec{x}/\hbar}, \tag{7.1}$$

$$v_\nu=-\frac{E_\nu-p_\nu^2/2m}{\sqrt{2E_\nu p_\nu^2/m}}\frac{1}{\sqrt{V}}e^{i\vec{p}_\nu\cdot\vec{x}/\hbar}, \tag{7.2}$$

with

$$E_\nu=\sqrt{\left(\frac{p_\nu^2}{2m}+\mu\right)^2-\mu^2}, \tag{7.3}$$

and $\vec{p}_\nu=\hbar(2\pi/L)\vec{n}_\nu$, with integer vector $\vec{n}_\nu$.

[6] In the general case the coupling of the condensate mode to the noncondensate modes differs from the coupling between the noncondensate modes and the relation between $\Gamma_0+\Gamma'$, and the γ_ν is less direct.

A. Transport coefficients

The squares of the relevant matrix elements for $E_\kappa = E_\nu + E_\mu$ become

$$|(M^{(1)}_{\kappa,\nu\mu})^* - M^{(2)}_{\nu\mu,\kappa}|^2 = \left(\frac{U_0}{V}\right)^2 \frac{G(E_\nu,E_\mu,\mu)}{E_\nu E_\mu E_\kappa}\,\delta_{\vec n_\kappa,\vec n_\nu+\vec n_\mu}, \tag{7.4}$$

$$|(M^{(1)}_{\kappa,\nu\mu})^* + M^{(2)}_{\nu\mu,\kappa}|^2 = \left(\frac{U_0}{V}\right)^2 \frac{G(E_\nu,E_\mu,-\mu)}{E_\nu E_\mu E_\kappa}\,\delta_{\vec n_\kappa,\vec n_\nu+\vec n_\mu}, \tag{7.5}$$

with

$$\begin{aligned} G(x,y,\alpha) = {} & \sqrt{\alpha^2+(x+y)^2}\,(3\sqrt{\alpha^2+x^2}\sqrt{\alpha^2+y^2} - xy \\ & + 2\alpha(\sqrt{\alpha^2+x^2}+\sqrt{\alpha^2+y^2}+\alpha)) \\ & + (x+y)(x(\sqrt{\alpha^2+y^2}+\alpha) + y(\sqrt{\alpha^2+x^2}+\alpha)) \\ & + 2\alpha(\sqrt{\alpha^2+x^2}+\alpha)(\sqrt{\alpha^2+y^2}+\alpha) \\ & + \alpha(x^2+y^2+\alpha^2). \end{aligned} \tag{7.6}$$

The transport coefficients are then expressed as the simple result

$$\Gamma'' = 0, \tag{7.7}$$

and

$$\Gamma_0 = \frac{2}{\pi}\left(\frac{a}{L}\right)^2 \sum_{\vec n_\nu,\vec n_\mu} \frac{\delta(\varepsilon_\nu+\varepsilon_\mu-\varepsilon_{\nu+\mu})}{\varepsilon_\nu\varepsilon_\mu(\varepsilon_\nu+\varepsilon_\mu)} \times G(\varepsilon_\nu,\varepsilon_\mu,\alpha)\,F\!\left(\varepsilon_\nu,\varepsilon_\mu,\frac{k_BT}{\hbar\omega_0}\right), \tag{7.8}$$

$$\Gamma_0 + \Gamma' = \frac{2}{\pi}\left(\frac{a}{L}\right)^2 \sum_{\vec n_\nu,\vec n_\mu} \frac{\delta(\varepsilon_\nu+\varepsilon_\mu-\varepsilon_{\nu+\mu})}{\varepsilon_\nu\varepsilon_\mu(\varepsilon_\nu+\varepsilon_\mu)} \times G(\varepsilon_\nu,\varepsilon_\mu,-\alpha)\,F\!\left(\varepsilon_\nu,\varepsilon_\mu,\frac{k_BT}{\hbar\omega_0}\right), \tag{7.9}$$

with

$$F\!\left(\varepsilon_\nu,\varepsilon_\mu,\frac{k_BT}{\hbar\omega_0}\right) = \frac{\hbar\omega_0}{k_BT}\,\frac{e^{\beta\hbar\omega_0(\varepsilon_\nu+\varepsilon_\mu)}}{(e^{\beta\hbar\omega_0(\varepsilon_\nu+\varepsilon_\mu)}-1)(e^{\beta\hbar\omega_0\varepsilon_\nu}-1)(e^{\beta\hbar\omega_0\varepsilon_\mu}-1)}. \tag{7.10}$$

Here we scaled the scattering length $a = mU_0/4\pi\hbar^2$ with L, and the energies E_ν and E_μ and μ and k_BT with the energy $\hbar\omega_0 = (2\pi\hbar)^2/2mL^2$, defining

$$\varepsilon_\nu = \sqrt{(n_\nu^2+\alpha)^2-\alpha^2}, \tag{7.11}$$

$$\varepsilon_{\nu+\mu} = \sqrt{((\vec n_\nu+\vec n_\mu)^2+\alpha)^2-\alpha^2}, \tag{7.12}$$

with $\alpha = \mu/\hbar\omega_0$.

The double sums over $\vec n_\nu$ and $\vec n_\mu$ start with $\vec n$ values with $|\vec n| = 1$. They are approximated by integrals according to

$$\sum_{\vec n_\nu,\vec n_\mu} \delta(\varepsilon_\nu+\varepsilon_\mu-\varepsilon_{\nu+\mu})(\cdots) = \pi^2 \int_{\sqrt{1+2\alpha}}^{\infty}\int_{\sqrt{1+2\alpha}}^{\infty} \frac{\varepsilon_\nu\varepsilon_\mu(\varepsilon_\nu+\varepsilon_\mu)(\cdots)\,d\varepsilon_\nu d\varepsilon_\mu}{\sqrt{(\varepsilon_\nu^2+\alpha^2)(\varepsilon_\mu^2+\alpha^2)((\varepsilon_\nu+\varepsilon_\mu)^2+\alpha^2)}}. \tag{7.13}$$

Here $(\cdots)$ is any smooth function of ε_ν and ε_μ. In all experiments so far, $\alpha \gg 1$ is satisfied, i.e. we can replace $\sqrt{1+2\alpha} \to \sqrt{2\alpha}$. This leaves us with the integral expressions

$$\Gamma_0 = 2\pi\left(\frac{a}{L}\right)^2 \int_{\sqrt{2\alpha}}^{\infty}\int_{\sqrt{2\alpha}}^{\infty} \frac{G(\varepsilon_\nu,\varepsilon_\mu,\alpha)\,F\!\left(\varepsilon_\nu,\varepsilon_\mu,\frac{k_BT}{\hbar\omega_0}\right) d\varepsilon_\nu d\varepsilon_\mu}{\sqrt{(\varepsilon_\nu^2+\alpha^2)(\varepsilon_\mu^2+\alpha^2)((\varepsilon_\nu+\varepsilon_\mu)^2+\alpha^2)}}, \tag{7.14}$$

$$\Gamma_0 + \Gamma' = 2\pi\left(\frac{a}{L}\right)^2 \int_{\sqrt{2\alpha}}^{\infty}\int_{\sqrt{2\alpha}}^{\infty} \frac{G(\varepsilon_\nu,\varepsilon_\mu,-\alpha)\,F\!\left(\varepsilon_\nu,\varepsilon_\mu,\frac{k_BT}{\hbar\omega_0}\right) d\varepsilon_\nu d\varepsilon_\mu}{\sqrt{(\varepsilon_\nu^2+\alpha^2)(\varepsilon_\mu^2+\alpha^2)((\varepsilon_\nu+\varepsilon_\mu)^2+\alpha^2)}}. \tag{7.15}$$

The expression for Γ_0 and the asymptotic behavior for $\varepsilon_\nu,\varepsilon_\mu \to 0$: $G(\varepsilon_\nu,\varepsilon_\mu,|\alpha|)F(\varepsilon_\nu,\varepsilon_\mu,k_BT/\hbar\omega_0) \to 18(k_BT/\hbar\omega_0)^2 1/[\varepsilon_\nu\varepsilon_\mu(\varepsilon_\nu+\varepsilon_\mu)]$ make it amply clear that the states with the smallest energies $\varepsilon_\nu \ll \alpha$ make a large contribution to Γ_0 (but *not* to $\Gamma_0+\Gamma'$). To calculate this contribution it is permitted[7] to use $\beta\hbar\omega_0\varepsilon_\nu \ll 1$ and $\beta\hbar\omega_0\varepsilon_\mu \ll 1$ under the integral, and to approximate

[7] We actually need the additional condition $\sqrt{\mu\hbar\omega_0} \ll k_BT$.

$$F\left(\varepsilon_\nu,\varepsilon_\mu,\frac{k_BT}{\hbar\omega_0}\right)=\left(\frac{k_BT}{\hbar\omega_0}\right)^2\frac{1}{(\varepsilon_\nu+\varepsilon_\mu)\varepsilon_\nu\varepsilon_\mu} \quad (7.16)$$

in addition to approximating

$$\sqrt{\varepsilon_\nu^2+\alpha^2}\sqrt{\varepsilon_\mu^2+\alpha^2}\sqrt{(\varepsilon_\nu+\varepsilon_\mu)^2+\alpha^2}\cong\alpha^3,$$

and neglecting terms of order $\varepsilon_\nu^2/\alpha^2$.

