Retrodiction for quantum optical communications

Stephen M. Barnett,[1] David T. Pegg,[2] John Jeffers,[1] Ottavia Jedrkiewicz,[3] and Rodney Loudon[3]
[1]*Department of Physics and Applied Physics, University of Strathclyde, Glasgow G4 ONG, Scotland*
[2]*Faculty of Science, Griffith University, Nathan, Brisbane 4111, Australia*
[3]*Department of Physics, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, England*
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Given the result of a measurement on the output of a quantum optical communication channel, we show how to calculate a retrodictive state at the input. This state can be used by the receiver to determine the probability that any one of a given set of states was selected by the transmitter. We establish the remarkably simple result that retrodicting the prepared input signal for an attenuating (amplifying) channel corresponds to predicting the measured output signal for an amplifying (attenuating) channel.

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I. INTRODUCTION

Quantum cryptography relies on communication by means of quantum states [1]. Quantum communication [2] will be of increasing importance as this technology, together with quantum information processing and computing, develops. Optics provides the most immediate potential for realizing practical quantum communication channels and hence the study of quantum optical devices forms an important part of quantum communication theory.

An essential part of the communication problem can be simply stated as determining the transmitted message from the received signal [3]. This is made more complicated by the presence of any noise sources that tend to corrupt the signal. With quantum communications we encounter two features not present in the classical theory. These are (i) the problem of which of a set of incompatible measurements to perform and (ii) the fact that environmental effects such as losses introduce an irreducible level of quantum noise into the communication channel [4]. The problem of assessing a quantum communication channel can been seen either from the perspective of the transmitter (Alice) or the receiver (Bob) of the signal. It is important to realize that Alice and Bob base their analyses on quite different prior knowledge. Alice knows the signal that was sent and tries to assess the likely information retrieved by Bob. Bob, in contrast, knows what he received and tries to retrieve the information transmitted by Alice. We will assume that both Alice and Bob know the form of the communication channel which may include features such as losses, amplification, and phase shifts. The natural approach for Alice is to evaluate the effect of the quantum communication channel on her prepared signal. This is a *predictive* problem involving the evolution of the signal state as it propagates through the channel. Alice can then calculate the probabilities for the outcome of any measurement chosen by Bob. Bob's natural approach is to begin with the known outcome of his single measurement and work backwards to calculate a *retrodictive* [5–7] state from which he can determine the probability that Alice selected any particular signal state.

In this paper we compare the predictive and retrodictive analyses of the important effects of attenuation and amplification. Ideal classical amplification and attenuation would be inverse processes: if Bob could receive a signal attenuated in this way, he could simply restore Alice's signal with an ideal amplifier. In quantum mechanics, on the other hand, such devices are impossible and both processes suffer a noise penalty. Thus, following an attenuator by an amplifier does not restore the original signal [8]. We derive, however, the interesting and surprising result that the retrodictive density operator for the input to an attenuating optical element is simply an amplified form of the density operator corresponding to Bob's measurement result. For an amplifying element the retrodictive input state is an attenuated form of Bob's measured output state. Hence quantum linear amplification and attenuation are predictive-retrodictive inverses.

II. PREDICTIVE AND RETRODICTIVE QUANTUM MECHANICS

Alice prepares, at time t_p, the optical field in one of a set of states $|A_i\rangle$ with prior probability $P(A_i)$. This set of states and the prior probabilities are known both to Alice and to Bob. At a later time, t_m, Bob measures a field observable $\hat{B}$ with nondegenerate eigenstates $|B_j\rangle$. This observable is known both to Alice and to Bob [9]. Alice knows the state selected and wishes to calculate the probability that Bob obtains the result B_j. Bob knows the result of his measurement and requires the probability that Alice prepared the state $|A_i\rangle$.

