Space-time description of photon emission from an atom

Ole Keller
Institute of Physics, Aalborg University, Pontoppidanstrade 103, DK-9220 Aalborg Øst, Denmark
(Received 14 December 1999; published 18 July 2000)

Starting from the postulate that the electromagnetic field appearing in the transverse set of microscopic Maxwell-Lorentz equations governing field-matter interactions, properly normalized, can be looked upon as describing one-photon-emission and -absorption processes in space and time, a first-quantized initiation of photon emission by a single atom is presented. The wave function for the emerging photon is introduced as a six-vector object constructed from the complex analytical signals of the Riemann-Silberstein vectors belonging to opposite photon helicities. When the atom is no longer electrodynamically active, the emitted photon is described in first quantization by the so-called energy wave function well known for photons in free space. From the momentum representation of the emerging photon wave function a condition on the analytical part of the transverse atomic current density is established which ensures that precisely one photon is emitted. A propagator description of the emerged photon dynamics in the coordinate representation is established. The photon propagator is introduced as a two-component spinor, where upper and lower tensor components are constructed, respectively, from positive and negative helicity combinations of the propagators describing the time-space evolution of the transverse electric and magnetic fields. It is shown that the emission region for the photon coincides with the region in space where the transverse atomic current density is nonvanishing. For a photon emitted in an electric dipole transition the emission region essentially is the near-field zone of the atom, and this zone therefore determines the initial (and best) spatial confinement of the photon. The photon emerging from an atom active for a finite time necessarily is of the polychromatic sort and the associated wave packet essentially is confined between spherical shells moving outwards with the vacuum speed of light. To illustrate the main principles of the fundamental theory in a heuristic fashion we apply it to a study of the emission of a one-photon sinusoidal wavetrain from a pointlike atom. It is found that the atomic current density needed to create just one photon is independent of the oscillation period in the train and thus depends only on the number of periods in the wave train. An explicit expression for the one-photon energy is derived, and it is shown that only for extremely short pulse trains pronounced deviations from the textbook result, $E=\hbar\omega_0$, occur. The radial energy flow in the coupled atom-photon system in the near-field zone of the atom is investigated, and the cycle-averaged outwards energy transport carried by the emerging photon in a given distance from the atom is determined.

PACS number(s): 03.65.Ca, 42.50.−p, 42.50.Ct

I. INTRODUCTION

The electromagnetic interaction between an atom and a quantum field usually is described in the language of second quantization, and the photons are the quantum excitations of the field [1]. To the best of our knowledge quantum electrodynamics (QED) offers us a rigorous framework for studying all fundamental atom-field interactions, and over the years various mathematical techniques have been used to investigate the time development of the coupled photon-atom dynamics. Traditionally, one starts from the Heisenberg equations of motion for the atomic operators and the mode operators of the plane-wave components of the field, and the fingerprints of the photon-atom interaction are looked for in the properties of the radiated field in the far-field zone [2,3].

The development of near-field optics (NFO) within the last two decades has made it clear that an improved understanding of the matter-field interaction on a length scale (much) smaller than the optical wavelength(s) is needed. In attempts to improve the spatial resolution in NFO it is important to understand what kind of conceptual limitations QED sets for the spatial confinement of light. Recently some insight into this question has been obtained using an electromagnetic propagator picture to describe as time elapses the loss of spatial confinement of quantized light emitted from a single atom [4]. For a given optical transition the spatial extension of the related transverse current density of the atom gives us precisely the strongest confinement of the quantum field, and for an electric-dipole active transition the source region of the field extends over the entire near-field zone of the atom.

Let us imagine now that just one photon is emitted from the atom. As long as the time derivative of the atomic current density is nonvanishing the photon is in the process of being emitted, and this process takes place over the near-field zone of the atom. To investigate theoretically the emission process of a single and necessarily polychromatic photon in *both* space and time I have found it useful to seek a first-quantized description of the process as a forerunner to a second-quantized QED theory. A first-quantized theory for the photon is attractive in the present context because a photon wave function in the coordinate representation may be introduced.

The concept of a photon wave function in direct space was suggested by Landau and Peierls in 1930 [5], and has more recently been investigated and used by Cook [6–8] and Inagaki [9]. The Landau-Peierls wave function has a number of less attractive properties. Hence, it bears a nonlocal relation to the local electromagnetic field and is apparently not a good candidate for understanding the detection of partially localized photons. In the present work we shall use the six-vector wave function advocated by Bialynicki-Birula [10–12]. This, as well as the closely related three-vector wave function used by Sipe [13] arise from the Riemann-Silberstein vector [14–17] introduced in the beginning of the 20th century to rewrite Maxwell's equations in complex form. It appears that Oppenheimer [18] suggested the use of the Riemann-Silberstein vector as the wave function of the photon in direct space. Since the information carried by the negative-frequency components of the Riemann-Silberstein wave function is already contained in the positive-frequency part of the wave function only the positive-energy (frequency) part enters the Bialynicki-Birula definition [12] and the one used below. By this choice a useful connection to the so-called analytical signal of importance in both classical and quantum field theory [3] is induced in the formalism. Quite recently another attractive description of the photon in free space was established by Hawton [19,20], who argued for a photon state vector proportional to the four-vector potential in order to base photon quantum mechanics on number density [1,3] as is usual. A good review of the almost century-old history of the photon wave function has recently been given by Bialynicki-Birula [12], and readers interested in the connection between the photon wave function concepts in the coordinate representation and the so-called coarse-grained detection theory [7] may start from the book by Mandel and Wolf [3].

The emission process in space-time of a single photon generated by an electrodynamically active atom is at focus in the present theory, and this necessitates that the matter-field coupling is involved in the formalism, and this aspect makes the Bialynicki-Birula-Sipe approach adequate. In fact, Sipe [13] relates his photon wave-function description to the spontaneous-emission process via the Power-Zinau-Woolley Lagrangian [21–25], but since the photon field is assumed to be confined to within $c_0\tau$ of the atom at a time τ after the atom has started to decay, the near-field confinement dynamics studied by the present author recently [4] was lost. As we shall realize the near-field zone plays an important role in photon emission. Also Bialynicki-Birula stresses the importance of going beyond the free-space photon description, and does this by studying the photon wave equation in an inhomogeneous medium and in optical fibers [10,12]. In both cases, however, the matter dynamics is described in the framework of macroscopic electrodynamics, and the phenomenological nature of such an approach as well as the complications arising when dealing with a material many-body system [26,27] make such systems less attractive in attempts to understand the emission and absorption processes for single photons in a first-quantized description.

In Sec. II B, we briefly review the Riemann-Silberstein-Bialynicki-Birula description of the photon wave function in free space, paying attention to those aspects which are of particular importance for the subsequent development of the theory. The need for dividing the electromagnetic field correctly into genuine transverse and longitudinal vector field parts in both matter and matter-free regions is emphasized [28], and the importance of this appears in full scale in Secs. II C and II D. In Sec. II C, single-photon emission in momentum space is studied and a description of the initiation of photon emission is introduced on the basis of a six-vector wave-function object. It is suggested that the transverse set of microscopic Maxwell-Lorentz equations, properly normalized, describe the photon-emission (and -absorption) process in space-time. In Sec. II D, photon emission is examined in direct space, and a propagator description of the emission initiation dynamics is established. The propagator description appears particularly useful because it allows us to investigate the spatial localization of the emerging photon in a direct manner, and because the propagation speed of the photon field, i.e., the vacuum speed of light, appears so explicitly in the formalism. In the coordinate representation the photon emission process is studied in two equivalent propagator pictures (views) [4]: The view from the photon's perspective in which the source region is identified with the region where the transverse atomic current density is different from zero, and the view from the electron's perspective in which the photon source region is imagined to be compressed to coincide with the region of nonvanishing electron density (essentially).

In Sec. III the general theory is applied in a model calculation. Thus, a sinusoidal wave train (of finite length) emitted from a pointlike atom is taken as an heuristic paradigm. The atomic current density needed to emit precisely one photon is calculated in Sec. III B for a point dipole, and in Sec. III C finite-size corrections to the current density are studied for the hydrogen $1s\leftrightarrow 2p_z$ transition. The one-photon energy is determined in Sec. III D, and a simple explicit formula, derived. From this the textbook result for the energy $\hbar\omega_0$ is regained as the length of the photon wave train is increased beyond a few cycles. In Sec. III E the cycle-averaged radial energy flows in the near-field zone of the atom are investigated for the wave-train paradigm, in both the photon and electron perspectives, and we determine how the energy flows between the emerging photon and the atom as a function of the distance from the point particle.

In the Appendixes, the photon-antiphoton interference is studied, the magnetic-field propagators relevant in the photon and electron perspectives calculated, and the transverse and longitudinal parts of the hydrogen $1s\leftrightarrow 2p_z$ current density determined in wave-vector space.

In a forthcoming paper it will be demonstrated that the present first-quantized theory can be used to establish a rigorous one-photon theory for optical tunneling [29,30]. In photon-tunneling processes the conceptual limitation in our ability to localize photons in space plays a crucial role [30], and in a space-time description of the tunneling of single photons studies of the dynamics of the emerging photon therefore turns out to be indispensable.

II. FIRST-QUANTIZED THEORY OF THE EMERGING PHOTON

A. Preliminary considerations

In a simple first-quantized description a photon cannot be created nor can it be annihilated, and therefore the single-photon wave function is a concept of the free electromagnetic field. Even in a nonrelativistic treatment of matter-field interactions, where the number of charged particles is fixed, the number of elementary excitations in the radiation field, the photons, inevitably will change as a function of time. Thus, spontaneously emitting atoms generate photons, in other processes photons are absorbed (since with no rest mass they cannot be stopped), and field propagation, e.g., in condensed media consists of a succession of photon absorption and emission processes. At first sight one might therefore be inclined to think that the photon wave-function concept is of limited usefulnes in studies of light-matter interactions in quantum optics. As we shall realize in the following, this conception is not correct.

The notion of a photon wave function in momentum representation has been well founded for many years, whereas the coordinate representation has stirred much controversy over the years [11]. It seems, however, that a good candidate for a position-representation wave function is (the) one which relates to the probability amplitude for the photon energy to be located (detected) at the various space points at a given time [10–13]. This so-called photon-energy wave function [3] is proportional to the transversely polarized electric field prevailing in free space.

Once accepted that the single-photon wave function is intimately connected with the transverse part of the classical electromagnetic field a search for a conceptual framework for understanding the transverse set of microscopic Maxwell-Lorentz equations, describing the interaction of matter with transverse electromagnetic fields from the photon point of view, seems unavoidable. I suggest here that these equations, properly normalized, in the one-photon case can be looked upon as describing the photon emission (or absorption) process in space and time. Since it is legitimate to claim that field-matter interaction occurs in every place in space where the time derivative of the transverse part of the particle current density is different from zero the domain occupied by these places constitutes the emission (or absorption) region of the photon. The process of emitting (or absorbing) the photon with certainty lasts as long as the time derivative of the particle current is different from zero.

In Secs. II B–II D we shall study the theoretical considerations which lay the foundation for the above-mentioned point of view, and describe some of the perspectives emerging in the wake. By incorporating the photon emission and absorption processes as an integral part of the first-quantized one-photon theory it appears to me that this theory might be quite useful. As an extra bonus the theory helps bridging the gap to the second-quantized time-space description of the near-field electrodynamics of atoms, a subject to be studied in detail in a forthcoming paper.

B. Single-photon wave function

We begin our study with a summary of the theory for the single-photon energy wave function in empty space paying attention to those aspects of the formulation which are of particular importance for the subsequent description of a photon emerging from an electrodynamically active atom.

We know from the microscopic Maxwell equations that the magnetic field $\mathbf{B}(\mathbf{r},t)$ is a *transverse vector field* since in *every* space-time point $(\mathbf{r},t)$ it satisfies the condition

$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=0. \tag{1}$$

Since matter inevitably is present in portions of space, the electric field $\mathbf{E}(\mathbf{r},t)$ is not so simple, but we can always divide it uniquely into a transverse (subscript T) vector-field part, $\mathbf{E}_T(\mathbf{r},t)$, and a longitudinal ($L$) part, $\mathbf{E}_L(\mathbf{r},t)$. Though the division is unique in a given inertial frame it will in general be different in another frame. This lack of relativistic invariance is of no importance here, however. Despite the fact that the electric field fulfils the condition $\nabla\cdot\mathbf{E}(\mathbf{r},t)=0$ in those regions of space where the particle charge density is zero, we are not entitled to claim that only the transverse part of the electric vector field is present in charge-free regions. This is so because also the condition $\nabla\times\mathbf{E}(\mathbf{r},t)=\mathbf{0}$ is obeyed in certain parts of the particle-empty space, namely in the near-field zone of matter [4,27,28]. Technically the extension of this zone is identified with the region of matter-empty space where the transverse (or equivalently longitudinal) part of the particle charge current density is nonvanishing. As I shall demonstrate later this region in fact also is the emission region of the photon, a spatially extended object already from the outset of the emission process. In every space-time point of the abstract matter-free space, in which only a transverse electric field, $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_T(\mathbf{r},t)$, exists, in addition to

$$\nabla\cdot\mathbf{E}_T(\mathbf{r},t)=0, \tag{2}$$

and also the two relations

$$\nabla\times\mathbf{E}_T(\mathbf{r},t)=-\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{3}$$

$$\nabla\times\mathbf{B}(\mathbf{r},t)=\frac{1}{c_0^2}\frac{\partial\mathbf{E}_T(\mathbf{r},t)}{\partial t}, \tag{4}$$

where c_0 is the vacuum speed of light, hold. Together, Eqs. (1)–(4) constitute the (microscopic) Maxwell equations in empty space. Only in this idealized world a photon is a robust object.