This contribution to Γ_0, which we shall denote as Γ_{00}, then reduces to

$$\Gamma_{00}=36\pi\left(\frac{a}{L}\right)^2\left(\frac{k_BT}{\hbar\omega_0}\right)^2 \times\int_{\sqrt{2\alpha}}^{\infty}d\varepsilon_\nu\int_{\sqrt{2\alpha}}^{\infty}d\varepsilon_\mu\,\frac{1}{\varepsilon_\nu\varepsilon_\mu(\varepsilon_\nu+\varepsilon_\mu)}. \quad (7.17)$$

The double integral can be evaluated as $\sqrt{2/\alpha}\ln 2$, which yields the final result

$$\Gamma_{00}=36\pi\sqrt{2}\ln 2\left(\frac{a}{L}\right)^2\left(\frac{\hbar\omega_0}{\mu}\right)^{1/2}\left(\frac{k_BT}{\hbar\omega_0}\right)^2 =1.59\ldots\left(\frac{T}{T_c}\right)^2\left(\frac{N}{\langle N_0\rangle}\right)^{1/2}N^{1/3}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{3/4}. \quad (7.18)$$

The second form of this expression is obtained by eliminating $V=L^3$ in favor of the critical temperature of the equivalent ideal Bose gas at the same density via

$$V=(N/\zeta(3/2))(2\pi\hbar^2/k_BT_cm)^{3/2}. \quad (7.19)$$

There is yet another particularly important contribution to Γ_0 due to the infrared singularity of the integrand; we shall denote it as Γ_{01}, where only *one* of the two excitation frequencies, say E_ν, is small compared to μ, while the other is larger, of order μ or even k_BT. With respect to ϵ_ν a low-energy asymptotics may then still be used. We then obtain

$$\Gamma_{01}=\frac{4\pi}{\alpha}\left(\frac{a}{L}\right)^2\int_{\sqrt{2\alpha}}^{q\alpha}d\varepsilon_\nu\int_{\sqrt{2\alpha}}^{\infty}d\varepsilon_\mu \times\frac{G(0,\varepsilon_\mu,\alpha)-G(0,0,\alpha)}{\varepsilon_\nu(\varepsilon_\mu^2+\alpha^2)}\,\frac{e^{\beta\hbar\omega_0\varepsilon_\mu}}{(e^{\beta\hbar\omega_0\varepsilon_\mu}-1)^2}, \quad (7.20)$$

where we have set an upper cutoff for the small energy at a fraction q of the chemical potential. Most of the ε_μ integral comes from a range around α, and we may therefore replace the thermal function by its asymptotics for $\beta\hbar\omega_0\varepsilon_\mu\to 0$, which is $(k_BT/\hbar\omega_0\varepsilon_\mu)^2$. The integrals can then be performed, with the result

$$\Gamma_{01}=16\pi\left(\frac{a}{L}\right)^2\ln\left(\frac{q^2\mu}{2\hbar\omega_0}\right)\frac{(k_BT)^2}{\hbar\omega_0\mu} =1.22\ldots\left(\frac{T}{T_c}\right)^2\frac{N}{\langle N_0\rangle}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{1/2}\ln\left(\frac{q^2\mu}{2\hbar\omega_0}\right). \quad (7.21)$$

We conclude that this contribution to Γ_0 is smaller than the leading term Γ_{00} by the order of magnitude $(\hbar\omega_0/\mu)^{1/2}\ln(q^2\mu/2\hbar\omega_0)$.

Let us now turn to the expressions for $\Gamma_0-\Gamma_{00}-\Gamma_{01}$ and $\Gamma_0+\Gamma'$. They can be simplified for $\alpha\gg 1$ by rescaling ε_ν and ε_μ by α, and taking the limit $\sqrt{2/\alpha}\to 0$ for the lower boundaries of the rescaled integrals. In this way we find

$$\Gamma_0-\Gamma_{00}-\Gamma_{01}=2\pi\left(\frac{a}{L}\right)^2\left(\frac{\mu}{\hbar\omega_0}\right)\int_0^\infty dx\int_0^\infty dy\left[\frac{G(x,y,1)F\left(x,y,\frac{k_BT}{\mu}\right)}{\sqrt{(x^2+1)(y^2+1)((x+y)^2+1)}},\right. \left.-\left(\frac{k_BT}{\mu}\right)^2\left(\frac{18}{xy(x+y)}+8\frac{x(x^2+1)(y^2+\sqrt{y^2+1}-1)+y(y^2+1)(x^2+\sqrt{x^2+1}-1)}{x^2y^2(x^2+1)(y^2+1)}\right)\right], \quad (7.22)$$

$$\Gamma_0+\Gamma'=2\pi\left(\frac{a}{L}\right)^2\left(\frac{\mu}{\hbar\omega_0}\right)\int_0^\infty dx\int_0^\infty dy\frac{G(x,y,-1)F\left(x,y,\frac{k_BT}{\mu}\right)}{\sqrt{(x^2+1)(y^2+1)((x+y)^2+1)}}, \quad (7.23)$$

where we used the scaling property

$$F\left(\varepsilon_\nu,\varepsilon_\mu,\frac{k_BT}{\hbar\omega_0}\right)=\frac{1}{\alpha}F\left(\frac{\varepsilon_\nu}{\alpha},\frac{\varepsilon_\mu}{\alpha},\frac{k_BT}{\hbar\omega_0\alpha}\right). \quad (7.24)$$

Equations (7.22) and (7.23) are the complete result for the temperature-dependent transport parameters of the condensate for boxlike traps. In general, the integrals have to be done numerically. We shall here consider some asymptotic results only.

First we consider these expressions asymptotically for $k_BT/\mu\gg 1$. Then the integrals receive important contributions from x and y of the order of 1, i.e., from quasiparticle-energies of the order of the chemical potential, and also from values of x and y large compared to 1, i.e., quasiparticle energies of order k_BT. The contributions $\Gamma_0^{(>)}$ and $\Gamma'^{(>)}$ from large energies can be determined in leading power in (k_BT/μ) by approximating

$$F\left(x,y,\frac{k_BT}{\mu}\right)\simeq\frac{\mu}{k_BT}e^{-(\mu/k_BT)(x+y)}, \tag{7.25}$$

and rescaling x and y by k_BT/μ. In the integrals[8] for Γ_0 and $\Gamma_0+\Gamma'$, we can then let $\mu/k_BT\to 0$ without any problem, using the property $G(x,y,0)=4xy(x+y)$, whereupon they are easily evaluated with the asymptotic results

$$\Gamma_0^{(>)}+\Gamma'^{(>)}\simeq\Gamma_0^{(>)}\simeq 8\pi\left(\frac{a}{L}\right)^2\frac{k_BT}{\hbar\omega_0}=1.27\ldots\frac{T}{T_c}\frac{k_BT_ca^2m}{\hbar^2}. \tag{7.26}$$

We see that the result for $\Gamma'^{(>)}$ vanishes to this order.

For the contributions $\Gamma_0^{(\mu)}$ and $\Gamma'^{(\mu)}$ from quasiparticles with energies around μ, we can approximate $F(x,y,k_BT/\mu)$ according to Eq. (7.16), and find

$$\Gamma_0^{(\mu)}\simeq B_0^{(\mu)}\left(\frac{a}{L}\right)^2\frac{(k_BT)^2}{\hbar\omega_0\mu} =0.0243\ldots B_0^{(\mu)}\left(\frac{T}{T_c}\right)^2\frac{N}{\langle N_0\rangle}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{1/2}, \tag{7.27}$$

$$\Gamma_0^{(\mu)}+\Gamma'^{(\mu)}\simeq B'^{(\mu)}\left(\frac{a}{L}\right)^2\frac{(k_BT)^2}{\hbar\omega_0\mu} =0.0243\ldots B'^{(\mu)}\left(\frac{T}{T_c}\right)^2\frac{N}{\langle N_0\rangle}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{1/2}, \tag{7.28}$$

with the numbers $B_0^{(\mu)}$ and $B'^{(\mu)}$ defined by the integrals

$$B_0^{(\mu)}=2\pi\int_0^\infty dx\int_0^\infty dy\left[\frac{G(x,y,1)}{\sqrt{(x^2+1)(y^2+1)((x+y)^2+1)}xy(x+y)} -\frac{18}{xy(x+y)}-8\frac{x(x^2+1)(y^2+\sqrt{y^2+1}-1)+y(y^2+1)(x^2+\sqrt{x^2+1}-1)}{x^2y^2(x^2+1)(y^2+1)}\right], \tag{7.29}$$

$$B'^{(\mu)}=2\pi\int_0^\infty dx\int_0^\infty dy\frac{G(x,y,-1)}{\sqrt{(x^2+1)(y^2+1)((x+y)^2+1)}xy(x+y)}. \tag{7.30}$$

We can conclude that the contribution from quasiparticles at energies of order μ is larger (for Γ_0 by an order of magnitude k_BT/μ) than the contribution from energies of order k_BT, but $\Gamma_0^{(\mu)}$ is, in large condensates, still subdominant to Γ_{00} by the order of magnitude $\sqrt{\hbar\omega_0/\mu}$.