Alice can use the predictive density operator $\hat{\rho}_i^{\text{pred}}(t_m)$, which may be defined by the requirement that its projection onto $|B_j\rangle\langle B_j|$ determines the probability that Bob obtains the result B_j conditioned on Alice preparing $|A_i\rangle$:

$$P(B_j|A_i) = \text{Tr}(|B_j\rangle\langle B_j|\hat{\rho}_i^{\text{pred}}(t_m)), \tag{2.1}$$

where the vertical stroke means ''if.'' Bob may perform a more general measurement than can be described by projection onto an eigenstate of a field observable. In such cases we can describe the measurement by means of a probability operator measure (POM) with elements $\hat{\Pi}_j$ [10]. These sum to the identity operator and the conditional probability (2.1) then becomes

$$P(B_j|A_i)=\mathrm{Tr}(\hat{\Pi}_j\hat{\rho}_i^{\mathrm{pred}}(t_m)). \tag{2.2}$$

Likewise, Bob requires a retrodictive density operator $\hat{\rho}_j^{\mathrm{retr}}(t_p)$ whose projection onto $|A_i\rangle\langle A_i|$ determines the probability that Alice selected $|A_i\rangle$ conditioned on Bob detecting the result B_j in accord with [7]

$$P(A_i|B_j)=\frac{\mathrm{Tr}[\hat{\Lambda}_i\hat{\rho}_j^{\mathrm{retr}}(t_p)]}{\sum_k \mathrm{Tr}[\hat{\Lambda}_k\hat{\rho}_j^{\mathrm{retr}}(t_p)]}. \tag{2.3}$$

Here $\hat{\Lambda}_i=P(A_i)|A_i\rangle\langle A_i|$ is an element of a set of operators which describes a biased preparation [7]. It is biased in the sense that Bob has some *a priori* information about the preparation. In the case where the preparation device is *unbiased*, the elements $\hat{\Lambda}_i$ become equal to $\hat{\Xi}_j/D$, where $\hat{\Xi}_j$ are the elements of a preparation POM and D is the dimension of the state space. In this case, Eq. (2.3) has a similar form to Eq. (2.2). The conditional probability can be obtained from the more familiar predictive approach using Eq. (2.2) in conjunction with Bayes' theorem [7].

The action of the quantum channel on the state, $\hat{\rho}_i^{\mathrm{pred}}(t_p)=|A_i\rangle\langle A_i|$, selected by Alice is represented by the unitary operator $\hat{U}$. Alice calculates the predictive reduced density operator for the channel to be

$$\hat{\rho}_i^{\mathrm{pred}}(t_m)=\mathrm{Tr}_E(\hat{U}\hat{\rho}_i^{\mathrm{pred}}(t_p)\otimes\hat{\rho}_E(t_p)\hat{U}^\dagger), \tag{2.4}$$

where $\hat{\rho}_E(t_p)$ is the density matrix representing the state of the channel environment at the preparation time and the trace is evaluated over all states of the environment.

The POM element associated with the field measurement result B_j is $\hat{\Pi}_j$. This POM element can be extended to act on the spaces of both the field and the channel environment. As Bob's measurement provides no information about the environment, the single POM element for the environment is the identity operator on the environment space, $\hat{1}_E$. The retrodictive density operator at time t_m is the normalized POM element associated with the result [7]. Hence the retrodictive density operator for the field and the environment at time t_m is

$$\hat{\rho}_{j,E}^{\mathrm{retr}}(t_m)=\frac{\hat{\Pi}_j\otimes\hat{1}_E}{\mathrm{Tr}_{ES}(\hat{\Pi}_j\otimes\hat{1}_E)}, \tag{2.5}$$

where the trace is evaluated over both the environment and the signal-field states [11]. The corresponding retrodictive density matrix at the preparation time t_p is

$$\hat{\rho}_{j,E}^{\mathrm{retr}}(t_p)=\frac{\hat{U}^\dagger\hat{\Pi}_j\otimes\hat{1}_E\hat{U}}{\mathrm{Tr}_{ES}(\hat{\Pi}_j\otimes\hat{1}_E)}. \tag{2.6}$$

The retrodictive state of the signal field at this time is conditioned by the known state of the channel environment $\hat{\rho}_E(t_p)$ at the preparation time. Hence the retrodictive density operator for the signal field at the preparation time, conditioned on $\hat{\rho}_E(t_p)$ being the state of the environment at time t_p, is

$$\hat{\rho}_j^{\mathrm{retr}}(t_p)=\frac{\mathrm{Tr}_E[\hat{\rho}_E(t_p)\hat{U}^\dagger\hat{\Pi}_j\otimes\hat{1}_E\hat{U}]}{\mathrm{Tr}_{ES}[\hat{\rho}_E(t_p)\hat{U}^\dagger\hat{\Pi}_j\otimes\hat{1}_E\hat{U}]}. \tag{2.7}$$

Bob can then determine the probability that Alice selected any given state $|A_i\rangle$ by substituting Eq. (2.7) into Eq. (2.3).