We denote the solution to the set of empty-space Maxwell equations, which below will be related to the energy wave function of a single photon, by $(\mathbf{e}_T(\mathbf{r},t),\mathbf{b}(\mathbf{r},t))$. Other solutions $(\mathbf{E}_T(\mathbf{r},t),\mathbf{B}(\mathbf{r},t))$ to those homogeneous Maxwell equations can be found by multiplying this solution with an arbitrary constant (α), i.e.,

$$(\mathbf{E}_T(\mathbf{r},t),\mathbf{B}(\mathbf{r},t))=\alpha(\mathbf{e}_T(\mathbf{r},t),\mathbf{b}(\mathbf{r},t)). \tag{5}$$

We now introduce the specific Riemann-Silberstein vectors [14–17]

$$\mathbf{f}_\pm(\mathbf{r},t)=\sqrt{\frac{\epsilon_0}{2}}[\mathbf{e}_T(\mathbf{r},t)\pm ic_0\mathbf{b}(\mathbf{r},t)], \tag{6}$$

where ϵ_0 is the vacuum permittivity. By means of Eqs. (3) and (4) these vectors are seen to satisfy the differential equations

$$i\hbar \frac{\partial \mathbf{f}_{\pm}(\mathbf{r},t)}{\partial t} = \pm c_0 \hbar \nabla \times \mathbf{f}_{\pm}(\mathbf{r},t). \quad (7)$$

To prepare for the quantum description the Planck constant divided by 2π, i.e., $\hbar$, has been put in. Following the suggestion by Birula-Bialynicki [10–12], I define the one-photon wave function in the space-time domain, $\mathbf{\Phi}(\mathbf{r},t)$, as the six-component object

$$\mathbf{\Phi}(\mathbf{r},t) \equiv \begin{pmatrix} \mathbf{f}_{+}^{(+)}(\mathbf{r},t) \\ \mathbf{f}_{-}^{(+)}(\mathbf{r},t) \end{pmatrix}, \quad (8)$$

where

$$\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t) = \sqrt{\frac{\epsilon_0}{2}}[\mathbf{e}_T^{(+)}(\mathbf{r},t) \pm ic_0 \mathbf{b}^{(+)}(\mathbf{r},t)]$$
$$= \sqrt{\frac{\epsilon_0}{2}} \int_0^\infty [\mathbf{e}_T(\mathbf{r};\omega) \pm ic_0 \mathbf{b}(\mathbf{r};\omega)] e^{-i\omega t} d\omega \quad (9)$$

are the positive-frequency (ω) parts of the respective Riemann-Silberstein vectors. In the theory of classical and quantum coherence, as well as in quantum detection theory, the positive-frequency parts of the various fields $(\mathbf{E}_T^{(+)}, \mathbf{B}^{(+)})$, called the (complex) analytic signals, play a more prominent role than the fields themselves [3], and as emphasized by Birula-Bialynicki [12] the analytic signal, for consistency in the broader framework of particles and antiparticles, must be related to the photon and the negative-frequency parts $\mathbf{f}_{\pm}^{(-)}(\mathbf{r},t)$ of the Riemann-Silberstein vectors to the antiphoton. Since the antiphoton is identical to the photon itself, the technical bonus of working with the analytical signal is that this removes a redundancy in the description.

By multiplying $\mathbf{\Phi}(\mathbf{r},t)$ with its Hermitian conjugate (row) vector $\mathbf{\Phi}^\dagger(\mathbf{r},t) = ([\mathbf{f}_{+}^{(+)}(\mathbf{r},t)]^*, [\mathbf{f}_{-}^{(+)}(\mathbf{r},t)]^*)$ one obtains upon integration over the entire space

$$E = \int_{-\infty}^{\infty} \mathbf{\Phi}^\dagger(\mathbf{r},t) \cdot \mathbf{\Phi}(\mathbf{r},t) d^3r = \int_{-\infty}^{\infty} \mathbf{f}_{+}(\mathbf{r},t) \cdot \mathbf{f}_{-}(\mathbf{r},t) d^3r. \quad (10)$$

Since

$$w_T(\mathbf{r},t) = \mathbf{f}_{+}(\mathbf{r},t) \cdot \mathbf{f}_{-}(\mathbf{r},t)$$
$$= \frac{\epsilon_0}{2}[\mathbf{e}_T(\mathbf{r},t) \cdot \mathbf{e}_T(\mathbf{r},t) + c_0^2 \mathbf{b}(\mathbf{r},t) \cdot \mathbf{b}(\mathbf{r},t)] \quad (11)$$

precisely is the energy density in the classical electromagnetic vacuum field, the quantity E above is identified as the energy of the photon. Though a deeper analysis appears needed, taking into account the coupling of the photon to the particle field, we may tentatively say that $\mathbf{\Phi}^\dagger(\mathbf{r},t) \cdot \mathbf{\Phi}(\mathbf{r},t)$ represents the energy density in the photon field in our first-quantized description; hence the name photon-energy wave function for $\mathbf{\Phi}(\mathbf{r},t)$. One may upgrade the formalism to second quantization and suggest [3,10–13] that $\mathbf{\Phi}^\dagger(\mathbf{r},t) \cdot \mathbf{\Phi}(\mathbf{r},t) d^3r$ represents in the statistical sense the (unnormalized) probability that the photon energy is localized in the infinitesimal volume d^3r around $\mathbf{r}$ at time t. The issue above cannot be elucidated further without involving matter, in relation for instance to (i) the photon detection process [3] and (ii) a discussion of the fundamental limitations which quantum electrodynamics (QED) forces upon us when we seek to obtain an extremely strong spatial localization of a photon [4]. Although the overwhelming majority of papers dealing with the conceptual possibilities of localizing a photon have dealt with the free-space dynamics [31–38,3,19], I hold the point of view that field-matter coupling is needed to understand the spatial localization process of photons [4]. Seen from this perspective, a description of the photon-emission process as the one presented in Secs. II C and II D might be useful as a first step towards a second-quantized formulation. In Appendix A a derivation of Eq. (10) is given, and the underlying physics is addressed in more detail than hitherto in the literature, in particular the photon versus antiphoton aspect. Since it follows from the dynamical equations (7), or from the energy balance equation for the electromagnetic field, that

$$\frac{d}{dt} \int_{-\infty}^{\infty} \mathbf{\Phi}^\dagger(\mathbf{r},t) \cdot \mathbf{\Phi}(\mathbf{r},t) d^3r = 0, \quad (12)$$

a photon once introduced in empty space never disappears.

Before proceeding a comment on the definition I have chosen in Eq. (8) for the photon wave function should be made. Historically, the use of the Riemann-Silberstein vectors $\mathbf{f}_{\pm}(\mathbf{r},t)$ as the wave function of the photon in $\mathbf{r}$ space has been advocated first by Oppenheimer [18], and subsequently by a number of other physicists [39,40,10–13]. The choice of the plus sign [in Eq. (6)] in fact means that only photons of positive helicity are considered. Photons of negative helicity are treated by means of $\mathbf{f}_{-}(\mathbf{r},t)$. If we do not want to address the photon-emission process it is sufficient to use a formalism in which the photon wave function for a given helicity is a three-component object $[\mathbf{f}_{\pm}(\mathbf{r},t)]$, remembering the particle $\leftrightarrow$ antiparticle redundancy hidden in the relations $\mathbf{e}_T^*(\mathbf{r};-\omega) = \mathbf{e}_T(\mathbf{r};\omega)$ and $\mathbf{b}^*(\mathbf{r};-\omega) = \mathbf{b}(\mathbf{r};\omega)$. In the emission (absorption) process of the photon linear superpositions of the two helicity states occur as we shall realize later on, and therefore, it is profitable to consider the two helicity states as the upper and lower components of the same wave function. Bialynicki-Birula argues for the need of a six-component wave function in order to deal with the one-photon concept in inhomogeneous media in an effective manner. Basically, I agree with this point of view, but an inhomogeneous medium which dynamics is described in a phenomenological manner by a (space-dependent) dielectric constant is not well suited for understanding the basic features of the coupling of one photon to matter. When extended to many-particle electronic wave functions the emission process formalism established in Secs. II C and II D, together with a similar description of the photon absorption process, appears to me to constitute a better framework for understanding, in a first-quantized version, single-photon dynamics in condensed matter systems. Using microscopic local-field calculation techniques I shall address this problem in a later publication. In the theory of Sipe [13], a three-component wave function including both helicities is defined. Essentially, his choice for the photon wave function in direct space is a properly normalized sum of the two Riemann-Silberstein vectors, i.e., $\mathbf{f}_+(\mathbf{r},t)+\mathbf{f}_-(\mathbf{r},t)$. This gives a free-space photon wave function proportional to the transverse part of the electric field, $\mathbf{e}_T$. In Sipe's treatment of the one-photon emission from a spontaneously decaying atom the transverse electric field is replaced (via a Power-Zinau-Woolley transformation [21–25]) by the displacement field, $\mathbf{d}(\mathbf{r},t)$, in order to preserve the causality of the outgoing photon field. In Sipe's description the photon is initially, i.e., when the emission process starts, completely localized in space, or at least to a region identical with the electronic size of the atom. The complete localization is not found in the present analysis, and yet no violation of Einstein causal photon propagation appears, see Ref. [4] and Sec. II D of this paper.

Following the standard approach [12,13] the one-photon wave function in momentum ($\mathbf{p}$) representation is introduced starting from the Fourier transformations

$$\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)=(2\pi\hbar)^{-3}\int_{-\infty}^{\infty}\sqrt{c_0 p}\,\mathbf{g}_{\pm}(\mathbf{p},t)e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}d^3p \tag{13}$$

and their inverse

$$\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)=\sqrt{c_0 p}\,\mathbf{g}_{\pm}(\mathbf{p},t)=\int_{-\infty}^{\infty}\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}d^3r. \tag{14}$$

If one then transforms Eq. (7) to the frequency (ω) domain, and thereupon integrates the resulting equation over the positive frequencies it is realized that the $\mathbf{f}_{\pm}^{(+)}$'s satisfy the differential equations

$$i\hbar\frac{\partial\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)}{\partial t}=\pm c_0\hbar\boldsymbol{\nabla}\times\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t). \tag{15}$$

By inserting the expressions in Eqs. (13) into Eqs. (15) one gets

$$\hbar\frac{\partial\mathbf{g}\pm(\mathbf{p},t)}{\partial t}=\pm c_0\mathbf{p}\times\mathbf{g}_{\pm}(\mathbf{p},t). \tag{16}$$

With the help of the helicity unit vectors $\hat{\boldsymbol{\epsilon}}_{\pm}(\hat{p})$, given by

$$\hat{\boldsymbol{\epsilon}}_{\pm}(\hat{\mathbf{p}})=\frac{1}{\sqrt{2}}[\hat{\boldsymbol{\epsilon}}_1(\hat{\mathbf{p}})\pm i\hat{\boldsymbol{\epsilon}}_2(\hat{\mathbf{p}})], \tag{17}$$

where the unit vectors $\hat{\boldsymbol{\epsilon}}_1(\hat{\mathbf{p}})$, $\hat{\boldsymbol{\epsilon}}_2(\hat{\mathbf{p}})$, and $\hat{\mathbf{p}}=\mathbf{p}/p$ (in this order) form a right-handed triad, the vector amplitudes, $g_{\pm}(\mathbf{p},t)$, introduced via

$$\mathbf{g}_{\pm}(\mathbf{p},t)=g_{\pm}(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_{\pm}(\hat{\mathbf{p}}), \tag{18}$$

are seen to satisfy the Schrödinger-like equation

$$i\hbar\frac{\partial g_{\pm}(\mathbf{p},t)}{\partial t}=pc_0 g_{\pm}(\mathbf{p},t). \tag{19}$$

The single-photon wave function in the momentum representation then is defined as the six vector

$$\boldsymbol{\Phi}(\mathbf{p},t)\equiv(2\pi\hbar)^{-3/2}\begin{pmatrix}g_+(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{p}})\\ g_-(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_-(\hat{\mathbf{p}})\end{pmatrix}. \tag{20}$$

Although the notations $\boldsymbol{\Phi}(\mathbf{r},t)$ and $\boldsymbol{\Phi}(\mathbf{p},t)$ have been used for the photon wave function in, respectively, the space and momentum representation, one must remember that the two functions do not form a pair of Fourier transforms, cf. Eqs. (13) and (14). The probability density $P(\mathbf{p},t)$ in momentum space

$$P(\mathbf{p},t)\equiv\boldsymbol{\Phi}^{\dagger}(\mathbf{p},t)\cdot\boldsymbol{\Phi}(\mathbf{p},t), \tag{21}$$

with $\boldsymbol{\Phi}^{\dagger}(\mathbf{p},t)=(2\pi\hbar)^{-3/2}[g_+^*(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_+^*(\hat{\mathbf{p}}),g_-^*(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_-^*(\hat{\mathbf{p}})]$, thus is given by

$$P(\mathbf{p},t)=(2\pi\hbar)^{-3}[|g_+(\mathbf{p},t)|^2+|g_-(\mathbf{p},t)|^2], \tag{22}$$

as one readily realizes since $\hat{\boldsymbol{\epsilon}}_{\pm}^*(\hat{\mathbf{p}})\cdot\hat{\boldsymbol{\epsilon}}_{\pm}(\hat{\mathbf{p}})=1$. From the assumption that we are dealing with just one photon follows the normalization condition

$$\int_{-\infty}^{\infty}\boldsymbol{\Phi}^{\dagger}(\mathbf{p},t)\cdot\boldsymbol{\Phi}(\mathbf{p},t)d^3p=1, \tag{23}$$

and if Eq. (23) is satisfied at one time, the Schrödinger-like time evolutions for $g_{\pm}(\mathbf{p},t)$, given in Eqs. (19), guarantee that it holds at all later times. In fact, since no coupling is present between the two helicity components, the much sharper conditions $\partial|g_{\pm}(\mathbf{p},t)|^2/\partial t=0$ hold, as one may realize with the help of Eqs. (19), and therefore the probability density in momentum space is time independent, i.e., $P=P(\mathbf{p})$. The homogeneity of the dynamical equations in Eq. (19) leaves the amplitudes of $g_{\pm}(\mathbf{p},t)$ undetermined but the normalization condition in Eq. (23) fixes them. In turn the amplitudes of $\mathbf{f}_{\pm}^{(+)}$ are determined via Eqs. (13), and finally the amplitudes of the transverse electric ($\mathbf{e}_T$) and magnetic ($\mathbf{b}$) fields entering the single-photon Riemann-Silberstein vectors [see Eqs. (9)] are uniquely determined.