Now let us consider also the low-temperature limit, namely, $k_BT/\mu\ll 1$ or, equivalently, $T/T_c\ll(k_BT_ca^2m/\hbar^2)^{1/2}$. In this region it is not necessary to distinguish N and $\langle N_0\rangle$. The integrals now receive their contributions for x and y both small compared to 1, but we can still use approximation (7.25). For small x and y we can expand

$$G(x,y,1)\simeq 18+3(x^2+y^2+xy)$$
$$G(x,y,-1)\simeq\frac{9}{32}x^2y^2(x+y)^2. \tag{7.31}$$

To obtain the leading term it is enough to keep only the smallest powers of x and y in the integrands. The integrals are easily evaluated with the asymptotic low-temperature results

$$\Gamma_0-\Gamma_{00}=36\pi\left(\frac{a}{L}\right)^2\frac{k_BT}{\hbar\omega_0}=5.72\ldots\frac{T}{T_c}\frac{k_BT_ca^2m}{\hbar^2}, \tag{7.32}$$

$$\Gamma_0+\Gamma'=\frac{189}{2}\pi\left(\frac{a}{L}\right)^2\frac{\mu}{\hbar\omega_0}\left(\frac{k_BT}{\mu}\right)^7 =0.366\ldots\left(\frac{T}{T_c}\right)^7\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{-2}. \tag{7.33}$$

As long as the temperature is high enough to satisfy $k_BT\gg\sqrt{\mu\hbar\omega_0}$, the part Γ_{00} still dominates the value of Γ_0.

B. Particle-number fluctuations

We follow the procedure of Giorgini *et al.* [10(a)], and deduce the particle-number fluctuations in the condensate

[8] In the high-energy regime the subtractions of the infrared-divergent terms in the integrand of Eq. (7.22) are of no importance.

from the number fluctuation in the thermal cloud. This leads to Eq. (2.28), which we evaluate using the expressions for u_ν, v_ν, and E_ν. We obtain

$$\langle \Delta N_0^2 \rangle = \sum_\nu \frac{2\bar{n}_\nu(\bar{n}_\nu+1)(E_\nu^2+2\mu^2)+\mu^2}{2E_\nu^2}. \tag{7.34}$$

Approximated by an integral, this becomes

$$\langle \Delta N_0^2 \rangle = \pi \int_{\sqrt{1+2\alpha}}^{\infty} \frac{d\varepsilon}{\varepsilon} \sqrt{\frac{\sqrt{\varepsilon^2+\alpha^2}-\alpha}{\varepsilon^2+\alpha^2}} \times \left(\alpha^2 + \frac{\varepsilon^2+2\alpha^2}{2\left(\sin h \frac{\beta\hbar\omega_0\varepsilon}{2}\right)^2} \right). \tag{7.35}$$

The dominant contribution comes from the lower boundary of the integration [10] which contributes, for $\alpha \gg 1$,

$$\langle \Delta N_0^2 \rangle \simeq 2\pi \left(\frac{k_B T}{\hbar\omega_0}\right)^2 = A' \left(\frac{m k_B T}{\hbar^2}\right)^2 V^{4/3}, \tag{7.36}$$

with

$$A' = \frac{1}{2\pi^3} = 0.0161 \ldots . \tag{7.37}$$

More precisely the dominant contribution to $\langle \Delta N_0^2 \rangle$ is given by the discrete sum [10]

$$\langle \Delta N_0^2 \rangle = 2\mu^2 (k_B T)^2 \sum_\nu \frac{1}{E_\nu^4}, \tag{7.38}$$

which gives the same expression as Eq. (7.36), but with the prefactor[9]

$$A = \frac{2}{(2\pi)^4} \sum_{n_\nu \neq 0} \frac{1}{n_\nu^4} = 0.021 \ldots . \tag{7.39}$$

If we eliminate the volume in favor of the critical temperature of the ideal Bose gas of the same density via Eq. (7.19) we obtain

$$\langle \Delta N_0^2 \rangle = A \frac{(2\pi)^2}{\zeta(3/2)^{4/3}} \left(\frac{T}{T_c}\right)^2 N^{4/3}. \tag{7.40}$$

At temperature $T=0$ a similar evaluation of Eq. (2.28) gives

$$\langle \Delta N_0^2 \rangle|_{T=0} = 2\sqrt{\pi}(aN)^{3/2} V^{-1/2}. \tag{7.41}$$

C. Particle-number relaxation rate

We are now in a position to evaluate the rate τ_c^{-1} from Eq. (3.17) using the results for $\langle \Delta N_0^2 \rangle$ (numbers are calculated with the prefactor A') and $\Gamma_0 \simeq \Gamma_{00}$. We obtain

$$\gamma_c = \frac{1}{\tau_c} = 18.0 \ldots \frac{T}{T_c} \sqrt{\frac{\langle N_0 \rangle}{N}} \left(\frac{k_B T_c a^2 m}{\hbar^2}\right)^{3/4} \frac{k_B T_c}{\hbar}. \tag{7.42}$$

This result also applies in the low-temperature region, because it makes use only of the results for $\langle \Delta N_0^2 \rangle$ and Γ_0, which also hold in that region.

To obtain an idea of order of magnitudes, we compare this and the following results with the damping rate γ_0 of the lowest-lying modes, which are given by [16–18]

$$\gamma_0 = \frac{3\pi^2}{4} \left(\frac{a}{L}\right) \frac{k_B T}{\hbar} = 4.06 \ldots \frac{T}{T_c} N^{-1/3} \left(\frac{k_B T_c a^2 m}{\hbar^2}\right)^{1/2} \frac{k_B T_c}{\hbar}. \tag{7.43}$$

We see that the relaxation rate γ_c is of the order

$$\gamma_c \sim N^{1/3} \left(\frac{k_B T_c a^2 m}{\hbar^2}\right)^{1/4} \gamma_0. \tag{7.44}$$

The proportionality factor is of the order of $\sqrt{\mu/\hbar\omega_0}$, and is large in large and strongly interacting condensates. Thus the relaxation of the condensate to its equilibrium is faster than the relaxation of the low-lying collective modes, but slower than the frequency of the lowest-lying modes, which is $\sqrt{2\omega_0\mu/\hbar}$.

D. Phase collapse rate

The phase collapse rate is given by Eq. (3.23), and requires only the results for $\langle \Delta\ N_0^2 \rangle$ and $\Gamma''=0$. At zero temperature it reduces to

$$\gamma_{collapse}|_{T=0} = \frac{1}{\hbar} \frac{\partial\mu}{\partial\langle N_0 \rangle} \sqrt{\langle \Delta N_0^2 \rangle|_{T=0}}, \tag{7.45}$$

from which we obtain

$$\gamma_{collapse}|_{T=0} = \frac{23.6 \ldots}{\sqrt{V}} (an_0)^{3/4} \frac{\hbar a}{m} = 2.50 \ldots \frac{k_B T_c}{\hbar} \left(\frac{k_B T_c a^2 m}{\hbar^2}\right)^{7/8} \frac{1}{\sqrt{N}}. \tag{7.46}$$

For finite temperature we obtain

$$\gamma_{collapse} = 0.876 \ldots \frac{k_B T}{\hbar} N^{-1/3} \left(\frac{k_B T_c a^2 m}{\hbar^2}\right)^{1/2}. \tag{7.47}$$

By comparison with Eq. (7.43), we see that $\gamma_{collapse}$ is of the order of γ_0, and is therefore large large condensates smaller

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[9] Formula (7.39) differs from the one given in Ref. [10(a)] by a factor 2^{-4} whereas the numerical result differs by yet another factor; the formula (7.41) differs from the one in Ref. [10(a)] by a factor 2.

than γ_c. In summary, the phase collapse is not effective in large condensates because it occurs with a rate $\gamma_{collapse} < \gamma_c$ and is at the same time restricted to a time interval $\Delta t < 1/\gamma_c$, since for larger times phase diffusion takes over.

E. Phase-diffusion rate

The phase-diffusion coefficient is a somewhat complicated quantity because it receives contributions from several processes which are physically distinct. We consider the different contributions separately, and also distinguish the two temperature regimes of high temperature, $k_BT > \mu$, for which we give the result first, and low temperature, $k_BT < \mu$.

1. Low-frequency condensate number fluctuations

From Eq. (3.19) we infer, with $\Gamma_0 = \Gamma_{00}$,

$$D_\phi^{(\alpha)} = \frac{1}{\hbar k_B T \langle N_0 \rangle \Gamma_{00}} \left(\langle \Delta N_0^2 \rangle \frac{\partial \mu}{\partial \langle N_0 \rangle} \right)^2, \quad (7.48)$$

which is evaluated as

$$D_\phi^{(\alpha)} = 0.0853 \ldots \frac{T}{T_c} \left(\frac{k_B T_c m a^2}{\hbar^2} \right)^{1/4} \frac{1}{\langle N_0 \rangle^{1/2} N^{1/6}} \frac{k_B T_c}{\hbar}. \quad (7.49)$$

The same result holds in the low-temperature regime $k_BT < \mu$. In comparison with γ_0 [Eq. (7.43)], it is of the order

$$D_\phi^{(\alpha)} \sim N^{-1/3} \left(\frac{k_B T_c m a^2}{\hbar^2} \right)^{-1/4}, \quad \gamma_0 \sim \sqrt{\frac{\hbar \omega_0}{\mu}} \gamma_0, \quad (7.50)$$

and is much smaller in large and strongly interacting condensates. Still this contribution to the phase-diffusion rate is always the dominant one at low temperatures and may dominate even at higher temperatures (see below).