III. AMPLIFIERS AND ATTENUATORS

The simplest fully quantum description of an attenuator or an amplifier comprises two field modes: the signal mode, with annihilation operator $\hat{a}$, and an environment mode with annihilation operator $\hat{b}$ [12]. The unitary transformations associated with the attenuating and amplifying channels are

$$\hat{U}_{\mathrm{att}}=\exp[i\theta(\hat{a}^\dagger\hat{b}+\hat{b}^\dagger\hat{a})], \tag{3.1}$$

$$\hat{U}_{\mathrm{amp}}=\exp[i\phi(\hat{a}^\dagger\hat{b}^\dagger+\hat{b}\hat{a})], \tag{3.2}$$

respectively. The attenuation factor is $K=\cos^2\theta$ and the amplifier gain is $G=\cosh^2\phi$. The noise characteristics of the channel are determined by the initial state of the environment mode. It has recently been proven that the unitary operators (3.1) and (3.2) are related by [13]

$$_b\langle\chi|\hat{U}_{\mathrm{att}}|\psi\rangle_b=\cosh\phi\, _b\langle\psi^*|\hat{U}_{\mathrm{amp}}|\chi^*\rangle_b \tag{3.3}$$

for all states $|\psi\rangle$ and $|\chi\rangle$ if we set $\sin\theta=\tanh\phi$ and $\cos\theta=\mathrm{sech}\,\phi$. This choice makes the amplifier gain equal to the reciprocal of the attenuator loss. The number-state coefficients of $|\psi^*\rangle$ and $|\chi^*\rangle$ are the complex conjugates of the corresponding coefficients of $|\psi\rangle$ and $|\chi\rangle$, respectively. It follows immediately from the form of Eq. (3.2) that we can obtain $\hat{U}_{\mathrm{amp}}^\dagger$ from $\hat{U}_{\mathrm{amp}}$ by changing ϕ to $-\phi$ and hence that

$$_b\langle\chi|\hat{U}_{\mathrm{att}}|\psi\rangle_b=\cosh\phi\, _b\langle\psi^*|\hat{U}_{\mathrm{amp}}^\dagger|\chi^*\rangle_b \tag{3.4}$$

if we set $\sin\theta=-\tanh\phi$ and $\cos\theta=\mathrm{sech}\,\phi$. It should be noted that the change in sign makes no difference to K or G.

The density operator for the initial state of the environment mode can be written in the general form

$$\hat{\rho}_E(t_p)=\sum_l P_l|\psi_l\rangle_b\, _b\langle\psi_l|. \tag{3.5}$$

If we evaluate the traces in Eqs. (2.4) and (2.7) using the environment-mode number states, then Alice's predicted output state and Bob's retrodicted input state will be

$$\hat{\rho}_i^{\mathrm{pred}}(t_m)=\sum_n\sum_l P_l\, _b\langle n|\hat{U}|\psi_l\rangle_b\hat{\rho}_i^{\mathrm{pred}}(t_p)\, _b\langle\psi_l|\hat{U}^\dagger|n\rangle_b, \tag{3.6}$$

$$\hat{\rho}_j^{\rm retr}(t_p)=N\sum_n \sum_l P_l \, {}_b\langle\psi_l|\hat{U}^\dagger|n\rangle_b\hat{\rho}_j^{\rm retr}(t_m) \, {}_b\langle n|\hat{U}|\psi_l\rangle_b\,, \tag{3.7}$$

respectively, where N is a normalization factor. The unitary operator Eq. (3.1) or Eq. (3.2) is substituted into both Eqs. (3.6) and (3.7) for an attenuating or amplifying channel, respectively. If we identify the states $|\psi_l\rangle$ and $|n\rangle$ with $|\psi\rangle$ and $|\chi\rangle$ in Eq. (3.4) and its Hermitian conjugate, then we see that for the amplifier