The one-photon energy density in momentum space, $w_T(\mathbf{p},t)$, can be obtained taking as a starting point the Parseval-Plancherèl relation

$$\int_{-\infty}^{\infty}\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)\cdot[\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)]^*d^3r$$
$$=(2\pi\hbar)^{-3}\int_{-\infty}^{\infty}\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)\cdot[\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)]^*d^3p. \tag{24}$$

Utilizing that $\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)=\sqrt{c_0 p}\,\mathbf{g}_{\pm}(\mathbf{p},t)$ [Eqs. (14)], the photon energy can be written in the form

$$E=\int_{-\infty}^{\infty}\{\mathbf{f}_{+}^{(+)}(\mathbf{r},t)\cdot[\mathbf{f}_{+}^{(+)}(\mathbf{r},t)]^{*}+\mathbf{f}_{-}^{(+)}(\mathbf{r},t)\cdot[\mathbf{f}_{-}^{(+)}(\mathbf{r},t)]^{*}\}d^{3}r$$
$$=(2\pi\hbar)^{-3}\int_{-\infty}^{\infty}c_{0}p[\mathbf{g}_{+}(\mathbf{p},t)\cdot\mathbf{g}_{+}^{*}(\mathbf{p},t)+\mathbf{g}_{-}(\mathbf{p},t)\cdot\mathbf{g}_{-}^{*}(\mathbf{p},t)]d^{3}p, \tag{25}$$

or equivalently

$$E=\int_{-\infty}^{\infty}c_{0}pP(\mathbf{p})d^{3}p, \tag{26}$$

with $P(\mathbf{p})$ given by Eq. (22). It appears from Eq. (26) that the one-photon energy density $w_T(\mathbf{p})$ equals the product of the $\mathbf{p}$-space photon energy c_0p and probability $P(\mathbf{p})$, i.e.,

$$w_{T}(\mathbf{p})=c_{0}pP(\mathbf{p}), \tag{27}$$

as desired.

C. Single-photon emission process in momentum space; emergence of a photon

Let us now turn our attention towards the establishment of a so-called photon perspective of the transverse microscopic Maxwell equations in the presence of matter. If we denote the transverse part of the prevailing current density, $\mathbf{J}(\mathbf{r},t)$, by $\mathbf{J}_T(\mathbf{r},t)$, Eq. (4) must be replaced by

$$\nabla\times\mathbf{B}(\mathbf{r},t)=\mu_{0}\mathbf{J}_{T}(\mathbf{r},t)+\frac{1}{c_{0}^{2}}\frac{\partial\mathbf{E}_{T}(\mathbf{r},t)}{\partial t}, \tag{28}$$

and again I emphasize that in order for $\mathbf{J}_T(\mathbf{r},t)$ to be a genuine transverse vector field the condition

$$\nabla\cdot\mathbf{J}_{T}(\mathbf{r},t)=0 \tag{29}$$

must be fulfilled in the entire space, i.e., inside as well as outside the region where the particle charge density is different from zero at the given time. The presence of a matter field does not change the two Maxwell equations in Eqs. (1) and (3), and the addition to the final one [Eq. (2)] does not enter the dynamics of the transverse photons directly but is of importance for the energy flow in the coupled photon-atom system, see Sec. III E.

In order that precisely one photon comes out of the emission process the transverse current density must have a specific amplitude. We shall determine this amplitude below, and by assuming here that this has been done, let us denote the resulting transverse current density by $\boldsymbol{\mathcal{J}}_T(\mathbf{r},t)$.

The positive-frequency parts of the Riemann-Silberstein vectors $\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)$ now fulfill the inhomogeneous differential equations

$$i\hbar\frac{\partial\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)}{\partial t}=\pm c_{0}\hbar\nabla\times\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)-\frac{i\hbar}{\sqrt{2\epsilon_{0}}}\boldsymbol{\mathcal{J}}_{T}^{(+)}(\mathbf{r},t) \tag{30}$$

in direct space, as one easily realizes from Eqs. (3), (6), (9), and (28) [with $\mathbf{J}_T(\mathbf{r},t)$ replaced by $\boldsymbol{\mathcal{J}}_T(\mathbf{r},t)$]. The quantity

$$\boldsymbol{\mathcal{J}}_{T}^{(+)}(\mathbf{r},t)=\int_{0}^{\infty}\boldsymbol{\mathcal{J}}_{T}(\mathbf{r};\omega)e^{-i\omega t}d\omega \tag{31}$$

is the analytical signal belonging to $\boldsymbol{\mathcal{J}}_T(\mathbf{r},t)$. In momentum space the equivalent equations read

$$i\hbar\frac{\partial\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)}{\partial t}=\pm ic_{0}\mathbf{p}\times\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)-\frac{i\hbar}{\sqrt{2\epsilon_{0}}}\boldsymbol{\mathcal{J}}_{T}^{(+)}(\mathbf{p},t), \tag{32}$$

where $\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{p},t)$ is the Fourier transform of $\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r},t)$. To obtain the dynamical equations for the scalar functions $f_{\pm}^{(+)}(\mathbf{p},t)$, given via

$$\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)=f_{\pm}^{(+)}(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_{\pm}(\hat{\mathbf{p}}), \tag{33}$$

as it readily appears from Eqs. (14) and (18), we use the dyadic expansion

$$\overleftrightarrow{U}=\hat{\boldsymbol{\epsilon}}_{+}(\hat{\mathbf{p}})\hat{\boldsymbol{\epsilon}}_{+}(\hat{\mathbf{p}})+\hat{\boldsymbol{\epsilon}}_{-}(\hat{\mathbf{p}})\hat{\boldsymbol{\epsilon}}_{-}(\hat{\mathbf{p}})+\hat{\mathbf{p}}\hat{\mathbf{p}} \tag{34}$$

of the unit tensor $\overleftrightarrow{U}$, and the condition that $\boldsymbol{\mathcal{J}}_T^{(+)}$ is divergence-free, i.e., $\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}}_T^{(+)}=0$, to resolve the current density generating a single photon in the form

$$\boldsymbol{\mathcal{J}}_{T}^{(+)}(\mathbf{p},t)=\hat{\boldsymbol{\epsilon}}_{+}(\hat{\mathbf{p}})\mathcal{J}_{T,+}^{(+)}(\mathbf{p},t)+\hat{\boldsymbol{\epsilon}}_{-}(\hat{\mathbf{p}})\mathcal{J}_{T,-}^{(+)}(\mathbf{p},t), \tag{35}$$

where

$$\mathcal{J}_{T,\pm}^{(+)}(\mathbf{p},t)\equiv\hat{\boldsymbol{\epsilon}}_{\mp}(\hat{\mathbf{p}})\cdot\boldsymbol{\mathcal{J}}_{T}^{(+)}(\mathbf{p},t). \tag{36}$$

An insertion of Eq. (35) into Eq. (32), a subsequent multiplication of the resulting upper- and lower-sign equations by $\hat{\boldsymbol{\epsilon}}_-(\hat{\mathbf{p}})$ and $\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{p}})$, respectively, and use of the relations $\hat{\boldsymbol{\epsilon}}_{\mp}(\hat{\mathbf{p}})\cdot\hat{\boldsymbol{\epsilon}}_{\pm}(\hat{\mathbf{p}})=1$, give after a few algebraic steps

$$i\hbar\frac{\partial f_{\pm}^{(+)}(\mathbf{p},t)}{\partial t}=c_{0}pf_{\pm}^{(+)}(\mathbf{p},t)-\frac{i\hbar}{\sqrt{2\epsilon_{0}}}\mathcal{J}_{T,\pm}^{(+)}(\mathbf{p},t). \tag{37}$$

Under the assumption that the Riemann-Silberstein vectors vanish in the remote past, the solution of Eq. (37) is

$$f_{\pm}^{(+)}(\mathbf{p},t)=-\frac{1}{\sqrt{2\epsilon_{0}}}e^{-ic_{0}pt/\hbar}\int_{-\infty}^{t}\mathcal{J}_{T,\pm}^{(+)}(\mathbf{p},t')e^{ic_{0}pt'/\hbar}dt'. \tag{38}$$

To characterize the initiation of the photon-emission process in momentum-time space the six-vector object

$$\boldsymbol{\Psi}(\mathbf{p},t)=h^{-3/2}\begin{pmatrix}\mathbf{g}+(\mathbf{p},t)\\ \mathbf{g}-(\mathbf{p},t)\end{pmatrix}=h^{-3/2}(c_{0}p)^{-1/2}\begin{pmatrix}f_{+}^{(+)}(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_{+}(\hat{\mathbf{p}})\\ f_{-}^{(+)}(\mathbf{p},t)\hat{\boldsymbol{\epsilon}}_{-}(\hat{\mathbf{p}})\end{pmatrix} \tag{39}$$

is introduced. With the $f_{\pm}^{(+)}(\mathbf{p},t)$'s given by Eq. (38), $\mathbf{\Psi}(\mathbf{p},t)$ can be written in the explicit form

$$\mathbf{\Psi}(\mathbf{p},t)=-h^{-3/2}(2\epsilon_0 c_0 p)^{-1/2} \times e^{-ic_0pt/\hbar}\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{p}})\hat{\boldsymbol{\epsilon}}_-(\hat{\mathbf{p}})\\ \hat{\boldsymbol{\epsilon}}_-(\hat{\mathbf{p}})\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{p}})\end{pmatrix} \cdot \int_{-\infty}^{t}\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{p},t')e^{ic_0pt'/\hbar}dt', \tag{40}$$

where the notation $\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_+\hat{\boldsymbol{\epsilon}}_-\\ \hat{\boldsymbol{\epsilon}}_-\hat{\boldsymbol{\epsilon}}_+\end{pmatrix}\cdot\boldsymbol{\alpha}\equiv\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_+\hat{\boldsymbol{\epsilon}}_-\cdot\boldsymbol{\alpha}\\ \hat{\boldsymbol{\epsilon}}_-\hat{\boldsymbol{\epsilon}}_+\cdot\boldsymbol{\alpha}\end{pmatrix}$ has been used for brevity. In the limit $t\to\infty$ the photon has emerged, and therefore

$$\lim_{t\to\infty}\mathbf{\Psi}(\mathbf{p},t)=\mathbf{\Phi}(\mathbf{p},t), \tag{41}$$

where $\mathbf{\Phi}(\mathbf{p},t)$ is the relevant free-space single-photon wave function in the momentum representation [Eq. (20)]. Since $\mathbf{\Psi}(\mathbf{p},t)$ describes a not fully emerged photon, we may call $\mathbf{\Psi}(\mathbf{p},t)$ the emerging-photon wave function (in momentum representation). By means of the Fourier transform

$$\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)=\int_{-\infty}^{\infty}\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{p},t)e^{ic_0pt/\hbar}dt, \tag{42}$$

we thus have

$$\mathbf{\Phi}(\mathbf{p},t)=-h^{-3/2}(2\epsilon_0c_0p)^{-1/2}e^{-ic_0pt/\hbar} \times\begin{pmatrix}\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{p}})\hat{\boldsymbol{\epsilon}}_-(\hat{\mathbf{p}})\\ \hat{\boldsymbol{\epsilon}}_-(\hat{\mathbf{p}})\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{p}})\end{pmatrix}\cdot\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right). \tag{43}$$

Above we have integrated the transverse source current density from $-\infty$ to ∞. Usually, the current density is only nonvanishing over a finite time interval, say $0\leqslant t\leqslant t_0$, and the photon is hence described by the wave function in Eq. (43) after the time t_0 where the source current has stopped.

We are now in a position where the positive-frequency amplitude of the transverse source current density, $\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{p},c_0p/\hbar)$, can be determined so that precisely one photon is emitted. Thus, from the free-space Schrödinger equation in Eq. (19), one obtains the general solution

$$g_{\pm}(\mathbf{p},t)=g_{\pm}^0(\mathbf{p})e^{-c_0pt/\hbar}, \tag{44}$$

and upon a comparison to Eqs. (39), (41), and (43) it appears that

$$g_{\pm}^0(\mathbf{p})=-(2\epsilon_0c_0p)^{-1/2}\mathcal{J}_{T,\pm}^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right). \tag{45}$$

In turn this means that the probability density in momentum space given by Eq. (22) becomes

$$P(\mathbf{p})=\frac{1}{2h^3\epsilon_0c_0p}\left[\left|\mathcal{J}_{T,+}^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)\right|^2 +\left|\mathcal{J}_{T,-}^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)\right|^2\right], \tag{46}$$

and hence the amplitude of $\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{p},c_0p/\hbar.)$ [and thereafter $\boldsymbol{\mathcal{J}}_T(\mathbf{p},c_0p/\hbar.)$] can be determined from the normalization condition

$$\frac{1}{2h^3\epsilon_0c_0}\int_{-\infty}^{\infty}\frac{1}{p}\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)\cdot\left[\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)\right]^*d^3p=1, \tag{47}$$

cf. Eqs. (23) and (35). Remembering that we, strictly speaking, performed a Fourier transformation from direct space ($\mathbf{r}$) to wave-vector ($\mathbf{q}=\mathbf{p}/\hbar.$) space earlier, the transverse current density is

$$\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)\equiv\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{q},\omega), \tag{48}$$

where $\omega=E/\hbar$ is the cyclic frequency of a monochromatic photon of energy $E=pc_0$ (see also Sec. III D). In the momentum representation the positive-frequency part of the transverse current density in the $(\mathbf{q},\omega)$ domain thus plays an important role, as one might have anticipated.