2. Condensate number fluctuations due to quasiparticles around energies μ

Splitting $\Gamma_0 = \Gamma_{00} + (\Gamma_0 - \Gamma_{00})$ and expanding to first order,

$$\frac{1}{\Gamma_0} = \frac{1}{\Gamma_{00}} - \frac{\Gamma_0 - \Gamma_{00}}{\Gamma_{00}^2}, \quad (7.51)$$

we estimate as contribution $D_\phi^{(\beta)}$ from the higher-frequency condensate number fluctuations described by $\Gamma_0 - \Gamma_{00}$, as given by Eq. (7.26),

$$D_\phi^{(\beta)} \sim -\frac{T}{T_c} \frac{1}{\langle N_0 \rangle} \frac{k_B T_c}{\hbar}, \quad (7.52)$$

which is in absolute value smaller than the contribution $D_\phi^{(\alpha)}$ from low-energy excitations by the order of magnitude factor $\sqrt{\hbar \omega_0 / \mu}$. This contribution is therefore negligible in very large condensates. In not so large condensates the complete integral in the result for Γ_0 needs to be evaluated.

In the low-temperature regime $k_BT < \mu$, we instead obtain

$$D_\phi^{(\beta)} = -0.306 \ldots \frac{1}{N} \left(\frac{k_B T_c a^2 m}{\hbar} \right)^{1/2} \frac{k_B T_c}{\hbar}. \quad (7.53)$$

This is much smaller than $D_\phi^{(\alpha)}$, by an order of magnitude factor $(\mu / k_B T_c)^{1/2} / N^{1/3}$.

3. Fluctuations in the thermal cloud at energies of order μ

From Eq. (3.19) this contribution is given by

$$D_\phi^{(\gamma)} = \frac{k_B T}{\hbar \langle N_0 \rangle} (\Gamma_0 + \Gamma'), \quad (7.54)$$

which is evaluated to

$$D_\phi^{(\gamma)} = 0.0243 \ldots B'^{(\mu)} \left(\frac{T}{T_c} \right)^3 \frac{N}{\langle N_0 \rangle^2} \left(\frac{k_B T_c a^2 m}{\hbar^2} \right)^{1/2} \frac{k_B T_c}{\hbar}. \quad (7.55)$$

This contribution differs from $D_\phi^{(\alpha)}$ by the order of magnitude factor $(T/T_c)^2 \sqrt{\mu \hbar \omega_0} / k_B T_c$, and is therefore much smaller.

For temperatures $k_BT < \mu$ we instead find

$$D_\phi^{(\gamma)} = 0.366 \ldots \left(\frac{T}{T_c} \right)^8 \frac{1}{\langle N_0 \rangle} \left(\frac{k_B T_c a^2 m}{\hbar^2} \right)^{-2} \frac{k_B T_c}{\hbar}, \quad (7.56)$$

which is again negligibly small compared to $D_\phi^{(\alpha)}$.

In summary, the phase diffusion is caused dominantly by the low-frequency particle-number fluctuations in the condensate, and the phase-diffusion constant is given by Eq. (7.49). It is proportional to temperature, and scales proportional to $N^{-2/3}$ for fixed T_c, or proportional to $N^{-1/2}$ for a fixed volume of the trap.

VIII. EVALUATION OF THE TRANSPORT PARAMETERS FOR AN ISOTROPIC HARMONIC TRAP

In this section we consider the more realistic case of condensates in a parabolic trapping potential $m\omega_0^2 x^2/2$, which we assume to be isotropic for simplicity. In order to analyze the noise $\mathrm{Im}(\xi(t))$ driving the fluctuations of $|\alpha_0|^2$, we must consider in detail the relevant linear combination of matrix elements:

$$(M_{\kappa,\nu\mu}^{(1)})^* - M_{\nu\mu,\kappa}^{(2)} = 2U_0 \int d^3x \, \psi_0 \{ (\tilde{u}_\kappa - \tilde{v}_\kappa)(\tilde{u}_\mu^* \tilde{v}_\nu^* + \tilde{v}_\mu^* \tilde{u}_\nu^*) - \tilde{u}_\kappa \tilde{u}_\mu^* \tilde{u}_\nu^* + \tilde{v}_\kappa \tilde{v}_\mu^* \tilde{v}_\nu^* \}. \quad (8.1)$$

In the following we shall make use of the local density and Thomas-Fermi approximation, restricting ourselves to large condensates. For high-lying states we can then use the local energies in the Thomas-Fermi approximation,

$$E(p,\boldsymbol{x})=\sqrt{\left(\frac{p^2}{2m}+|U_0n_0(\boldsymbol{x})|\right)^2-U_0^2n_0^2(\boldsymbol{x})\Theta(\mu-V(\boldsymbol{x}))},\tag{8.2}$$

with the condensate density

$$n_0(\boldsymbol{x})=\langle N_0\rangle|\tilde{\psi}_0(\boldsymbol{x})|^2=(\langle\mu\rangle/U_0)(1-(x/r_{TF})^2),\tag{8.3}$$

and the Thomas-Fermi radius

$$r_{TF}=\sqrt{2\langle\mu\rangle/m\omega_0^2}=\left(\frac{15U_0\langle N_0\rangle}{8\pi\langle\mu\rangle}\right)^{1/3}.\tag{8.4}$$

The high-lying quasiparticle modes can be represented similarly to the spatially homogeneous case as

$$u_\kappa(\boldsymbol{x})=\frac{E_\kappa+p_\kappa^2/2m}{\sqrt{2E_\kappa p_\kappa^2/m}}e^{i\boldsymbol{p}_\kappa\cdot\boldsymbol{x}/\hbar},\qquad v_\kappa(\boldsymbol{x})=-\frac{E_\kappa-p_\kappa^2/2m}{\sqrt{2E_\kappa p_\kappa^2/m}}e^{i\boldsymbol{p}_\kappa\cdot\boldsymbol{x}/\hbar}.\tag{8.5}$$

The low-lying collective modes can be represented as

$$u_\nu(\boldsymbol{x})=\left(\sqrt{\frac{U_0n_0(\boldsymbol{x})}{2\hbar\tilde{\omega}_\nu}}+\frac{1}{2}\sqrt{\frac{\hbar\tilde{\omega}_\nu}{2U_0n_0(\boldsymbol{x})}}\right)\chi_\nu(\boldsymbol{x}),\qquad v_\nu(\boldsymbol{x})=\left(-\sqrt{\frac{U_0n_0(\boldsymbol{x})}{2\hbar\tilde{\omega}_\nu}}+\frac{1}{2}\sqrt{\frac{\hbar\tilde{\omega}_\nu}{2U_0n_0(\boldsymbol{x})}}\right)\chi_\nu(\boldsymbol{x}),\tag{8.6}$$

with

$$\int d^3x|\chi_\nu(\boldsymbol{x})|^2=1.\tag{8.7}$$

The mode functions $\chi_\nu(\boldsymbol{x})$ are known in the hydrodynamic (long-wavelength) and Thomas-Fermi approximation [19–24] by analytic solutions of the Bogoliubov equations. In spatially isotropic parabolic traps they have the form [19]

$$\chi_\nu(\boldsymbol{x})=\frac{1}{r_{TF}^{3/2}}P_{l_\nu}^{(2n_\nu)}(x/r_{TF})(x/r_{TF})^lY_{lm}(\theta,\varphi)\Theta(1-x/r_{TF}).\tag{8.8}$$

The polynomials $P_l^{(2n)}(x)$ of degree $2n$ are the normalized solutions of the radial part of the Bogoliubov-Fetter equations in the Thomas-Fermi and long-wavelength limit [19,21] given by [21]

$$P_l^{(2n)}(x)=\frac{\sqrt{4n+2l+3}}{n!}x^{-2l-1}\frac{d^n}{d(x^2)^n}\times[x^{2n+2l+1}(1-x^2)^n],\tag{8.9}$$

with the normalization

$$\int_0^1dxx^{2l+2}[P_l^{(2n)}(x)]^2=1.\tag{8.10}$$

In the phonon part of the excitation spectrum, we have $u_\lambda\simeq-v_\lambda\sim\tilde{\omega}_\lambda^{-1/2}$. Furthermore, in that low-energy region the statistical factor in Eqs. (5.13)–(5.15) is well approximated by $\overline{n}_\nu\overline{n}_\mu\overline{n}_\kappa\approx(k_BT)^3/\hbar^3\tilde{\omega}_\kappa\tilde{\omega}_\nu\tilde{\omega}_\mu$. Just as in the case of boxlike traps, the frequency factors in the denominator, together with similar further factors in the denominator coming from the matrix elements, make the phonon contribution to the sums in Eq. (5.13) the dominant one, at least in large condensates, and we shall therefore concentrate on this contribution in the following. This frequency range has a natural upper cutoff at $\langle\mu\rangle/\hbar$, where the collective phonons go over smoothly into particlelike excitations.