$$\hat{\rho}_j^{\rm retr}(t_p)=\frac{{\rm Tr}_E(\hat{\rho}_E(t_p)\hat{U}_{\rm amp}^\dagger\hat{\rho}_j^{\rm retr}(t_m)\hat{U}_{\rm amp})}{{\rm Tr}_{Es}(\hat{\rho}_E(t_p)\hat{U}_{\rm amp}^\dagger\hat{\rho}_j^{\rm retr}(t_m)\hat{U}_{\rm amp})}$$
$$={\rm Tr}_E(\hat{U}_{\rm att}\hat{\rho}_j^{\rm retr}(t_m)\otimes\hat{\rho}_E^*(t_p)\hat{U}_{\rm att}^\dagger), \tag{3.8}$$

where $\hat{\rho}_E^{(*)}(t_p)$ is formed by replacing $|\psi_l\rangle$ with $|\psi_l^*\rangle$ in Eq. (3.5).

It is clear from Eq. (3.8) that the state *retrodicted* by Bob for an *amplifier* with gain G given $\hat{\rho}_j^{\rm retr}(t_m)$ is the same as that which Alice would *predict* for the action of an *attenuator*, with $K=G^{-1}$, given the input state $\hat{\rho}_j^{\rm retr}(t_m)$ and an initial environment state $\hat{\rho}_E^{(*)}(t_p)$. It is straightforward to show the corresponding result: the retrodicted state for an attenuator with loss K given $\hat{\rho}_j^{\rm retr}(t_m)$ is the same as that which Alice would predict for the action of an amplifier, with $G=K^{-1}$, given the input state $\hat{\rho}_j^{\rm retr}(t_m)$ and an initial environment state $\hat{\rho}_E^{(*)}(t_p)$.

For most attenuators and amplifiers the initial environment state is phase-independent and so the initial environment density matrix is diagonal in the energy basis. For such devices $\hat{\rho}_E^{(*)}(t_p)=\hat{\rho}_E(t_p)$, which further simplifies the above correspondence. It is possible, however, to prepare phase-dependent environments [14] and in such cases retrodiction using the above correspondence will involve changing the initial environment state to its ‘‘conjugate’’ state.

IV. AN EXAMPLE

As a very simple example of the attenuator-amplifier correspondence, consider an attenuating channel with an environment prepared in its ground state so that light will only be absorbed. Both Alice and Bob know that Alice sends some photon number state. They also know that Bob's detector perfectly counts photons. If Alice prepares a one-photon state to send through the channel, then she predicts that Bob will detect either one or zero photons. Similarly, if Bob detects a single photon at the output of an ideal amplifier (in which there is no absorption), then he can retrodict that either one or zero photons were sent.

Let Alice choose to prepare either the vacuum state with probability $P(A_0)$ or a one-photon state with probability $P(A_1)=1-P(A_0)$. She sends her signal to Bob through an attenuating channel with attenuation factor K. She can calculate the two corresponding predictive density operators:

$$\hat{\rho}_0^{\rm pred}(t_m)=|0\rangle\langle 0|, \tag{4.1a}$$

$$\hat{\rho}_1^{\rm pred}(t_m)=|1\rangle\langle 1|K+|0\rangle\langle 0|(1-K). \tag{4.1b}$$

She can use these to find the probabilities for Bob to register either one or zero photocounts in his detector, conditioned on her knowledge of the state prepared, by projection onto the appropriate POM elements $|1\rangle\langle 1|$ and $|0\rangle\langle 0|$:

$$P(B_1|A_0)=0, \tag{4.2a}$$

$$P(B_0|A_0)=1, \tag{4.2b}$$

$$P(B_1|A_1)=K, \tag{4.2c}$$

$$P(B_0|A_1)=1-K. \tag{4.2d}$$

Bob performs a measurement and finds either one or zero photocounts. He wishes to retrodict the state sent by Alice. To do so, he needs to calculate the corresponding two retrodictive density operators. According to the above predictive-retrodictive correspondence, he can find these by calculating the predictive density operators that would result if either a single photon or vacuum signal were sent through an amplifier with gain $G=1/K$. A straightforward calculation gives the output (predictive) density operator of an amplifier (with gain G) for an input m-photon number state, $|m\rangle$, as [15]

$$\hat{\rho}_m^{\rm pred}({\rm out})=\sum_{n=m}^{\infty}\binom{n}{m}\frac{(G-1)^{n-m}}{G^{n+1}}|n\rangle\langle n|. \tag{4.3}$$