D. Single-photon emission process in direct space; propagator description of the emerging photon

1. Photon perspective

To describe the emergence of the photon in space-time we must now introduce the relevant direct space emerging-photon wave function $\mathbf{\Psi}(\mathbf{r},t)$. Despite the notation, $\mathbf{\Psi}(\mathbf{r},t)$ is not the Fourier transform of $\mathbf{\Psi}(\mathbf{p},t)$ given in Eq. (40). In terms of the analytical part of the Riemann-Silberstein vectors, $\mathbf{\Psi}(\mathbf{r},t)$ still has the six-component form displayed in Eq. (8), but now the $\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)$'s have to satisfy the inhomogeneous first-order partial differential equations in Eq. (30). The relations between the positive-frequency parts of the Riemann-Silberstein vectors and the analytical signal related to the transverse (atomic) current density can be written in a number of physically equivalent ways. Thus, one may start from the momentum-time relations between the $\mathbf{f}_{\pm}^{(+)}(\mathbf{p},t)$'s and $\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{p},t)$, given by Eq. (38) with Eq. (36) inserted, and then perform the inverse Fourier transformations to obtain the $\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)$'s as functions of $\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r},t)$. Here, we will establish instead an electromagnetic propagator relation between the analytical parts of the Riemann-Silberstein vectors and (the time-derivative of) the transverse current density. The reason for choosing such a procedure is twofold. Hence, first the speed of light is introduced in the space-time dynamics in such a manner that the role of the Einstein causality in the photon-emission process appears explicitly. Second, the propagator formalism offers us a direct way of following the time development of the loss of spatial confinement of the photon wave function during its generation.

By combining Eqs. (3) and (28) and limiting ourselves to the complex analytical signals one can obtain the following wave equation for the electric field of the emerging photon:

$$\left(\nabla^2-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{e}_T^{(+)}(\mathbf{r},t)=\mu_0\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r},t)}{\partial t}, \tag{49}$$

and by means of the isotropic electromagnetic propagator

$$\overleftrightarrow{d}(R,\tau)=-\frac{1}{4\pi R}\,\delta\!\left(\frac{R}{c_0}-\tau\right)\overleftrightarrow{U}, \tag{50}$$

with $R=|\mathbf{R}|$, $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, and $\tau=t-t'$, the Einstein-causal relation between $\mathbf{e}_T^{(+)}(\mathbf{r},t)$ and $\partial\mathcal{J}_T^{(+)}(\mathbf{r}',t')/\partial t'$ reads

$$\mathbf{e}_T^{(+)}(\mathbf{r},t)=\mu_0\int_{-\infty}^{\infty}\overleftrightarrow{d}(|\mathbf{r}-\mathbf{r}'|,t-t')\cdot\frac{\partial\mathcal{J}_T^{(+)}(\mathbf{r}',t')}{\partial t'}d^3r'dt'. \tag{51}$$

In Eq. (51) the source domain of the electric field of the photon is identified with the region occupied by the (time derivative) of the transverse part, $\mathcal{J}_T^{(+)}(\mathbf{r}',t')$, of the analytical atomic current density, $\mathcal{J}^{(+)}(\mathbf{r}',t')$. Although the electron motion is confined to a region of essentially exponential extension around the nucleus (the decay length being of the order of the Bohr radius), the transverse current density exhibits a much weaker spatial confinement of the R^{-3} type. The view given us of the electrodynamics starting from Eqs. (51) I have called the photon perspective, and a detailed account of the picture it offers us can be found in Ref. [4]. In this reference the propagator description of the magnetic field emitted from the atom was not studied but this is necessary here in order to develop the propagator description of the emerging-photon energy wave function and the emitted photon.

The magnetic field of the photon $\mathbf{b}^{(+)}(\mathbf{r},t)$ may be obtained from the expression

$$\mathbf{b}^{(+)}(\mathbf{r},t)=-\int_{-\infty}^{t}\nabla\times\mathbf{e}_T^{(+)}(\mathbf{r},t')dt' \tag{52}$$

by the help of Eq. (51), as one readily infers from Eq. (3). In setting up Eq. (52) we have assumed that the magnetic field vanishes in the remote past. In the far-field ($\sim R^{-1}$) zone of the atom the propagator description of the magnetic field has a form closely resembling the one given for the transverse electric field in Eq. (51). Thus, the source density is still $\partial\mathcal{J}_T^{(+)}(\mathbf{r}',t')/\partial t'$, and in the propagator we just need to replace $\overleftrightarrow{U}$ by $\overleftrightarrow{U}\times\hat{\mathbf{R}}$, where $\hat{\mathbf{R}}=\mathbf{R}/R$. Finally, the replacement $\mu_0\rightarrow\mu_0/c_0$ is needed. In the near- and midfield zones of the atom, zones which are of particular importance for the photon-emission process, the modifications are more pronounced, and, in fact, both propagator and nonpropagator formalisms, equivalent from a physical point of view, can be established. In the present context it is convenient to picture the physics in propagator form. In this form the driving term is the same as for the electric field, i.e., $\partial\mathcal{J}_T(\mathbf{r}',t')/\partial t'$. As shown in Appendix B 1, the magnetic field of the photon then becomes

$$\mathbf{b}^{(+)}(\mathbf{r},t)=\frac{\mu_0}{c_0}\int_{-\infty}^{\infty}\overleftrightarrow{m}(\mathbf{r}-\mathbf{r}',t-t')\cdot\frac{\partial\mathcal{J}_T^{(+)}(\mathbf{r}',t')}{\partial t'}d^3r'dt', \tag{53}$$

where

$$\overleftrightarrow{m}(\mathbf{R},\tau)=\left[-\frac{1}{4\pi R}\,\delta\!\left(\frac{R}{c_0}-\tau\right)-\frac{c_0}{4\pi R^2}\,\theta\!\left(\tau-\frac{R}{c_0}\right)\right]\times[\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})-\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})] \tag{54}$$

is the relevant propagator for the magnetic field in the photon perspective. In Eq. (54), the magnetic-field propagator, $\overleftrightarrow{m}(\mathbf{R},\tau)$, is expressed in polar-coordinate form and the unit vectors $\hat{\mathbf{R}},\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})$, and $\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})$, which form a right-handed triad, are the local ones. Since,

$$\overleftrightarrow{U}\times\hat{\mathbf{R}}=\hat{\mathbf{R}}\times\overleftrightarrow{\mathbf{U}}=\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})-\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})\hat{\mathbf{\Phi}}(\hat{\mathbf{R}}), \tag{55}$$

the far-field ($\sim R^{-1}$) part of the propagator has precisely the form cited above. Although the far-field contribution to $\mathbf{e}_T^{(+)}(\mathbf{r},t)$ which originates in the differential source $[\partial\mathcal{J}_T^{(+)}(\mathbf{r}',t')/\partial t']d^3r'dt'$ is different from zero only on the light shell, $R=c_0\tau$, near the atom the magnetic field is nonvanishing for timelike ($\tau>R/c_0$) source-observation point couplings also, cf. the presence of the term with the Heaviside unit step function, $\theta(\tau-R/c_0)$, in Eq. (54). The timelike couplings vanish as R^{-2} (midfield dependence) with the distance from the local source point. Only in the far field the electromagnetic field stemming from the in time and space infinitesimally extended source $[\partial\mathcal{J}_T(\mathbf{r}',t')/\partial t']d^3r'dt'$ is located entirely on the light cone. In the photon perspective, the Einstein causality is thus never violated.

The final steps towards a propagator description of the photon emission process and the subsequent free-space evolution can now be taken. Thus, I introduce what one may call the photon-energy wave-function propagators in the photon perspective, $\overleftrightarrow{F}_{\pm}^{T}(\mathbf{R},\tau)$, for the two (+, −) helicities by the definitions

$$\overleftrightarrow{F}_{\pm}^{T}(\mathbf{R},\tau)\equiv\overleftrightarrow{d}(R,\tau)\pm i\overleftrightarrow{m}(\mathbf{R},\tau), \tag{56}$$

or in explicit forms

$$\begin{aligned}\overleftrightarrow{F}_{\pm}^{T}(\mathbf{R},\tau)=&-\frac{1}{4\pi R}\,\delta\!\left(\frac{R}{c_0}-\tau\right)\times\{\overleftrightarrow{U}\pm i[\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})-\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})]\}\\&\mp\frac{ic_0}{4\pi R^2}\,\theta\!\left(\tau-\frac{R}{c_0}\right)[\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})-\hat{\mathbf{\Theta}}(\hat{\mathbf{R}})\hat{\mathbf{\Phi}}(\hat{\mathbf{R}})].\end{aligned} \tag{57}$$

The superscript T put on the propagators just is meant to remind us that these are related to the transverse photon dynamics. To describe the photon-emission process in compact form, a two-component spinor propagator

$$\overleftrightarrow{\boldsymbol{\mathcal{F}}}^T(\mathbf{R},\tau)\equiv\begin{pmatrix}\overleftrightarrow{F}^T_+(\mathbf{R},\tau)\\ \overleftrightarrow{F}^T_-(\mathbf{R},\tau)\end{pmatrix} \tag{58}$$

is defined. By means of this the six-vector energy wave function of the emerging photon may now be written in the propagator form

$$\boldsymbol{\Psi}(\mathbf{r},t)=\begin{pmatrix}\mathbf{f}^{(+)}_+(\mathbf{r},t)\\ \mathbf{f}^{(+)}_-(\mathbf{r},t)\end{pmatrix} =\mu_0\sqrt{\frac{\epsilon_0}{2}}\int_{-\infty}^{\infty}\overleftrightarrow{\boldsymbol{\mathcal{F}}}^T(\mathbf{R},\tau)\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')}{\partial t'}d^3r'\,dt', \tag{59}$$

with the abbreviation

$$\overleftrightarrow{\boldsymbol{\mathcal{F}}}^T\cdot\boldsymbol{\alpha}\equiv\begin{pmatrix}\overleftrightarrow{F}^T_+\cdot\boldsymbol{\alpha}\\ \overleftrightarrow{F}^T_-\cdot\boldsymbol{\alpha}\end{pmatrix}.$$

Some important conclusions about the spatial localization of the photon wave function during the emission process can be made on the basis of Eqs. (57)–(59), but before doing this, let us carry out the t' integration for the magnetic midfieldlike contribution. Essentially, this amounts to

$$\int_{-\infty}^{\infty}\theta\left(t-\frac{R}{c_0}-t'\right)\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')}{\partial t'}dt'=\boldsymbol{\mathcal{J}}^{(+)}_T\left(\mathbf{r}',t-\frac{R}{c_0}\right), \tag{60}$$

assuming that the transverse current density vanishes in the remote past, i.e., $\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',-\infty)=\mathbf{0}$. In passing, I note that this assumption follows once the principle of causality is adopted. To be specific, let us consider the situation where the transverse current density and its first-order time derivative vanish identically outside the time interval $0\leqslant t'\leqslant t_0$ in every space point $\mathbf{r}'$. Hence, the photon-emission process takes a time t_0 and begins at $t'=0$. At the end points of the interval we must demand $\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',0)=\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t_0)=\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')/\partial t'|_{t'=0}=\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')/\partial t'|_{t'=t_0}=\mathbf{0}$ for a physically acceptable atom dynamics. It readily appears from Eq. (60) that the magnetic midfieldlike contribution to the photon wave function, $\boldsymbol{\Psi}(\mathbf{r},t)$, from the infinitesimal source term $\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')d^3r'$ located at $\mathbf{r}'$ is different from zero inside a sphere of radius $|\mathbf{r}-\mathbf{r}'|=c_0t$ during the emission in process $(0<t<t_0)$. In the limit $t\to 0^+$, the radius shrinks to zero, and therefore it follows from Eq. (59) that the strongest spatial confinement of the magnetic midfieldlike contribution is found when the emission process starts, and is given by the spatial extension of the transverse current-density distribution of the atom at these early times, i.e., by $\boldsymbol{\mathcal{J}}^{(+)}_T\times(\mathbf{r}',0^+)$. The far-field contribution to the photon wave function, which is located on the light shell $|\mathbf{r}-\mathbf{r}'|=c_0t$ does not change this conclusion in any essential manner, since the best localization of this contribution is given by $\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',0^+)/\partial t'$. At times $t'>t_0$, or equivalently $t>t_0+R/c_0$, $\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t-R/c_0)=\mathbf{0}$. This means that the magnetic midfieldlike term does not contribute to the photon wave function $\boldsymbol{\Psi}(\mathbf{r},t)$ after the finish of the emission process, and in consequence of this the contribution to the photon wave function from the infinitesimal source $[\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')/\partial t']d^3r'$ is different from zero in the spatial region between the two spherical shells $|\mathbf{r}-\mathbf{r}'|=c_0(t-t_0)$ and $|\mathbf{r}-\mathbf{r}'|=c_0t$ which both move outwards with the vacuum speed of light. A schematic illustration of the emission process in the photon perspective is presented in Fig. 1.

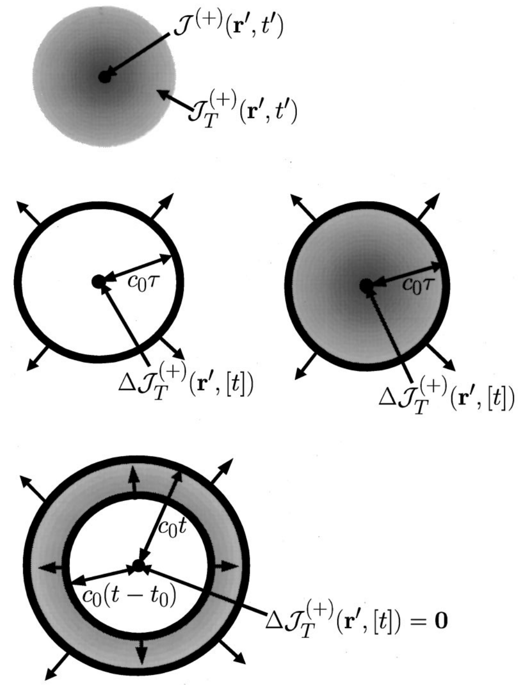

FIG. 1. Schematic illustration of the photon-emission process in the photon perspective. Upper part: For an electrodynamically active atom with its strongly localized analytical current density distribution, $\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')$, the source region of the photon, $\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r},t')$, extends over the atomic near-field zone. Middle parts: The field of the photon emerging from the infinitesimal source region, $\Delta\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',[t])$, has an electric component (figure to the left) located entirely on the light shell, $R=c_0\tau$, and a magnetic component (figure to the right) located not only on the light shell but also behind it (a timelike response). Lower part: When the photon is created the electromagnetic field from the $\Delta\boldsymbol{\mathcal{J}}^{(+)}_T(\vec{r}',[t])$ source is located between the light shells $R=c_0(t-t_0)$ and $R=c_0t$, the atom being active in the time interval $(0|t_0)$.