For $E_\kappa,E_\nu,E_\mu\ll\langle\mu\rangle$ the matrix elements $(M^{(1)}_{\kappa,\nu\mu})^*$ and $M^{(2)}_{\nu\mu,\kappa}$ are given by the integral

$$(M^{(1)}_{\kappa,\nu\mu})^*\approx-M^{(2)}_{\nu\mu,\kappa}\tag{8.11}$$

$$\approx-\sqrt{\frac{15}{8\pi}}\frac{3U_0\langle\mu\rangle^{3/2}\delta_{m_\kappa,m_\nu+m_\mu}}{r_{TF}^3\sqrt{2E_\nu E_\mu(E_\nu+E_\mu)}}\times J(n_\kappa,n_\nu,n_\mu;l_\kappa,l_\nu,l_\mu)C(l_\kappa|l_\mu m_\mu,l_\nu m_\nu),\tag{8.12}$$

where J denotes the integral

$$J(n_\kappa,n_\nu,n_\mu;l_\kappa,l_\nu,l_\mu)=\int_0^1dxx^2(1-x^2)^2x^{l_\kappa+l_\nu+l_\mu}P_{l_\kappa}^{(2n_\kappa)}\times(x)P_{l_\nu}^{(2n_\nu)}(x)P_{l_\mu}^{(2n_\mu)}(x)$$

and the Clebsch-Gordan coefficients $C(l_\kappa|l_\mu m_\mu,l_\nu m_\nu)$ are given by the angle integral

$$C(l_\kappa|l_\mu m_\mu,l_\nu m_\nu)=\int d\Omega Y^*_{l_\kappa,m_\nu+m_\mu}(\theta,\varphi)Y_{l_\mu,m_\mu}(\theta,\varphi)Y_{l_\nu,m_\nu}(\theta,\varphi)$$

if $|l_\mu-l_\nu|\le l_\kappa\le l_\mu+l_\nu$, otherwise they vanish. Below we shall have to calculate, e.g., $\Sigma_{m_\nu,m_\mu}|(M^{(1)}_{\kappa,\nu\mu})^*-M^{(2)}_{\nu\mu,\kappa}|^2$, where we can make use of the sum rule for $|l_\mu-l_\nu|\le l_\kappa\le l_\mu+l_\nu$,

$$\sum_{m_\nu,m_\mu}|C(l_\kappa|l_\mu m_\mu,l_\nu m_\nu)|^2=1,$$

so that the Clebsch-Gordan coefficients need actually not be used explicitly.

In order to have well-defined expressions for the rate coefficients, we again need to smooth the δ function expressing energy conservation, which is done physically by experimental imperfections or limitations in resolution. Here this can be done by replacing the discrete sum over the ''quantum number'' l_κ by an integral

$$\sum_{l_\kappa} \delta(E_\kappa - E_\nu - E_\mu)(\cdots)$$

$$\approx \int dE_\kappa \frac{1}{dE_\kappa/dl_\kappa} \delta(E_\kappa - E_\nu - E_\mu)(\cdots)$$

$$= \int dE_\kappa \frac{E_\nu + E\mu}{(\hbar\omega_0)^2(n_\kappa+1/2)} \delta(E_\kappa - E_\nu - E_\mu)(\cdots) \quad (8.13)$$

where we used an expression for the excitation energies [19]:

$$E_\kappa = \hbar\omega_0 e_\kappa,$$

$$e_\kappa = \sqrt{2n_\kappa^2 + 2n_\kappa l_\kappa + 3n_\kappa + l_\kappa}. \qquad (8.14)$$

We introduced the dimensionless eigenvalues $e_{\kappa,\nu,\mu}$ which will appear in the ensuing expressions from now on. The integration over E_κ with the δ function then picks out the energy value $E_\kappa = E_\nu + E_\mu$, so that l_κ becomes a function $l_\kappa^{(0)}$ of the other quantum numbers:

$$l_\kappa^{(0)} = \frac{(e_\nu + e_\mu)^2 - 2n_\kappa^2 - 3n_\kappa}{2n_\kappa + 1}.$$

The inequalities

$$|l_\nu - l_\mu| \leq l_\kappa^{(0)} \leq l_\nu + l_\mu$$

then imply that n_κ must lie in the interval

$$n_{\kappa-} \leq n_\kappa \leq n_{\kappa+},$$

with

$$n_{\kappa\pm} = \frac{1}{2}\left(\sqrt{|l_\nu \pm l_\mu|^2 + |l_\nu \pm l_\mu| + 9/4 + 2(e_\nu + e_\mu)^2} - |l_\nu \pm l_\mu| - 3/2\right).$$

Using all this, from Eq. (5.13) we obtain

$$\Gamma_0 = B_{00}\left(\frac{a}{d_0}\frac{k_B T}{\hbar\omega_0}\right)^2 = B_{00}\frac{T^2}{T_c^2}\frac{k_B T_c a^2 m}{\hbar^2}\left(\frac{N}{\zeta(3)}\right)^{1/3}, \qquad (8.15)$$

where $\hbar\omega_0$ is now eliminated in favor of $k_B T_c$ via the relation $\hbar\omega_0 = k_B T_c(\zeta(3)/N)^{1/3}$, and where the temperature- and particle-number-independent positive real number B_{00} is defined by the multiple sums

$$B_{00} = \frac{135\pi^2}{2} \sum_{n_\nu} \sum_{n_\mu} \sum_{l_\nu} \sum_{l_\mu} \sum_{n_\kappa = n_{\kappa-}}^{n_{\kappa+}} \frac{(2l_\kappa^{(0)}+1) J^2(n_\kappa, n_\nu, n_\mu; l_\kappa^{(0)}, l_\nu, l_\mu)}{e_\nu^2 e_\mu^2 (e_\nu + e_\mu)(2n_\kappa+1)}. \qquad (8.16)$$

The result for Γ_0 agrees, except for the numerical prefactor, with the result of Ref. [3] which was evaluated there using the local-density approximation and imposing a lower cutoff for the excitation frequencies at the geometrical mean trap frequency $\overline{\omega}$. It can also be compared with the corresponding result [Eq. (7.18)] for the boxlike trap, which shows the same dependence on temperature and particle number (if we stipulate $\langle N_0 \rangle \sim N$), but the comparison of the prefactor is problematic because the condensate in the parabolic trap has two length scales d_0 and r_{TF}, whereas in the boxlike trap only the length scale L is relevant.

Property (8.11) of the matrix elements implies that *low-lying* excitations do not contribute to $\Gamma_0 + \Gamma'$. The reality of the matrix elements furthermore implies that Γ'' vanishes. These remarkably simple results mean that the noise source $\xi(t)$ introduced in Eq. (3.6) is purely imaginary, corresponding to a total squeezing in the direction of the phase ϕ. In other words, the coupling of the condensate to the collective excitations introduces a *direct* Langevin noise source only for the number fluctuations δN_0, not the phase variable ϕ.[10]

The fact that $\Gamma_0 + \Gamma' = 0$ for the contribution from the low-lying states implies that the contributions from the higher-lying states must also be considered in order to evaluate the small (compared to Γ_0) but finite value of this quantity. For this purpose we need to consider the matrix element $(M'^{(1)}_{\kappa,\nu\mu})^* + M'^{(2)}_{\nu\mu,\kappa}$. It differs from the matrix elements we have considered so far by the replacement $\tilde{\psi}_0(x) \to \tilde{\psi}_0(x) + 2\langle N_0 \rangle \partial\tilde{\psi}_0(x)/\partial\langle N_0 \rangle$ in the matrix element. In the Thomas-Fermi approximation this is tantamount to the replacement

$$\tilde{\psi}_0(x) \to (2/5)\tilde{\psi}_0(x)/(1 - x^2/r_{TF}^2). \qquad (8.17)$$

Physically this implies a reduced coupling of the thermal fluctuations with the center of the condensate and a strongly enhanced coupling at its boundary, as one would expect for fluctuations located in the thermal cloud. A mathematical consequence is the fact that the integrals defining these matrix elements diverge at the boundary in the Thomas-Fermi approximation, meaning that here we encounter the limitations of that approximation. Instead of a full-fledged extension of the theory beyond the Thomas-Fermi, approximation it will be sufficient for our purposes here to cure its deficiencies by substituting as a cutoff the finite thickness of the boundary layer given by [25]

$$d = \frac{1}{2} r_{TF}\left(\frac{\hbar\omega_0}{\langle \mu \rangle}\right)^{2/3}.$$

[10] The latter is of course affected by the noise-source indirectly, because the fluctuations of δN_0 driven by the latter cause fluctuations in the chemical potential.