For the one-photon and vacuum state input, these reduce to

$$\hat{\rho}_1^{\rm pred}({\rm out})=G^{-2}\sum_{n=1}^{\infty} n(1-1/G)^{n-1}|n\rangle\langle n|, \tag{4.4a}$$

$$\hat{\rho}_0^{\rm pred}({\rm out})=G^{-1}\sum_{n=0}^{\infty} (1-1/G)^{n}|n\rangle\langle n|. \tag{4.4b}$$

Hence the retrodictive density operators calculated for the attenuating channel are

$$\hat{\rho}_1^{\rm retr}(t_p)=K^2\sum_{n=1}^{\infty} n(1-K)^{n-1}|n\rangle\langle n|, \tag{4.5a}$$

$$\hat{\rho}_0^{\rm retr}(t_p)=K\sum_{n=0}^{\infty} (1-K)^{n}|n\rangle\langle n|. \tag{4.5b}$$

Combining these with $\hat{\Lambda}_0=P(A_0)|0\rangle\langle 0|$ and $\hat{\Lambda}_1=P(A_1)|1\rangle\langle 1|$, in accord with Eq. (2.3), gives the probabilities that Alice prepared either one or zero photons conditioned on the result of Bob's measurement:

$$P(A_0|B_0)=\frac{P(A_0)K}{P(A_0)K+P(A_1)K(1-K)}, \tag{4.6a}$$

$$P(A_1|B_0)=\frac{P(A_1)K(1-K)}{P(A_0)K+P(A_1)K(1-K)}, \tag{4.6b}$$

$$P(A_0|B_1)=0, \tag{4.6c}$$

$$P(A_1|B_1)=1. \tag{4.6d}$$

It is straightforward to show that these probabilities are the same as would be obtained by use of conventional (predictive) quantum mechanics for the attenuator in conjunction with Bayes' theorem, which states

$$P(B_j,A_i)=P(B_j|A_i)P(A_i)=P(A_i|B_j)P(B_j), \tag{4.7}$$

where the comma means "and." In order to show this, we require the *a priori* probabilities that Bob detects one and zero photons. From the conditional probabilities (4.2), these are

$$P(B_1)=P(B_1|A_0)P(A_0)+P(B_1|A_1)P(A_1)=KP(A_1), \tag{4.8a}$$

$$P(B_0)=P(B_0|A_0)P(A_0)+P(B_0|A_1)P(A_1) =P(A_0)+(1-K)P(A_1). \tag{4.8b}$$

Substituting these and the conditional probabilities (4.2) into Bayes' theorem (4.7) gives precisely the retrodictive conditional probabilities (4.6). We have previously established the more general result that all conditional probabilities calculated within the retrodictive formalism are in accord with Bayes' theorem [7].

V. CONCLUSION

Until now, retrodiction has been little more than an interesting philosophical concept associated with the problem of time asymmetry in quantum mechanics [5]. With the rapid development of quantum communications, however, it is likely that retrodictive quantum mechanics will become increasingly important. It is the receiver's natural approach to the communication problem of establishing the signal sent on the basis of the message received. Equations (2.3) and (2.7) provide all that is needed, for both closed and open systems, to perform retrodictive calculations of the probabilities that any one of a known set of quantum states was prepared given the outcome of a subsequent measurement. The first implementations of quantum communications have been optical and it is likely that quantum optical communications will continue to play a leading role. Any quantum optical communications channel will involve losses and perhaps also some amplification. For this reason we have concentrated, in this paper, specifically on the problem of retrodiction for optical attenuators and amplifiers.

We have established a remarkably simple correspondence between predicting the output of a quantum optical attenuator (amplifier) and retrodicting the input for an amplifier (attenuator). Moreover, Eqs. (2.3) and (2.7) apply generally to retrodiction for open systems and have important applications beyond single-mode amplifiers and attenuators including multimode devices and decohering qubits. We shall explore some of these applications elsewhere.

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