The energy wave function of the emitted photon, i.e., $\boldsymbol{\Psi}(\mathbf{r},t'>t_0)\equiv\boldsymbol{\Phi}(\mathbf{r},t)$, as it appears in the photon perspective, can now be determined. Hence, with the help of the result

$$\int_{-\infty}^{\infty}\delta\left(\frac{R}{c_0}-t+t'\right)\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')}{\partial t'}dt' =\left.\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}_T(\mathbf{r}',t')}{\partial t'}\right|_{t'=t-R/c_0}\equiv\dot{\boldsymbol{\mathcal{J}}}^{(+)}_T(\mathbf{r}',[t]), \tag{61}$$

where $[t]=t-R/c_0$ is the retarded time, one gets

$$\boldsymbol{\Phi}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\sqrt{\frac{\epsilon_0}{2}}\int_{-\infty}^{\infty}\frac{1}{R}\begin{pmatrix}\overleftrightarrow{U}+i(\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Theta}}-\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{\Phi}})\\ \overleftrightarrow{U}-i(\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Theta}}-\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{\Phi}})\end{pmatrix}\cdot\dot{\boldsymbol{\mathcal{J}}}_T^{(+)}(\mathbf{r}',[t])d^3r', \quad t \geqslant t_0+\mathbf{R}_0/c_0. \tag{62}$$

In the local Cartesian coordinate system, where $\hat{\mathbf{R}}$, $\hat{\boldsymbol{\Theta}}$, and $\hat{\boldsymbol{\Phi}}$ are unit vectors along the one, two, and three axes, the tensors appearing in the upper and lower parts of the spinor propagator have the explicit forms

$$\overleftrightarrow{U}\mp i(\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Theta}}-\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{\Phi}}) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \mp i\\ 0 & \pm i & 1\end{pmatrix}. \tag{63}$$

In ending this section, and in contrast to the claim in Ref. [13], we may thus conclude that even in the moment of emission (or infinitesimally short time after) the photon is not completely confined in space.

2. Electron perspective

In Sec. II D 1, the photon emission process was studied using a propagator formalism in which the source region was identified with the domain occupied by the transverse part of the analytical atomic current density distribution and its first-order time derivative. Though such a formalism is quite easy to establish starting from the transverse set of microscopic Maxwell equations and intuitively appealing because the electromagnetic fields generated from every one of the infinitesimal sources constituting $\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r}',t')$ propagate in an Einstein causal manner, it has the disadvantage that the source domain is spread over a region of space much larger than the region occupied by the electron(s) of the atom. In the photon-emission process, the interference between the fields produced by the various differential sources therefore always plays a crucial role. In the so-called electron perspective discussed below one identifies the source domain with the domain occupied by (the time derivative of) the atomic current density itself. The positive-frequency part of the current density needed for the emission of precisely one photon we denote by $\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r},t)$. The analytical parts of the total atomic current density and its transverse part are related by the spatially nonlocal linear relation

$$\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r},t) = \int_{-\infty}^{\infty}\overleftrightarrow{\delta}_T(\mathbf{r}-\mathbf{r}')\cdot\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t)d^3r', \tag{64}$$

where $\overleftrightarrow{\delta}_T(\mathbf{R})$ is the transverse delta function. As indicated, the relation is local in time, and this has the consequence that the new near-field propagator of the atom attains a spacelike component in the electron perspective. For certain analyses of the photon emission process the electron perspective may have the advantage that the interference problem related to the field emission from the various differential sources in the atomic current density domain is easier to tackle due to the fact that the $\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r},t)$ distribution is much better localized than its transverse $[\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r},t)]$ part. In the heuristic point-particle analysis to be discussed in Sec. III, the interference problem does not exist at all in the electron perspective approach, for instance. See Fig. 2.

By inserting Eq. (64) into Eq. (51), a calculation aiming at obtaining a propagator formalism in which the time derivative of the total atomic current density distribution plays the role of the source leads to the result [4,26]

$$\mathbf{e}_T^{(+)}(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_{T,\mathrm{SF}}^{(+)}(\mathbf{r},t) + \mathbf{e}_{T,R}^{(+)}(\mathbf{r},t), \tag{65}$$

where

$$\boldsymbol{\mathcal{E}}_{T,\mathrm{SF}}^{(+)}(\mathbf{r},t) = -\frac{1}{3\epsilon_0}\int_{-\infty}^{t}\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r},t')dt', \tag{66}$$

is a transverse self-field (SF) contribution, and

$$\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t) = \mu_0\int_{-\infty}^{\infty}\overleftrightarrow{D}_T(\mathbf{r}-\mathbf{r}',t-t')\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}d^3r'dt' \tag{67}$$

is the retarded (R) part of the transverse electric field. The transverse self-field is nonvanishing only in the near-field ($\sim R^{-3}$) zone of the atom, and furthermore, it is different

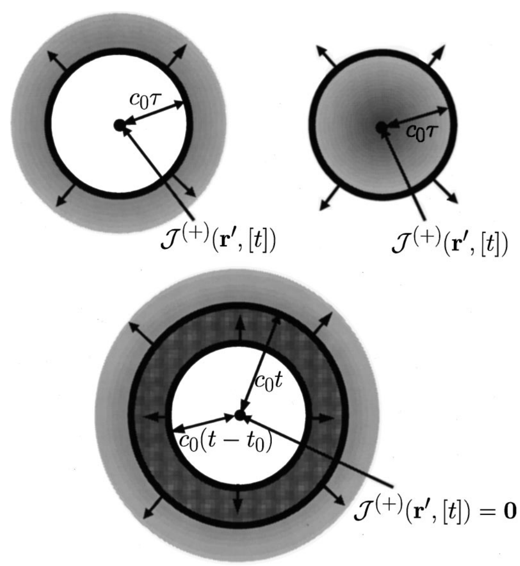

FIG. 2. Schematic illustration showing the photon-emission process in the electron perspective. Upper parts: The emergent-photon field generated by the analytical atomic current density distribution, $\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',[t])$, has an electric part (figure to the left) located on and in front of (spacelike response) the light shell, $R=c_0\tau$, and a magnetic part different from zero on and behind (timelike response) the light shell. Once the photon is emitted the main part of the electromagnetic field is located between the light shells $R=c_0t$ and $R=c_0(t-t_0)$, under the assumption that the atom is active in the time interval $(0|t_0)$. A small spacelike electric field yet still is present.

from zero only in the time interval where the photon-emission process takes place [because the mean value (over time) of $\mathcal{J}_T^{(+)}$ is zero]. Also a longitudinal self-field is present in the near-field zone, cf., e.g., Refs. [4], [28] and the analysis in Sec. III, and in the quantum electrodynamic description removal of redundancy requires that the entire self-field operator is eliminated in favor of the particle-position variable(s). In consequence the self-field dynamics is transferred to the particle Hamiltonian and only the retarded part, $\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t)$, of the transverse field is subjected to the canonical quantization procedure leading to the photon concept [4]. In our first-quantized description of the photon-emission process in the electron perspective we therefore necessarily have to identify *only* $\mathbf{e}_{T,R}^{(+)}$ with the electric field of the photon. Since $\mathbf{e}_T^{(+)}(\mathbf{r},t)=\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t)$ once the photon is fully emitted i.e., for $t>t_0$, the elimination of the transverse self-field from the photon field does not change the normalization condition in Eq. (47). Once the amplitude of $\mathcal{J}_T^{(+)}$ has been obtained from this equation, Eq. (64) fixes the amplitude of $\mathcal{J}^{(+)}$ entering Eq. (67). The transverse electromagnetic propagator $\overleftrightarrow{D}_T(\mathbf{R},\tau)$, appearing in Eq. (67), is known to have the explicit form [27]

$$\overleftrightarrow{D}_T(\mathbf{R},\tau)=-\frac{1}{4\pi R}\,\delta\!\left(\frac{R}{c_0}-\tau\right)(\overleftrightarrow{U}-\hat{\mathbf{R}}\hat{\mathbf{R}})+\frac{c_0^2\tau}{4\pi R^3}\,\theta(\tau)\,\theta\!\left(\frac{R}{c_0}-\tau\right)(\overleftrightarrow{U}-3\hat{\mathbf{R}}\hat{\mathbf{R}}). \quad (68)$$

An elaborate discussion of the physics hidden in Eq. (68) can be found elsewhere [27] and need not be repeated here. The first term on the right-hand side of Eq. (68) represents the far-field ($\sim R^{-1}$) contribution to the propagator. It is anisotropic, with an anisotropy given by the tensor $\overleftrightarrow{U}-\hat{\mathbf{R}}\hat{\mathbf{R}}$, but besides this it exhibits the same form as the isotropic $\overleftrightarrow{d}(\mathbf{R},\tau)$ propagator; see Eq. (50). The other term, different from zero only for spacelike events, is present solely in the near-field zone of the atom, i.e., in the spatial region where the photon is created; see Fig. 2.

The retarded (R) magnetic field associated with the photon in the electron perspective, i.e.,

$$\mathbf{b}_R^{(+)}(\mathbf{r},t)=-\int_{-\infty}^{t}\nabla\times\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t')\,dt', \quad (69)$$

is obtained by combining Eqs. (67)–(69). Technically, this calculation is rather cumbersome and the interested reader may consult Appendix B 2 for a stepwise derivation. The final result is remarkably simple; however, viz.,

$$\mathbf{b}_R^{(+)}(\mathbf{r},t)=\frac{\mu_0}{c_0}\int_{-\infty}^{\infty}\overleftrightarrow{m}(\mathbf{r}-\mathbf{r}',t-t')\cdot\frac{\partial\mathcal{J}^{(+)}(\mathbf{r}',t')}{\partial t'}\,d^3r'\,dt', \quad (70)$$

with $\overleftrightarrow{m}(\mathbf{R},\tau)$ given by Eq. (54). The propagator describing the retarded magnetic response hence is the same in the photon and electron perspectives. See Fig. 2.

In the electron perspective description of the transverse electrodynamics also a magnetic self-field, $\mathcal{B}_{\mathrm{SF}}^{(+)}(\mathbf{r},t)$, enters. This field is different from zero only inside the atomic current density distribution, and is given by

$$\mathcal{B}_{\mathrm{SF}}^{(+)}(\mathbf{r},t)=\frac{1}{3\epsilon_0}\int_{-\infty}^{t}\int_{-\infty}^{t'}\nabla\times\mathcal{J}^{(+)}(\mathbf{r},t'')\,dt''\,dt', \quad (71)$$

as one readily realizes by combining Eqs. (3) and (66) and remembering that $\nabla\times\mathcal{J}_T^{(+)}=\nabla\times\mathcal{J}^{(+)}$.

By means of the two-component spinor propagator

$$\overleftrightarrow{\mathcal{H}}^T(\mathbf{R},\tau)\equiv\begin{pmatrix}\overleftrightarrow{D}_T(\mathbf{R},\tau)+i\overleftrightarrow{m}(\mathbf{R},\tau)\\ \overleftrightarrow{D}_T(\mathbf{R},\tau)-i\overleftrightarrow{m}(\mathbf{R},\tau)\end{pmatrix}, \quad (72)$$

the emerging photon is described by the six-vector energy wave function

$$\mathbf{\Psi}(\mathbf{r},t)=\mu_0\sqrt{\frac{\epsilon_0}{2}}\int_{-\infty}^{\infty}\overleftrightarrow{\mathcal{H}}^T(\mathbf{R},\tau)\cdot\frac{\partial\mathcal{J}^{(+)}(\mathbf{r}',t')}{\partial t'}\,d^3r'\,dt' \quad (73)$$

if observed from the electron perspective. In Eq. (73) the same compact notation as in Eq. (59) has been employed. It appears from Eqs. (72) and (73) that the energy wave function of the emitted photon, which we hitherto had written as displayed in Eq. (62), may be written in the alternative form

$$\mathbf{\Phi}(\mathbf{r},t)=-\frac{\mu_0}{4\pi}\sqrt{\frac{\epsilon_0}{2}}\int_{-\infty}^{\infty}\frac{1}{R}\begin{pmatrix}\overleftrightarrow{U}-\hat{\mathbf{R}}\hat{\mathbf{R}}+i(\hat{\mathbf{\Phi}}\hat{\mathbf{\Theta}}-\hat{\mathbf{\Theta}}\hat{\mathbf{\Phi}})\\ \overleftrightarrow{U}-\hat{\mathbf{R}}\hat{\mathbf{R}}-i(\hat{\mathbf{\Phi}}\hat{\mathbf{\Theta}}-\hat{\mathbf{\Theta}}\hat{\mathbf{\Phi}})\end{pmatrix}\cdot\dot{\mathcal{J}}^{(+)}(\mathbf{r}',[t])\,d^3r',\quad t\rightarrow\infty, \quad (74)$$

identifying the photon source with the analytical part of the entire atomic current density distribution.

III. HEURISTIC PARADIGM: SINUSOIDAL PHOTON WAVE TRAIN EMITTED FROM A POINTLIKE ATOM

A. Model

It is instructive to throw light on the general theory established in Secs. II C–II D by applying it in a model calculation. The obtained results are also of interest in their own right because they offer a simple qualitative picture of the photon-emission process and allow us to make contact to the textbook description of the photon emission from an atom.