The matrix element itself is then evaluated in the local-density approximation [25], where we can make use to good purpose of the analysis already performed in Sec. VII. The finite volume $V=L^3$ [and the associated $\hbar\omega_0=(2\pi\hbar)^2/2mL^2$, which is not to be confused with the trap frequency called ω_0 in the present section] is then an arbitrary local subvolume of the condensate, introduced merely as a technical device like a quantization volume. It must be sufficiently small so that the condensate within it can be treated as homogeneous, and sufficiently large that we can replace sums over local momenta by integrals. At the end we have to check for consistency whether the result is indeed independent of the choice of this volume. The result obtained in this way is the local average of result (7.28) for the homogneous case, which now becomes space dependent, because we have to substitute a space-dependent chemical potential $\mu\to\langle\mu\rangle(1-x^2/r_{TF}^2)$. This local result can be written

$$\Gamma_0(x)+\Gamma'(x)=\frac{B'^{(\mu)}}{2\pi^2}\frac{(k_BT)^2a^2m}{\hbar^2\langle\mu\rangle(1-x^2/r_{TF}^2)},$$

and is indeed independent of the choice of V. The local average has to be performed with the weight $(\tilde{\psi}_0(x)+2\langle N_0\rangle\partial\tilde{\psi}_0(x)/\partial\langle N_0\rangle)^2$ determined from Eq. (8.17). Doing the average and regulating the divergency of the integral at the boundary of the condensate by the physical cutoff, we obtain

$$\Gamma_0+\Gamma'=\frac{3}{10}\frac{2^{1/3}}{15^{2/15}\pi^2}B'^{(\mu)}\left(\frac{k_BT}{\hbar\omega_0}\right)^2\langle N_0\rangle^{-2/15}\left(\frac{a}{d_0}\right)^{28/15}$$
$$=0.024\ldots B'^{(\mu)}\left(\frac{T}{T_c}\right)^2\left(\frac{N}{\langle N_0\rangle}\right)^{2/15}$$
$$\times N^{2/9}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{14/15}. \tag{8.18}$$

In order to extract results for the relaxation rate of the condensate number and the phase-diffusion rate, it is also necessary to know the mean square of the number fluctuations $\langle\Delta N_0^2\rangle$. This can be evaluated from Eq. (2.28), using the fact that these fluctuations are also dominated by the low-lying modes [10(a)]. The result of this calculation to leading order in $(\hbar\omega_0/k_BT)$ is

$$\langle\Delta N_0^2\rangle=A\left(\frac{\langle N_0\rangle a}{d_0}\right)^{4/5}\left(\frac{k_BT}{\hbar\omega_0}\right)^2 \tag{8.19}$$

$$=\frac{A}{(\zeta(3))^{8/15}}\left(\frac{T}{T_c}\right)^2\left(\frac{\langle N_0\rangle}{N}\right)^{4/5}N^{4/3}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{2/5}, \tag{8.20}$$

with the number A given by the multiple sums

$$A=\frac{(15)^{4/5}}{2}\sum_n\sum_{n'}\sum_l\frac{2l+1}{(e(n,l)e(n',l))^2}$$
$$\times\left|\int_0^1dx(1-x^2)x^{2(l+1)}P_l^{(2n)}(x)P_l^{(2n')}(x)\right|^2. \tag{8.21}$$

In order to find the scaling of $\langle\Delta N_0^2\rangle$ in the thermodynamic limit $N\to\infty,\omega_0\to 0,k_BT_c=\hbar\omega_0(N/\zeta(3))^{1/3}$ *fixed*, it is necessary to use the form of the preceding results in which $\hbar\omega_0$ is eliminated in favor of k_BT_c and to use $\langle N_0\rangle\sim N$. Then the scaling $\langle\Delta N_0^2\rangle\sim N^{4/3}$ derived in Ref. [10(a)] is recovered. The particle-number relaxation rate now follows from Eqs. (3.17) and (8.15) as

$$\gamma_c=\frac{2B_{00}}{A}\langle N_0\rangle^{1/5}\left(\frac{a}{d_0}\right)^{6/5}\frac{k_BT}{\hbar}$$
$$=\frac{2(\zeta(3))^{1/5}B_{00}}{A}\frac{T}{T_c}\left(\frac{\langle N_0\rangle}{N}\right)^{1/5}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{3/5}\frac{k_BT_c}{\hbar}. \tag{8.22}$$

This is the largest of the various rates we calculate here, but is still small compared to ω_0, the inverse time scale of motion in the trap, by the order of magnitude $N^{-2/3}(Na/d_0)^{6/5}$.

The phase-collapse rate is obtained from Eq. (3.23). At $T\neq 0$ (more precisely, above a crossover temperature of order $\hbar\omega_0$), we find

$$\gamma_{collapse}=\frac{15^{2/5}A^{1/2}}{5}\langle N_0\rangle^{-1/5}\left(\frac{a}{d_0}\right)^{4/5}\frac{k_BT}{\hbar}$$
$$=\frac{15^{2/5}(\zeta(3))^{2/15}A^{1/2}}{5}\frac{T}{T_c}\left(\frac{N}{\langle N_0\rangle}\right)^{1/5}$$
$$\times N^{-1/3}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{2/5}\frac{k_BT_c}{\hbar}. \tag{8.23}$$

Apart from the numerical prefactor, this is the same asymptotic expression as obtained for the damping rate γ_0 of the low-lying collective modes (see, e.g., Ref. [15]). It is smaller than γ_c by the order of magnitude $(\langle N_0\rangle a/d_0)^{-2/5}$, i.e., the phase collapse remains inefficient before phase diffusion takes over.

The phase-diffusion constant $D_\phi^{(\alpha)}$, due to the exchange of particles between the condensate and low-lying excitations, is obtained by inserting the results for $\langle\Delta N_0^2\rangle$ and Γ_{00} in Eq. (3.19):

$$D_\phi^{(\alpha)}=\frac{(15)^{4/5}A^2}{25B_{00}}\langle N_0\rangle^{-3/5}\left(\frac{a}{d_0}\right)^{2/5}\frac{k_BT}{\hbar}$$
$$=\frac{15^{4/5}(\zeta(3))^{1/15}A^2}{25B_{00}}\frac{T}{T_c}\left(\frac{N}{\langle N_0\rangle}\right)^{3/5}$$
$$\times N^{-2/3}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{1/5}\frac{k_BT_c}{\hbar}. \tag{8.24}$$

It is smaller than $\gamma_{collapse}$, again by the order of magnitude of $(\langle N_0\rangle a/d_0)^{-2/5}$.

Finally, the contribution of the fluctuations in the thermal cloud to the phase diffusion is also obtained from Eq. (3.19) by inserting result (8.18) for $\Gamma_0+\Gamma'$:

$$D_\phi^{(\gamma)}=\frac{3}{10}\,\frac{2^{1/3}B'(\mu)}{15^{2/15}\pi^2}\langle N_0\rangle^{-17/15}\left(\frac{a}{d_0}\right)^{28/15}\left(\frac{k_BT}{\hbar\omega_0}\right)^2\frac{k_BT}{\hbar}$$

$$=\frac{3}{10}\,\frac{2^{1/3}(\zeta(3))^{-16/45}B'(\mu)}{15^{2/15}\pi^2}\left(\frac{T}{T_c}\right)^3\left(\frac{N}{\langle N_0\rangle}\right)^{17/15}$$

$$\times N^{-7/9}\left(\frac{k_BT_ca^2m}{\hbar^2}\right)^{14/15}\frac{k_BT_c}{\hbar}. \qquad (8.25)$$

The differs from the previous rates, which were all proportional to temperature, by the stronger temperature dependence $\sim T^3$. However, this contribution to D_ϕ remains smaller than $D_\phi^{(\alpha)}$ by an order of magnitude $N^{-1/9}(k_BT_ca^2m/\hbar^2)^{11/15}(T/T_c)^2$.

IX. DISCUSSION AND CONCLUSION

In this paper we have put forward a detailed theory of fluctuation and relaxation processes of the condensate in thermal equilibrium with the cloud of its excitations. For a given number of particles N_0 in the condensate, we have defined the condensate mode as the corresponding normalized solution of the Gross-Pitaevskii equation, at the same-time defining the N_0-dependent part of the chemical potential. The equilibrium value of $\langle N_0\rangle$ is distinguished as the value of N_0 for which the number of particles in the thermal cloud *in equilibrium* with the condensate plus N_0 is equal to N. We have calculated the fluctuations of N_0 around its equilibrium value, as well as the fluctuations of the phase of the complex amplitude α_0 of the condensate with $|\alpha_0|^2=N_0$. In a general phenomenological framework presented in the first part of this paper, we were able to separate the fluctuations of the complex condensate amplitude into several contributions, which have different physical origins.

(i) The fluctuation of the atom number in the condensate, which are driven by the exchange of atoms between the condensate and the thermal cloud.

(ii) The fluctuation of the chemical potential with two different contributions, namely, the fluctuations of μ due to number fluctuations in the condensate, and the faster fluctuations of μ at constant N_0 caused by number fluctuations in the excitations.

The importance of number fluctuations in the condensate, assumed at first in the phenomenological approach due to the importance of N_0 for the value of the chemical potential, but later born out by the microscopic calculations, leads to the appearance of the linear relaxation rates γ_c of the condensate number as an important characteristic inverse time scale of the problem. At times much shorter than γ_c^{-1}, a phase diffusion of the condensate phase due to the fast number fluctuations in the excitations can occur. In the same regime a process of collapse may also occur due to the reversible spreading of the phase caused by the static uncertainty in N_0, and the associated chemical potential. At times large compared to γ_c^{-1}, the number fluctuations in the condensate are dynamical and irreversible, and lead to the replacement of the reversible collapse by an irreversible phase diffusion with a larger diffusion rate than in the short-time regime.

The second and larger part of this paper was devoted to microscopic theory. First we provided a microscopic derivation of the phenomenological Langevin equation, established microscopic formulas for all phenomenological parameters, and also exhibited the relation between the short-time diffusion rate and fluctuation rates of the population numbers of excitations via a sum rule. Then a microscopic theory was used to evaluate the transport parameters and the various rates as functions of the temperature, particle number, and the scattering length of the interaction potential. The evaluation was done for two simple cases—the cubic box-like trap, where the form of the condensate mode does not depend on N_0 and the thermal cloud penetrates the condensate homogeneously; and the isotropic harmonic trap, where the form of the condensate-mode changes with N_0 and the thermal cloud is located preferentially near the boundaries of the condensate. The physically important results for both kinds of traps are similar, even though they have to differ, obviously, in the details of the scalings with the atom-numbers and the scattering length. A calculation of the transport parameters reveals some interesting physical results.

(i) The fluctuations driving the absolute value $|\alpha_0|$ and the phase ϕ of α_0 are quite different in strength, those driving $|\alpha_0|$ being much stronger. The reason for this is a pronounced squeezing of the bath of thermal excitations with respect to the instantaneous phase of the condensate. This squeezing reaches nearly 100% for the lowest-lying modes, which is the reason that fluctuations of ϕ are practically not driven by such modes. On the other hand, the contributions of high-lying modes to the fluctuating forces driving $|\alpha_0|$ and ϕ are nearly the same (after the obvious normalization with $|\alpha_0|$), i.e., there is no squeezing in this (much weaker) contribution to the noise.

(ii) The cross-correlation between the fluctuations driving $|\alpha_0|$ and ϕ are found to vanish exactly in a real condensate, where both the Gross-Pitaevskii equation and the Bogoliubov-Fetter equations are real, and all solutions can (but need not) be taken as real. This can also be understood as a general consequence of time-reversal symmetry: ϕ is a velocity potential and therefore odd under time reversal, while $|\alpha_0|$ is even under time reversal. Their fluctuating forces therefore transform oppositely. In a time reversal-symmetric condensate (no vortices), the cross-correlation between even and odd quantities under time reversal must vanish.

It turns out that the relaxation rate γ_c of the atom number in the condensate is the largest of the calculated rates. In particular it is larger than the collapse rate and the phase-diffusion rate which, like γ_c, are proportional to temperature in the regime $k_BT>\mu$. It is also larger than the decay rates of the lowest-lying collective modes γ_0, which might look surprising because at the same time the theory tells us that γ_c is dominated by particle-transfer rates between condensate and low-lying modes. However, it is clear that γ_c ought to be larger than γ_0 because the condensate couples to all low-lying modes in parallel, which increases the number of decay channels by a factor proportional to the ratio of the chemical potential and the lowest-lying mode frequency.

The next largest rate we find is the thermal phase-collapse rate $\gamma_{collapse}$. It turns out to have the same functional dependence on T, a, $\langle N_0\rangle$, and N as the decay rate of the lowest-lying collective modes. I cannot see any fundamental reason for this coincidence, and have to count it as just that. Physically the smallness of $\gamma_{collapse}/\gamma_c$ means that the phase collapse will not be observable at finite temperature, because it can only lead to a decay factor $\exp(-\frac{1}{2}(\gamma_{collapse}/\gamma_c)^2)$ very close to 1 before phase diffusion takes over.

Finally, the phase-diffusion rate D_ϕ is the smallest of the rates calculated here. We find the simple result that the ratios γ_c/γ_0 and γ_0/D_ϕ are of about equal orders of magnitude, given by the ratio of μ to the smallest excitation energy, which is $\hbar\omega_0$ for the harmonic trap and $\sqrt{\mu/m}(2\pi\hbar/L)$ for the boxlike trap. Instead of γ_0 we may also take γ_c in these ratios with the same conclusion. D_ϕ, like the rate γ_c, is found to be dominated by the atom-number exchange between the condensate and the low-lying modes.

This observation actually explains the coincidence of the two ratios we have just indicated and turns them into a precise relation: In Eq. (3.19), for D_ϕ, we put $\Gamma''=0$, which is exact for real condensate modes, and the neglect $\Gamma_0+\Gamma'$, which comes from high-lying excitations. Then multiplying the resulting expression for D_ϕ, with $\gamma_c=\tau_c^{-1}$ from Eq. (3.17), we readily find

$$\frac{1}{2}D_\phi\gamma_c=\gamma_{collapse}^2, \tag{9.1}$$

with $\gamma_{collapse}$ from Eq. (3.23) again with $\Gamma''=0$.

Let us now compare our results with related ones found in the literature. Most closely related to the present work in goal and scope is a paper by Jaksch *et al.* [26] on the intensity and amplitude fluctuations of a Bose-Einstein condensate at finite temperature, which builds on extensive earlier work by Gardiner and Zoller with collaborators (cf. the references given in Ref. [26]). Unlike the present paper, it also takes trap losses into account. The theory presented in Ref. [26] is based on a conceptual division of the Bose gas into two energy regions called the condensate band and the noncondensate band. In this construction the boundary between the two regions is chosen in such a way that the noncondensate band is not significantly affected by the mean field of the condensate, while the influence of excitations in the condensate band is neglected. Thus the main physical difference of Ref. [26] from to the present work is that it neglects fluctuations of particles from the condensate mode to quasiparticle modes, as well as to very low-lying one-particle excitations.

Conversely, in the present work we avoid the division of the energy region into two parts. We find, as we have discussed, that the exchange of particles between the condensate and the low-lying modes makes not only an important contribution, but in fact the dominant contribution, to the relaxation rate of the condensate number and the phase-diffusion rate, determining their dependence on temperature, atom number, population of the condensate, and scattering length.

The importance of the particle exchange between low-lying excitations for the phase diffusion of the condensate and the number-relaxation rate γ_c was first pointed out in Ref. [3], while for the *intensity* of the number fluctuations in the condensate this was already shown in Ref. [10(a)]. The theory put forward in Ref. [3] already proceeded along essentially the same lines we follow here, but it had some shortcomings which we overcome and correct in the present work: The squeezing of the noise from the thermal cloud with respect to the phase of the condensate was briefly remarked upon in Ref. [3], but was not taken into account in the calculation of the transport coefficients presented there and in the formula for phase diffusion. Moreover, in the conservative part of the Langevin equation (3.6) $\Delta_0\mu$ was replaced by $\partial\Delta F(|\alpha|^2)/\partial|\alpha_0|^2$ in Ref. [3], which, on scrutiny, appears questionable when used in conjunction with the fluctuation formulas (2.27) and (2.28) for $\langle\Delta N_0^2\rangle$. After all, neither $\Delta_0\mu$ nor ΔF are equilibrium quantities. The use of the aforementioned relation between them is therefore avoided here.

Even though in the present paper I have opted for the use of the fluctuation formulas (2.27) and (2.28), which in my opinion have a firm basis, it is only fair to mention that they are still under debate in the current literature; see, e.g., Ref. [13]. In another recent paper with some bearing on this topic Bergeman, *et al.* [27] used, as equilibrium distribution for the condensate number $P(N_0)\sim\exp[(\langle\mu\rangle N_0-\frac{5}{14}(15N_0a/d_0)^{2/5}N_0)/k_BT]$, [cf. the discussion after their Eq. (21)], which implies $\langle\Delta N_0^2\rangle\sim T\langle N_0\rangle^{3/5}$, a result which is rather different, both in temperature dependence and in scaling, from the particle number, from the result (8.19) on which our present calculations have been based. It is clear that not the method but the details of our results on the dynamics of the fluctuations of the condensate would change, if the results on the statics were to be changed. Needless to say, a resolution of the theoretical debate concerning the correct approach to the statics seems urgent, and would be highly welcome. Conversely, experimental results on the dynamics (i.e., on γ_c and D_ϕ) would also help to decide, by applying the theory presented here, which of the approaches to the statics of the number fluctuations in the condensate proposed in the literature describes the physics correctly.

A quantum kinetic theory of trapped atomic gases was also formulated by Stoof [28]. In Ref. [28] the general coupled Fokker-Planck equations of the condensate and the excited modes was presented and applied to the kinetics of the formation of a condensate. This problem was also studied by Gardiner *et al.* [29] as well as Kagan and Svistunov [30], where also earlier work by further authors is quoted.

By contrast the present work has focused on the fluctuations around the equilibrium state of the condensate, *after* it has been formed. However, the application of our approach to the kinetics of the formation of a condensate would be an interesting goal for future work.