In our model the atom is considered as a pointlike entity from an electronic point of view, and we assume that it electrodynamically behaves like an electric dipole (ED). In the ED approximation the atomic current density is given by

$$\mathcal{J}(\mathbf{r},t)=\mathcal{J}(t)\,\delta(\mathbf{r}), \tag{75}$$

assuming that the dipole is placed at the origin of the coordinate system. In the near-field zone of the point particle a (singular) current density distribution of the form given in Eq. (75) gives rise to both attached and radiated electromagnetic fields. In the far-field zone only the radiative part is left. To emphasize that the dipole current density has been normalized so that precisely one photon is emitted, calligraphic letters are used to denote the current density and its amplitude. The source of the emitted photon is related to the transverse part of the current density in Eq. (75), i.e.,

$$\boldsymbol{\mathcal{J}}_T(\mathbf{r},t)=\overleftrightarrow{\delta}_T(\mathbf{r})\cdot\boldsymbol{\mathcal{J}}(t), \tag{76}$$

where $\overleftrightarrow{\delta}_T(\mathbf{r})$ is the transverse delta function [see also Eq. (64)], a dyadic quantity. Since the range of $\overleftrightarrow{\delta}_T(\mathbf{r})$ is characterized by an r^{-3} dependence, the emission region of the photon is just the near-field zone of the point particle. A detailed semiclassical (field-quantized) study of the attached and radiated electromagnetic fields of an electric point dipole based on a Green-function approach may be found in Ref. [28].

Let us assume now that the atom is excited by a monochromatic field of finite duration and with a cyclic frequency ω_0 so far from any of the atomic transition frequencies that the Rabi oscillations in the current density can be neglected [3]. The fact that the atomic current density must be so small that only one photon is emitted in itself suppresses the Rabi sidebands except at resonance (or close to resonance). A time dependence of the form

$$\boldsymbol{\mathcal{J}}(t)=\boldsymbol{\mathcal{J}}_0[\theta(t)-\theta(t-T_0)]\sin\omega_0 t \tag{77}$$

hence is taken. The period of the harmonic oscillation is $T=2\pi/\omega_0$, the current density is different from zero in the time interval $0<t<T_0$, and lasts for an integer number of cycles, i.e., $T_0=nT$, where n is a positive integer.

B. One-photon atomic current density

We begin our investigation of the photon-emission and -absorption process with a calculation of the associated vectorial point particle current density amplitude $\boldsymbol{\mathcal{J}}_0$. Thus, by means of the inverse spatial Fourier transformation [1,26]

$$\int_{-\infty}^{\infty}\overleftrightarrow{\delta}_T(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r=\overleftrightarrow{U}-\frac{\mathbf{pp}}{p^2}, \tag{78}$$

where

$$\mathbf{p}=\hbar\mathbf{q}, \tag{79}$$

and $p^2=\mathbf{p}\cdot\mathbf{p}$, and the inverse Fourier transformation in time

$$\int_{-\infty}^{\infty}[\theta(t)-\theta(t-T_0)]\sin\omega_0 t e^{i\omega t}dt=\frac{\omega_0}{\omega^2-\omega_0^2}(e^{i\omega T_0}-1), \tag{80}$$

it appears that the positive-frequency part of the transverse current density in the ($\mathbf{q}$, ω) domain (momentum-energy representation) is given by

$$\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)=\left(\overleftrightarrow{U}-\frac{\mathbf{pp}}{p^2}\right)\cdot\boldsymbol{\mathcal{J}}_0\frac{\omega_0}{\left(\frac{c_0p}{\hbar}\right)^2-\omega_0^2}\left[\exp\left(i\frac{c_0pT_0}{\hbar}\right)-1\right], \tag{81}$$

having used also the energy-momentum constraint for the photon, namely,

$$\omega=\frac{c_0p}{\hbar}\quad(>0) \tag{82}$$

in Eq. (80). The analytical current density signal leading to the emission of just one photon in turn is given by

$$\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r},t)=h^{-3}\int_{-\infty}^{\infty}\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)\exp\left[\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-c_0pt)\right]d^3p, \tag{83}$$

remembering that $p>0$.

The point-particle current density, which is linearly polarized, is assumed to be directed along the z axis of our Cartesian coordinate system, i.e., $\boldsymbol{\mathcal{J}}_0=\mathcal{J}_0\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector in the z direction, and to determine the yet unknown amplitude $\boldsymbol{\mathcal{J}}_0$, Eq. (81) is inserted into the normalization condition given in Eq. (47). Since

$$\left|\boldsymbol{\mathcal{J}}_T^{(+)}\left(\mathbf{p},\frac{c_0p}{\hbar}\right)\right|^2=\frac{4\omega_0^2\sin^2\left(\frac{c_0pT_0}{2\hbar}\right)}{\left[\left(\frac{c_0p}{\hbar}\right)^2-\omega_0^2\right]^2}\boldsymbol{\mathcal{J}}_0\cdot(\overleftrightarrow{U}-\hat{\mathbf{p}}\hat{\mathbf{p}})\cdot\boldsymbol{\mathcal{J}}_0, \tag{84}$$

letting without restriction $\boldsymbol{\mathcal{J}}_0$ be a real quantity, it appears that the integral in Eq. (47) adequately is carried out in spherical coordinates. With the polar axis in the z direction one has $\boldsymbol{\mathcal{J}}_0\cdot(\overleftrightarrow{U}-\hat{\mathbf{p}}\hat{\mathbf{p}})\cdot\boldsymbol{\mathcal{J}}_0=\mathcal{J}_0^2\sin^2\theta$, where θ is the polar angle, and after having performed the trivial angular integrations the normalization condition reads

$$\mathcal{J}_0^2\int_0^{\infty}\frac{p\sin^2\left(\frac{c_0pT_0}{2\hbar}\right)}{\left[\left(\frac{c_0}{\hbar}\right)^2p^2-\omega_0^2\right]^2}dp=\frac{3h^3\epsilon_0c_0}{16\pi\omega_0^2}. \tag{85}$$

A substitution $p=(\hbar\omega_0/c_0)y$, followed by an integration by parts (taking $y/(y^2-1)^2=d[2(1-y^2)]^{-1}/dy$ as the one function) gives

$$\int_0^\infty \frac{p\sin^2\left(\frac{c_0 p T_0}{2\hbar}\right)}{\left[\left(\frac{c_0}{\hbar}\right)^2 p^2-\omega_0^2\right]^2}dp=-\left(\frac{\hbar}{c_0\omega_0}\right)^2\frac{n\pi}{2}\int_0^\infty\frac{\sin(2\pi n y)}{1-y^2}dy, \tag{86}$$

where $n=T_0/T$. Since n is a positive integer one has

$$\int_0^\infty\frac{\sin(2\pi n y)}{1-y^2}dy=-\int_0^{2\pi n}\frac{\sin x}{x}dx. \tag{87}$$

By combining Eqs. (85)–(87) it appears that the current density amplitude needed to ensure that precisely one photon is emitted from the atom in the point-particle approximation is

$$\mathcal{J}_0=\pi c_0\sqrt{6\hbar\epsilon_0 c_0}\left[2\pi n\int_0^{2\pi n}\frac{\sin x}{x}dx\right]^{-1/2}. \tag{88}$$

The result in Eq. (88) is remarkable because it shows that $\mathcal{J}_0$ is independent of the frequency ω_0, and thus depends only on the number of periods (n) in the wave train and on the fundamental quantities h and c_0. If one abandons the naïve point-particle model, the atomic current density amplitude will depend on the atomic length parameter (essentially the Bohr radius a_0) and the characteristic wavelength of the photon wave train, $\lambda_0=c_0T$. For photons emitted in electric dipole transitions the dependence of $\mathcal{J}_0$ on a_0 and λ_0, however, will be weak since the ratio a_0/λ_0 is small in the optical region. Finally, also the ratio e/m, between the electron charge $(-e)$ and mass (m), will appear in $\mathcal{J}_0$.

C. Finite-size correction to the transverse atomic current density: Hydrogen $1s\leftrightarrow 2p_z$ transition

Let us now make a pause in the analysis of the emergent photon and photon wave trains emitted from a pointlike atom and estimate the importance of the inevitable finite size of the atom by studying the one-photon wave train emitted from electron oscillations between the $1s$ and $2p_z$ states in hydrogen.

When the hydrogen atom is electrodynamically active the electron wave function, $\psi(\mathbf{r},t)$, is in a time-dependent superposition

$$\psi(\mathbf{r},t)=c_1(t)\psi_1(\mathbf{r})+c_2(t)\psi_2(\mathbf{r}) \tag{89}$$

of the $1s$ and $2p_z$ eigenstates, named $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$, respectively. Normalization requires that $|c_1(t)|^2+|c_2(t)|^2=1$. In the state $\psi(\mathbf{r},t)$ the atomic current density is given by

$$\boldsymbol{\mathcal{J}}(\mathbf{r},t)=c_1(t)c_2^*(t)\mathbf{j}_{1\to2}(\mathbf{r})+c_1^*(t)c_2(t)\mathbf{j}_{2\to1}(\mathbf{r}), \tag{90}$$

where the transition current densities can be obtained from the relations

$$i\mathbf{j}_{1\to2}(\mathbf{r})=i^{-1}\mathbf{j}_{2\to1}(\mathbf{r})\equiv\mathbf{j}(\mathbf{r}), \tag{91}$$

where in polar coordinates (r, θ, φ) [the polar axis coinciding with the z axis] with the local unit vectors denoted by $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$, the vector $\mathbf{j}(\mathbf{r})$ has the explicit form

$$\mathbf{j}(r)=B\left[\left(1+\frac{br}{3}\right)\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right]e^{-br}, \tag{92}$$

with $b=3/(2a_0)$ and $B=e\hbar/(8\pi\sqrt{2}ma_0^4)$. In passing we note that $\mathbf{j}(\mathbf{r})$ is independent of the azimuth angle (φ) and has no $\hat{\boldsymbol{\varphi}}$ component, as expected from the symmetry of the $1s$ and $2p_z$ orbitals. By writing the atomic current density in the form

$$\boldsymbol{\mathcal{J}}(\mathbf{r},t)=i[c_1^*(t)c_2(t)-c_1(t)c_2^*(t)]\mathbf{j}(\mathbf{r}), \tag{93}$$

one obtains, when Rabi sidebands effects are neglected,

$$i[c_1^*(t)c_2(t)-c_1(t)c_2^*(t)]=\mathcal{A}[\theta(t)-\theta(t-T_0)]\sin\omega_0 t, \tag{94}$$

when a sinusoidal excitation of finite length is applied to the atom. The amplitude $\mathcal{A}$ must be determined so that only a single photon is emitted from the hydrogen $1s\leftrightarrow 2p_z$ transition. By a comparison of the current densities of the hydrogen atom [Eq. (93) with Eq. (94) inserted] and the point-particle model [Eq. (75) with Eq. (77) inserted] it appears that the association is as follows:

$$\mathcal{A}\mathbf{j}(\mathbf{r})\Leftrightarrow\mathcal{J}_0\delta(\mathbf{r})\hat{\mathbf{z}}. \tag{95}$$

In order to calculate $\mathcal{A}$ the inverse Fourier transformation

$$\mathbf{j}(\mathbf{q})=\int_{-\infty}^{\infty}\mathbf{j}(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d^3q \tag{96}$$

needs to be carried out, and from the division

$$\mathbf{j}(\mathbf{q})=\mathbf{j}_T(\mathbf{q})+\mathbf{j}_L(\mathbf{q}), \tag{97}$$

the transverse part of the (effective) transition current density in the wave-vector (∼momentum) representation, $\mathbf{j}_T(\mathbf{q})$, can be obtained. The above-mentioned calculation is cumbersome, and the interested reader may find a few of the intermediate steps leading to the final result in Appendix C. The results for the transverse and longitudinal parts of $\mathbf{j}(\mathbf{q})$ conveniently are given in spherical coordinates (q,θ_q,φ_q) with the polar axis coincident with the one used in $\mathbf{r}$ space. The local unit vectors (in $\mathbf{q}$ space) are denoted by $\hat{\mathbf{q}}$, $\hat{\boldsymbol{\theta}}_q$, and $\hat{\boldsymbol{\varphi}}_q$. As one might have anticipated from Eq. (92), the independence of $\mathbf{j}(\mathbf{r})$ on the azimuth angle implies that $\mathbf{j}_T(\mathbf{q})$ only gets a component along the $\hat{\boldsymbol{\theta}}_q$ direction. Hence, in explicit form one obtains

$$\mathbf{j}_T(\mathbf{q})=-\frac{2^6}{3^5\sqrt{2}}\frac{e\hbar}{m}\frac{b^5}{(b^2+q^2)^2}\hat{\boldsymbol{\theta}}_q\sin\theta_q, \tag{98}$$

and for the longitudinal part $\mathbf{j}_L(\mathbf{q})$ of $\mathbf{j}(\mathbf{q})$ the final result reads

$$\mathbf{j}_L(\mathbf{q})=\frac{2^6}{3^5\sqrt{2}}\frac{e\hbar}{m}\frac{b^5\left[b^2+\left(\frac{q}{2}\right)^2\right]}{(b^2+q^2)^3}\hat{\mathbf{q}}\cos\theta_q, \tag{99}$$

and this of course only has a $\hat{\mathbf{q}}$ component. The association in $\mathbf{q}$ space between the transverse current density of the point-particle model and the hydrogen $1s\leftrightarrow 2p_z$ transition is

$$\mathcal{A}\jmath_T(\mathbf{q})\Leftrightarrow \mathcal{J}_0 \overleftrightarrow{\delta}_T(\mathbf{q})\cdot\hat{\mathbf{z}}, \tag{100}$$

and since $\delta_T(\mathbf{q})\cdot\hat{\mathbf{z}}=(\overleftrightarrow{U}-\hat{\mathbf{q}}\hat{\mathbf{q}})\cdot\hat{\mathbf{z}}=-\hat{\boldsymbol{\theta}}_q\sin\theta_q$, Eq. (98) shows that the transcription in Eq. (100) is equivalent to

$$\frac{2^6}{3^5\sqrt{2}}\frac{e\hbar}{m}\frac{b^5}{(b^2+q^2)^2}\mathcal{A}\Leftrightarrow\mathcal{J}_0. \tag{101}$$

To determine the one-photon value for $\mathcal{A}$ one may proceed as described in Sec. III B, and it now follows that an extra factor proportional to $[b^2+(p/\hbar)^2]^{-4}$ appears under the integral sign in Eq. (85). Since the wavelength(s) of light in the optical region is four orders of magnitude larger than the Bohr radius, this extra factor only gives a small correction to the p integral. By neglecting this correction, and thus setting $b^2+q^2\approx b^2$ in Eq. (101), the relation between $\mathcal{J}_0$ and $\mathcal{A}$ becomes

$$\mathcal{J}_0\Leftrightarrow\frac{2^5}{3^4\sqrt{2}}\frac{e\hbar}{ma_0}\mathcal{A}, \tag{102}$$

and with a current-density amplitude

$$\mathcal{A}=\left(\frac{3}{2}\right)^4\pi\sqrt{3}\,\frac{ma_0c_0}{e}\sqrt{\frac{\epsilon_0c_0}{\hbar}}\left[2\pi n\int_0^{2\pi n}\frac{\sin x}{x}dx\right]^{-1/2}, \tag{103}$$

precisely one photon is emitted in the hydrogen $1s\leftrightarrow 2p_z$ oscillation.