Experimentally, the rates γ_c and D_ϕ we have calculated should be measurable. The rate γ_c may be observable as the relaxation rate of the condensate back to its equilibrium state after creating a nonequilibrium state by a sudden small change of temperature via evaporative cooling. The sum of the phase-diffusion rates of two condensates could be measured by monitoring the phase difference between them after it was initially fixed by measurement or preparation at a reference-time $t=0$. Methods for measuring phase differences in Bose-Einstein condensates were recently demonstrated [4,31,32]. It is to be hoped, therefore, that phase diffusion in Bose-Einstein condensates—a fundamental process intimately linked to the spontaneously broken gauge symmetry in a finite system—will be measured in the near future.

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APPENDIX A

Here we wish to derive expression (2.13) for H_0. Using the Gross-Pitaevskii equation (2.4) we put Eq. (2.8) into the form

$$H_0=(\mu_0-\langle\mu\rangle)|\alpha_0|^2-\frac{U_0}{2}|\alpha_0|^4\int d^3x\,\psi_0^4. \tag{A1}$$

Taking the derivative with respect to $|\alpha_0|^2$, we obtain

$$\frac{\partial H_0}{\partial|\alpha_0|^2}=\mu_0-\langle\mu\rangle+|\alpha_0|^2\left(\frac{\partial\mu_0}{\partial|\alpha_0|^2}-U_0\int d^3x\,\psi_0^4-|\alpha_0|^2U_0\times\int d^3x\,\psi_0^2\frac{\partial\psi_0^2}{\partial|\alpha_0|^2}\right). \tag{A2}$$

To evaluate this further, we use Eq. (2.4) and its derivative with respect to $|\alpha_0|^2$:

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V-\mu_0+3U_0|\alpha_0|^2\psi_0^2\right)\frac{\partial\psi_0}{\partial|\alpha_0|^2}=\left(\frac{\partial\mu_0}{\partial|\alpha_0|^2}-U_0\psi_0^2\right)\psi_0. \tag{A3}$$

Multiplying Eq. (A3) with ψ_0 and integrating over space, using the Gross-Pitaevskii equation (2.4) after partial integration, we derive the identity

$$U_0|\alpha_0|^2\int d^3x\,\psi_0^2\frac{\partial\psi_0^2}{\partial|\alpha_0|^2}=\frac{\partial\mu_0}{\partial|\alpha_0|^2}-U_0\int \psi_0^4d^3x, \tag{A4}$$

which is used in Eq. (A2) to yield $\partial H_0/\partial|\alpha_0|^2=\mu_0-\langle\mu\rangle$, and upon integration results in Eq. (2.13).

APPENDIX B

Here we wish to derive Eq. (4.14). This is achieved if we succeed is showing that the coupling of the condensate and the thermal cloud via

$$\hat{H}_3=U_0\sqrt{\langle N_0\rangle}\int d^3x\,\tilde{\psi}_0\hat{\tilde{\chi}}^\dagger(e^{-i\phi}\hat{\tilde{\chi}}+e^{i\phi}\hat{\tilde{\chi}}^\dagger)\hat{\tilde{\chi}} \tag{B1}$$

gives rise to the systematic change of $\mathrm{Im}(\hat{\xi}(t))$, to first order in the interaction, of

$$\delta\langle\mathrm{Im}(\hat{\xi}(t))\rangle_\phi=-\frac{2\sqrt{\langle N_0\rangle}}{\hbar k_BT}\int_{-\infty}^t dt'\,S_{JJ}(t-t')\frac{\partial H_0(t')}{\partial|\alpha_0|^2}, \tag{B2}$$

because this can then be used in Eq. (4.7) to yield Eq. (4.14). In Eq. (B1) we could put $|\alpha_0|=\sqrt{\langle N_0\rangle}$, since we linearize around equilibrium and only wish to calculate the dissipation in $|\alpha_0|^2$ which is conjugate to ϕ, the variable we kept in Eq. (B1). Standard first-order perturbation theory with an adiabatic switch-on of the interaction gives, with $\epsilon\to+0$,

$$\delta\langle\mathrm{Im}(\hat{\xi}(t))\rangle_\phi=-\frac{i}{\hbar}\int_{-\infty}^t dt'\langle[\mathrm{Im}(\hat{\xi}(t)),\hat{H}_3(t')]\rangle_\phi e^{\epsilon t'}. \tag{B3}$$

We can rewrite this as

$$\delta\langle\mathrm{Im}(\hat{\xi}(t))\rangle_\phi=-2i\sqrt{\langle N_0\rangle}\int_{-\infty}^t dt'(\chi''_{J\hat{\xi}}(t,t')e^{-i\phi(t')}+\chi''_{J\hat{\xi}^+}(t,t')e^{i\phi(t')})e^{\epsilon t'}, \tag{B4}$$

where we introduced the response functions

$$\chi''_{J\hat{\xi}}(t,t')=\frac{1}{2\hbar}\langle[\mathrm{Im}(\hat{\xi}(t)),\hat{\xi}(t')]\rangle_\phi\Theta(t-t'),$$
$$\chi''_{J\hat{\xi}^+}(t,t')=\frac{1}{2\hbar}\langle[\mathrm{Im}(\hat{\xi}(t)),\hat{\xi}^\dagger(t')]\rangle_\phi\Theta(t-t'), \tag{B5}$$

with

$$\hat{\xi}(t)=U_0\int d^3x\,\tilde{\psi}_0\hat{\tilde{\chi}}^\dagger(t)\hat{\tilde{\chi}}(t)\hat{\tilde{\chi}}(t). \tag{B6}$$

Here $\Theta(t-t')$ is the Heaviside step function. We shall define $\Theta(0)=0$ without loss of generality. The fluctuation-dissipation theorem (in the classical frequency domain $\hbar\omega\ll k_BT$) ensures the relations

$$\chi''_{J\hat{\xi}}(t,t')=-\frac{i\Theta(t-t')}{2k_BT}\frac{\partial}{\partial t'}S_{J\hat{\xi}}(t,t'),$$
$$\chi''_{J\hat{\xi}^+}(t,t')=-\frac{i\Theta(t-t')}{2k_BT}\frac{\partial}{\partial t'}S_{J\hat{\xi}^+}(t,t'), \tag{B7}$$

with the correlation functions[11]

$$S_{J\hat{\xi}}(t,t')=\langle \mathrm{Im}(\hat{\xi}(t))\hat{\xi}(t')\rangle_{\phi},$$
$$S_{J\hat{\xi}^{+}}(t,t')=\langle \mathrm{Im}(\hat{\xi}(t))\hat{\xi}^{+}(t')\rangle_{\phi}. \tag{B8}$$

We can use Eq. (B7) in Eq. (B4), and apply a partial integration in t', to obtain

$$\delta\langle \mathrm{Im}(\hat{\xi}(t))\rangle_{\phi}=\frac{\sqrt{\langle N_0\rangle}}{ik_BT}\int_{-\infty}^{t}dt'\,(S_{J\hat{\xi}}(t,t')e^{-i\phi(t')}-S_{J\hat{\xi}^{+}}(t,t')e^{i\phi(t')})\frac{d\phi(t')}{dt'}e^{\epsilon t'}-\frac{\sqrt{\langle N_0\rangle}}{k_BT}(S_{J\hat{\xi}}(t,t)e^{-i\phi(t)}+S_{J\hat{\xi}^{+}}(t,t)e^{i\phi(t)}), \tag{B9}$$

which can be rewritten as

$$\delta\langle \mathrm{Im}(\hat{\xi}(t))\rangle_{\phi}=\frac{2\sqrt{\langle N_0\rangle}}{k_BT}\int_{-\infty}^{t}dt'\,S_{JJ}(t,t')\frac{d\phi(t')}{dt'}-\frac{2\sqrt{\langle N_0\rangle}}{k_BT}S_{JR}(t,t), \tag{B10}$$

with $S_{JJ}(t,t')$ defined by Eqs. (4.9) and (4.11) and

$$S_{JR}(t,t')=\frac{1}{4i}\langle(\hat{\xi}(t)e^{-i\phi(t)}-\hat{\xi}^{\dagger}(t)e^{i\phi(t)})(\hat{\xi}(t')e^{-i\phi(t')}+\hat{\xi}^{\dagger}(t')e^{i\phi(t')})\rangle_{\phi}. \tag{B11}$$

The constant term with $S_{JR}(t,t)$ amounts to a small shift of the equilibrium value of μ in the final result, which we shall neglect like other terms contributing to such shifts. Then, using Eq. (4.2), we set $\hbar\, d\phi(t')/dt'=-\partial H_0(t')/\partial|\alpha_0|^2$ in Eq. (B10) which establishes Eq. (B1) and hence Eq. (4.14).

[11] $\mathrm{Im}(\hat{\xi}(t))$ according to Eq. (4.9) contains an explicit external time dependence via $\phi(t)$, in addition to the internal time dependence of $\hat{\tilde{\chi}}(t),\hat{\tilde{\chi}}^{\dagger}(t)$ via their Heisenberg equations of motion. This explicit time dependence has to be taken into account when applying the fluctuation-dissipation theorem. We avoid this additional step by applying the time derivative in the fluctuation-dissipation relation (B7) directly to the *second* time argument t', with of course the appropriate extra minus sign.

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