D. One-photon energy

The emergent photon, which energy wave function $\boldsymbol{\Psi}(\mathbf{r},t)$ in the photon perspective is given by Eq. (59), does not possess a time-independent energy. Only after the emission of the photon the energy

$$E(t)=\int_{-\infty}^{\infty}\boldsymbol{\Psi}^\dagger(\mathbf{r},t)\cdot\boldsymbol{\Psi}(\mathbf{r},t)d^3r \tag{104}$$

becomes independent of t. The reason that the emergent photon energy is time dependent originates in the fact that the photon during the creation process is coupled to the atom, and only the entire atom-photon system therefore is in an eigenstate for the energy. The state of things is further complicated since the energy ascribed to the emerging photon at a given time will be different in the photon and electron perspectives. This is so because the self-field part of the transverse energy of the electromagnetic field in the electron perspective is considered as belonging to the particle energy, cf. the discussion in Sec. III D 2. In the subsequent section (III E) we shall study how the total energy, averaged over a wave-train period, is shared between the atom and the emerging photon within the framework of the point-particle model.

Once the photon-emission process has been completed all the energy released from the atom resides (stays) in the photon. In terms of the current density amplitude $\mathcal{J}_0$, the cycle-averaged energy $\langle\mathcal{E}\rangle$, emitted by a point particle performing a sinusoidal motion, is

$$\langle\mathcal{E}\rangle=\frac{\mu_0\omega_0\mathcal{J}_0^2}{6c_0}. \tag{105}$$

The total energy emitted by the atom therefore becomes

$$E=n\langle\mathcal{E}\rangle=\hbar\omega_0\left[\frac{2}{\pi}\int_0^{2\pi n}\frac{\sin x}{x}dx\right]^{-1}, \tag{106}$$

as one readily realizes by combining Eqs. (88) and (105). The energy of the photon wave train hence is given by Eq. (106). For finite n, one always has $E>\hbar\omega_0$, and in the limit $n\to\infty$ the textbook result $E=\hbar\omega_0$ is recovered. Since the integral in Eq. (106) converges rapidly towards $\pi/2$ with increasing n, only for extremely short pulse trains the deviations from $E=\hbar\omega_0$ are pronounced (see Fig. 3).

E. Radial energy flows in the near-field zone

During the photon-emission process the dynamics of the emerging photon and the atom are coupled. This implies that the energy flow in the near-field zone of the atom is shared between the photon and a cross coupling effect between the atom and emerging photon. In the far-field zone the energy transport is provided solely by the emergent photon. In this section, the cycle-averaged energy flow in the radial direction will be examined, paying particular attention to the conditions in the near-field zone of the point particle.

1. Energy balance equations

By taking the inner products of the transverse Maxwell equations in Eqs. (3) and (28) with $\mathbf{B}$ and $\mathbf{E}_T$, respectively,

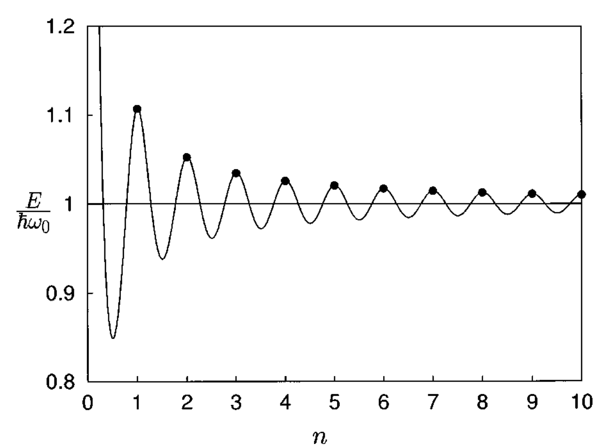

FIG. 3. Normalized one-photon energy, $E/(\hbar\omega_0)$, of a sinusoidal photon wave train (oscillation period $2\pi/\omega_0$) as a function of the number of periods n in the train. As indicated by the black dots only integer values of n are physically meaningful (The fully drawn $((2/\pi)\int_0^{2\pi n}x^{-1}\sin x dx)^{-1}$ curve is plotted to guide the eye.) Note that the one-photon energy E is always larger than $\hbar\omega_0$.

and afterwards subtracting the resulting equations, one readily obtains the following local energy balance relation:

$$\nabla\cdot\left(\frac{1}{\mu_0}\mathbf{E}_T\times\mathbf{B}\right)+\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E}_T\cdot\mathbf{E}_T+\frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right)=-\mathbf{E}_T\cdot\mathbf{J}_T, \tag{107}$$

where $\mu_0^{-1}\mathbf{E}_T\times\mathbf{B}$ is the Poynting vector of the transverse field, $(\epsilon_0/2)E_T^2+(2\mu_0)^{-1}B^2$ is the energy density in the transverse field, and $\mathbf{E}_T\cdot\mathbf{J}_T$ is the work carried out per unit time locally of the transverse field on the transverse degrees of freedom of the atom. A similar and well-known local energy balance equation holds for the flow and exchange of power in and between the total elecromagnetic field and the atom, namely [41],

$$\nabla\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)+\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E}+\frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right)=-\mathbf{E}\cdot\mathbf{J}, \tag{108}$$

where $\mathbf{J}$ and $\mathbf{E}$ denote the atomic current density and the total electric field. If one subtracts Eq. (28) from the Maxwell equation $\nabla\times\mathbf{B}=\mu_0\mathbf{J}+c_0^{-2}\partial\mathbf{E}/\partial t$, one obtains the relation

$$\mu_0\mathbf{J}_L(\mathbf{r},t)+\frac{1}{c_0^2}\frac{\partial\mathbf{E}_L(\mathbf{r},t)}{\partial t}=\mathbf{0} \tag{109}$$

between the longitudinal (L) parts of the atomic current density ($\mathbf{J}_L=\mathbf{J}-\mathbf{J}_T$) and the electric field ($\mathbf{E}_L=\mathbf{E}-\mathbf{E}_T$). By taking the scalar product of Eq. (109) and $\mathbf{E}_L$, an energy balance equation

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E}_L\cdot\mathbf{E}_L\right)=-\mathbf{E}_L\cdot\mathbf{J}_L \tag{110}$$

for the longitudinal atom-field dynamics emerges. Since the magnetic field has no longitudinal part [see Eq. (1)], the Poynting vector of the longitudinal electromagnetic field is zero, and this part of the field therefore cannot transport energy from one place to another. By subtracting Eqs. (107) and (108), and utilizing Eq. (110), one obtains an energy balance equation

$$\nabla\cdot\left(\frac{1}{\mu_0}\mathbf{E}_L\times\mathbf{B}\right)+\frac{\partial}{\partial t}(\epsilon_0\mathbf{E}_T\cdot\mathbf{E}_L)=-\mathbf{E}_T\cdot\mathbf{J}_L-\mathbf{E}_L\cdot\mathbf{J}_T \tag{111}$$

for the cross coupling between the longitudinal and transverse dynamics.

2. Cycle-averaged dynamics and outwards transport of energy

In a notation adequate when working with analytical signals (cf. e.g., Eq. (9) and Ref. [3]), the cycle-averaged product of two harmonically varying quantities

$$X_i(t)=X_i(\omega_0)e^{-i\omega_0 t}+X_i^*(\omega_0)e^{i\omega_0 t},\quad i=1,2 \tag{112}$$

of angular frequency $\omega_0(>0)=2\pi/T$ is given by

$$\langle X_1(t)X_2(t)\rangle\equiv\frac{1}{T}\int_{t_0}^{t_0+T}X_1(t)X_2(t)dt$$
$$=2\,\mathrm{Re}[X_1(\omega_0)X_2^*(\omega_0)]. \tag{113}$$

The result in Eq. (113) is independent of t_0, and for the field-atom dynamics associated with the sinusoidal wave train given in Eq. (77) one must choose t_0 in the interval $0\leqslant t_0\leqslant T_0-T$ to apply Eq. (113). By performing the cycle-average procedure to Eqs. (107), (108), (110), and (111), one obtains

$$\nabla\cdot\left\langle\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right\rangle=-\langle\mathbf{E}\cdot\mathbf{J}\rangle, \tag{114}$$

$$\nabla\cdot\left\langle\frac{1}{\mu_0}\mathbf{E}_T\times\mathbf{B}\right\rangle=-\langle\mathbf{E}_T\cdot\mathbf{J}_T\rangle, \tag{115}$$

$$\nabla\cdot\left\langle\frac{1}{\mu_0}\mathbf{E}_L\times\mathbf{B}\right\rangle=\langle\mathbf{E}_T\cdot\mathbf{J}_T\rangle-\langle\mathbf{E}\cdot\mathbf{J}\rangle, \tag{116}$$

and

$$\langle\mathbf{E}_L\cdot\mathbf{J}_L\rangle=0. \tag{117}$$

If we integrate Eqs. (114)–(116) over a spherical volume (V_0) of radius r_0 centered on the point particle position, and hereafter apply Gauss's theorem to the terms containing the ∇ operation, we get (denoting the surface element of the sphere by dS_0)

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right\rangle\cdot\hat{\mathbf{r}}\,dS_0=-\int_{V_0}\langle\mathbf{E}\cdot\mathbf{J}\rangle dV_0, \tag{118}$$

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}\mathbf{E}_T\times\mathbf{B}\right\rangle\cdot\hat{\mathbf{r}}\,dS_0=-\int_{V_0}\langle\mathbf{E}_T\cdot\mathbf{J}_T\rangle dV_0, \tag{119}$$

and

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}\mathbf{E}_L\times\mathbf{B}\right\rangle\cdot\hat{\mathbf{r}}\,dS_0$$
$$=\int_{V_0}(\langle\mathbf{E}_T\cdot\mathbf{J}_T\rangle-\langle\mathbf{E}\cdot\mathbf{J}\rangle)dV_0. \tag{120}$$

The sum of the two last equations gives Eq. (118), of course.

3. Energy flows in the photon perspective

The cycle-averaged energy balance equations in Eqs. (118)–(120) can only be fully understood if one (i) considers both the incident field ($\mathbf{E}^{\mathrm{inc}}$) acting on the atom and the scattered field ($\mathbf{E}^{\mathrm{scatt}}$) created by the particle, and (ii) takes into account the finite size of the atom. The field driving the atom hence consists of the sum of the incident field, which we necessarily must assume is transverse ($\mathbf{E}^{\mathrm{inc}}=\mathbf{E}_T^{\mathrm{inc}}$), and the transverse vector-field part ($\mathbf{E}_T^{\mathrm{scatt}}$) of the (yet unknown) scattered field (see Ref. [41]). If the incident field did have a longitudinal component on the site of the atom this would mean that the source of this field would be in the near-field zone of the atom, and consequently the source and the atom had to be considered as a single system from an electrodynamic point of view. Consequently, the problem would be more complicated than ''just'' a single-particle interaction with an electromagnetic field. The transverse local electric field

$$\mathbf{E}_T=\mathbf{E}^{\mathrm{inc}}+\mathbf{E}_T^{\mathrm{scatt}} \tag{121}$$

acting on the atom must be determined in a self-consistent manner from the combined Schrödinger (or Dirac) and microscopic Maxwell equations. In such a calculation finer details would be lost if the atom were considered as an electric point dipole. Once $\mathbf{E}_T$ has been determined (to a sufficient accuracy) the current density of the atom can be obtained and afterwards the split into $\mathbf{J}_T$ and $\mathbf{J}_L$ done. Finally, this allows a calculation of the right-hand sides of Eqs. (118)–(120). The radial energy flows appearing on the left-hand sides of these equations contain not only a scattered field contribution stemming from $\langle\mathbf{E}^{\mathrm{scatt}}\times\mathbf{B}^{\mathrm{scatt}}\rangle$, but also parts $(\sim\langle\mathbf{E}^{\mathrm{scatt}}\times\mathbf{B}^{\mathrm{inc}}\rangle,\langle\mathbf{E}^{\mathrm{inc}}\times\mathbf{B}^{\mathrm{scatt}}\rangle)$ associated with the interference of the incident and scattered fields. In agreement with the analysis presented in the previous parts of this paper, we here confine ourselves to a calculation of the radial flux associated with the scattered field, and we assume that the current density amplitude has such a magnitude that precisely one photon is emitted. The related scattered electric field we denote by $\mathbf{e}=\mathbf{e}_T+\mathbf{e}_L$, and the associated magnetic field by $\mathbf{b}$, as before.

In the space-frequency domain the electric field of the electric point dipole is given by (see, e.g., Ref. [28])

$$\mathbf{e}(\mathbf{r};\omega_0)=-i\mu_0\omega_0\overleftrightarrow{D}(\mathbf{r};\omega_0)\cdot\boldsymbol{\mathcal{J}}(\omega_0), \tag{122}$$

where, with the $q_0=\omega_0/c_0$,

$$\overleftrightarrow{D}(\mathbf{r};\omega_0)=\frac{q_0}{4\pi i}\left\{\frac{1}{iq_0r}(\overleftrightarrow{U}-\hat{\mathbf{r}}\hat{\mathbf{r}})-\left[\frac{1}{(iq_0r)^2}-\frac{1}{(iq_0r)^3}\right]\times(\overleftrightarrow{U}-3\hat{\mathbf{r}}\hat{\mathbf{r}})\right\}e^{iq_0r} \tag{123}$$

is the standard (textbook) propagator. With the current density in the z direction, the relevant tensor-vector products in Eq. (122) become in our spherical coordinates $(\overleftrightarrow{U}-\hat{\mathbf{r}}\hat{\mathbf{r}})\cdot\hat{\mathbf{z}}=-\hat{\boldsymbol{\theta}}\sin\theta$, and $(\overleftrightarrow{U}-3\hat{\mathbf{r}}\hat{\mathbf{r}})\cdot\hat{\mathbf{z}}=-\hat{\boldsymbol{\theta}}\sin\theta-2\hat{\mathbf{r}}\cos\theta$, and from the Maxwell equation $\boldsymbol{\nabla}\times\mathbf{e}(\mathbf{r};\omega_0)=i\omega_0\mathbf{b}(\mathbf{r};\omega_0)$ written in spherical coordinates a straightforward calculation gives

$$\mathbf{b}(\mathbf{r};\omega_0)=\frac{\mu_0q_0\mathcal{J}(\omega_0)}{4\pi i}\left(1-\frac{1}{iq_0r}\right)\frac{\sin\theta}{r}e^{iq_0r}\hat{\boldsymbol{\varphi}}. \tag{124}$$

Using distribution theory the same result may be obtained starting from the integral expression in Eq. (70). The cycle-averaged total energy flow per unit time in the radial direction through a sphere of radius r_0, i.e.,

$$\beta\equiv\oint_{S_0}\left\langle\frac{1}{\mu_0}\mathbf{e}\times\mathbf{b}\right\rangle\cdot\hat{\mathbf{r}}dS_0 \tag{125}$$

now easily can be calculated via Eqs. (113) and (122)–(124). If one remembers that $\boldsymbol{\mathcal{J}}(\omega_0)=\frac{1}{2}\boldsymbol{\mathcal{J}}_0$ [compare Eqs. (77) and (112)] we finally have

$$\beta=\frac{\mu_0\omega_0^2\mathcal{J}_0^2}{12\pi c_0}=\frac{\langle\mathcal{E}\rangle}{T}, \tag{126}$$

where the last relation follows from Eq. (105). The quantity β is independent of r_0 (the textbook result [42]) and represents the cycle-averaged power emitted by the atom. The energy of the emerged photon is $nT\beta$.

The energy transport associated with the cross coupling between the transverse and longitudinal fields we calculate next. The matter attached longitudinal electric field, $\mathbf{e}_L(\mathbf{r},t)$, accompanying the atomic one-photon current density, is given by

$$\mathbf{e}_L(\mathbf{r};\omega_0)=\frac{1}{i\epsilon_0\omega_0}\overleftrightarrow{\delta}_L(\mathbf{r})\cdot\boldsymbol{\mathcal{J}}(\omega_0),\quad \mathbf{r}\neq\mathbf{0} \tag{127}$$

in the space-frequency domain, except at the dipole position where the field is singular. With the help of the spherical-coordinate expression for the longitudinal delta function, i.e.,

$$\overleftrightarrow{\delta}_L(\mathbf{r})=\frac{1}{4\pi r^3}(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}+\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}-2\hat{\mathbf{r}}\hat{\mathbf{r}}),\quad \mathbf{r}\neq\mathbf{0} \tag{128}$$

the matter attached field becomes

$$\mathbf{e}_L(\mathbf{r};\omega_0)=\frac{i}{4\pi\epsilon_0\omega_0r^3}(\hat{\boldsymbol{\theta}}\sin\theta+2\hat{\mathbf{r}}\cos\theta)\mathcal{J}(\omega_0). \tag{129}$$

By combining Eqs. (124) and (129), and carrying out (in spherical coordinates) subsequently a number of elementary integrations it is found that the related energy flow per unit time through the sphere of radius r_0 is given by

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}\mathbf{e}_L\times\mathbf{b}\right\rangle\cdot\hat{\mathbf{r}}dS_0=\beta\alpha(q_0r_0), \tag{130}$$

where

$$\alpha(q_0r_0)=\frac{1}{(q_0r_0)^2}\left[\frac{\sin(q_0r_0)}{q_0r_0}-\cos(q_0r_0)\right]. \tag{131}$$

The coupling of the matter attached electric field to the magnetic field hence results in a cycle-averaged radial power transport which depends on the distance (r_0) from the point dipole. The power flow exhibits strongly damped spatial oscillations of period c_0T, and vanishes in the far field (but not in the midfield zone $[\sim(q_0r_0)^{-2}]$). For $r_0\to0$, the matter attached power flow equals $\beta/3$ since $\alpha(q_0r_0)|_{r_0\to0}=\frac{1}{3}$.

A subtraction of the results in Eqs. (125) and (130) shows that the radial power flow associated with the transverse electromagnetic field of the emerging photon is given by

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}\mathbf{e}_T\times\mathbf{b}\right\rangle\cdot\hat{\mathbf{r}}dS_0=\beta[1-\alpha(q_0r_0)] \tag{132}$$

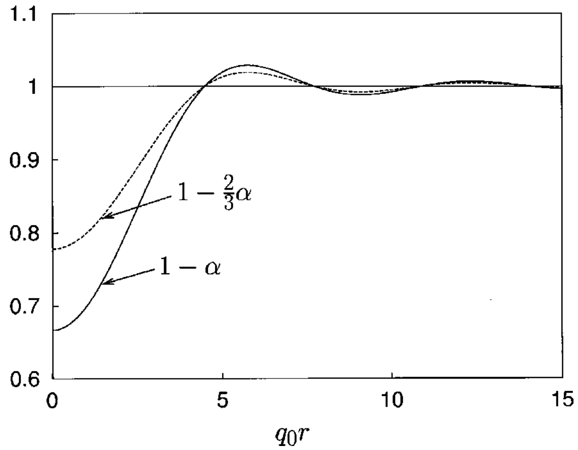

FIG. 4. Normalized and cycle-averaged radial power flows associated with the emergent photon as a function of the normalized distance, $q_0 r$, from the atom in the photon perspective $(1-\alpha)$ and electron perspective $[1-(2/3)\alpha]$. Far from the atom the outwards power flow is carried entirely in the photon field.

in the photon perspective (see Fig. 4). In the far field the photon attached field provides the entire energy transport in the outwards direction.

4. Energy flows in the electron perspective

In the space-frequency domain the relation between the transverse self-field and the single-photon current density has the form

$$\boldsymbol{\mathcal{E}}_{T,\mathrm{SF}}(\mathbf{r};\omega_0)=\frac{1}{3i\epsilon_0\omega_0}\overleftrightarrow{\delta}_T(\mathbf{r})\cdot\boldsymbol{\mathcal{J}}(\omega_0), \qquad (133)$$

and since $\overleftrightarrow{\delta}_T(\mathbf{r})=-\overleftrightarrow{\delta}_L(\mathbf{r})$ for $\mathbf{r}\neq\mathbf{0}$, it appears that

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}_{T,\mathrm{SF}}(\mathbf{r};\omega_0)=\mathbf{0}, \quad \mathbf{r}\neq\mathbf{0} \qquad (134)$$

in the point-particle approach. In turn this implies that $\boldsymbol{\nabla}\times\mathbf{e}_T(=\boldsymbol{\nabla}\times\mathbf{e})=\boldsymbol{\nabla}\times\mathbf{e}_{T,R}$, where $\mathbf{e}_{T,R}$ is the retarded part of the transverse electric field in the electron perspective. The magnetic field accompanying the emergent photon therefore is the same in the photon and electron perspectives, i.e., $\mathbf{b}=\mathbf{b}_R(\mathbf{b}^{(+)}=\mathbf{b}_R^{(+)})$, in the point-particle approximation.

By a comparison of Eqs. (127) and (133), one sees that

$$\boldsymbol{\mathcal{E}}_{T,\mathrm{SF}}(\mathbf{r};\omega_0)=-\tfrac{1}{3}\mathbf{e}_L(\mathbf{r};\omega_0), \quad \mathbf{r}\neq\mathbf{0}. \qquad (135)$$

By combining Eqs. (130) and (135) it appears that the coupling between the magnetic field and the transverse part of the matter attached field leads to a cycle-averaged power flow in the radial direction given by

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}\boldsymbol{\mathcal{E}}_{T,\mathrm{SF}}\times\mathbf{b}\right\rangle\cdot\hat{\mathbf{r}}dS_0=-\tfrac{1}{3}\beta\alpha(q_0r_0). \qquad (136)$$

If now one substracts the results in Eqs. (132) and (136), it is realized that the retarded part of the transverse electromagnetic field is responsible for a radial power transport (see also Fig. 4)

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}\mathbf{e}_{T,R}\times\mathbf{b}\right\rangle\cdot\hat{\mathbf{r}}dS_0=\beta[1-\tfrac{2}{3}\alpha(q_0r_0)]. \qquad (137)$$

The radial power flow associated with the coupling between the magnetic field and the total attached field is obtained by adding the results in Eqs. (130) and (136). Thus,

$$\oint_{S_0}\left\langle\frac{1}{\mu_0}(\mathbf{e}_L+\boldsymbol{\mathcal{E}}_{T,\mathrm{SF}})\times\mathbf{b}\right\rangle\cdot\hat{\mathbf{r}}dS_0=\tfrac{2}{3}\beta\alpha(q_0r_0). \qquad (138)$$

Close to the electric dipole, i.e., for $r_0\to 0$, Eq. (138) equals $2\beta/9$.

APPENDIX A: PHOTON AND ANTIPHOTON INTERFERENCE

In Sec. II B we defined the photon wave function in the space-time domain via the positive-frequency parts of the (normalized) Riemann-Silberstein vectors belonging to the positive and negative helicities [see Eq. (8)]. In analogy with this definition, the antiphoton energy wave function, denoted by $\boldsymbol{\Phi}_A(\mathbf{r},t)$, is introduced as the six-component object

$$\boldsymbol{\Phi}_A(\mathbf{r},t)=\begin{pmatrix}\mathbf{f}_+^{(-)}(\mathbf{r},t)\\ \mathbf{f}_-^{(-)}(\mathbf{r},t)\end{pmatrix}, \qquad (\mathrm{A1})$$

composed of the negative-frequency components of the two Riemann-Silberstein vectors. The antiphoton wave function may be obtained from the photon wave function by the particle-antiparticle conjugation operation

$$\boldsymbol{\Phi}_A(\mathbf{r},t)=\overleftrightarrow{\sigma}_1\cdot\boldsymbol{\Phi}^*(\mathbf{r},t), \qquad (\mathrm{A2})$$

$\overleftrightarrow{\sigma}_1$ being the Pauli spin matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. To show that Eq. (A2) is correct one may use the relations

$$[\mathbf{f}_\pm^{(-)}(\mathbf{r},t)]^*=\mathbf{f}_\mp^{(+)}(\mathbf{r},t). \qquad (\mathrm{A3})$$

These readily follow from the fact that since the frequency components of the real fields $\mathbf{e}_T(\mathbf{r},t)$ and $\mathbf{b}(\mathbf{r},t)$ must obey the conditions $\mathbf{e}_T(\mathbf{r};-\omega)=\mathbf{e}_T^*(\mathbf{r};\omega)$ and $\mathbf{b}(\mathbf{r};-\omega)=\mathbf{b}^*(\mathbf{r};\omega)$, one has $\mathbf{e}_T^{(-)}(\mathbf{r},t)=[\mathbf{e}_T^{(+)}(\mathbf{r},t)]^*$ and $\mathbf{b}^{(-)}(\mathbf{r},t)=[\mathbf{b}^{(+)}(\mathbf{r},t)]^*$.

If one considers the photon and antiphoton as different (orthogonal) eigenstates of a one-particle electromagnetic field superposition of these states are allowed. In the present context it is sufficient to consider a particular simple superposition, viz.,

$$\boldsymbol{\Xi}(\mathbf{r},t)=\frac{1}{\sqrt{2}}[\boldsymbol{\Phi}(\mathbf{r},t)+\boldsymbol{\Phi}_A(\mathbf{r},t)]. \qquad (\mathrm{A4})$$

Since

$$\boldsymbol{\Xi}(\mathbf{r},t)=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{f}_+(\mathbf{r},t)\\ \mathbf{f}_-(\mathbf{r},t)\end{pmatrix}, \qquad (\mathrm{A5})$$

and $\mathbf{f}_-(\mathbf{r},t)=\mathbf{f}_+^*(\mathbf{r},t)$, it appears that

$$\boldsymbol{\Xi}^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Xi}(\mathbf{r},t)=w_T(\mathbf{r},t) \tag{A6}$$

i.e., the local energy density of the transverse electromagnetic field, cf. Eq. (11). Thus, if one insists that $w_T(\mathbf{r},t)$ has physical reality, the photon and antiphoton must be able to interfere (self-photon-interference).

By means of Eq. (A6) it is easy to show that

$$\boldsymbol{\Phi}^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}(\mathbf{r},t)=\boldsymbol{\Phi}_A^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}_A(\mathbf{r},t), \tag{A7}$$

and the photon and antiphoton energy density distributions therefore are identical. It also follows with the use of Eq. (A3) that

$$[\boldsymbol{\Phi}^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}_A(\mathbf{r},t)]^*=\boldsymbol{\Phi}_A^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}(\mathbf{r},t). \tag{A8}$$

The total energy in the electromagnetic field E_{em} given by

$$E_{\mathrm{em}}=\int_{-\infty}^{\infty}\boldsymbol{\Xi}^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Xi}(\mathbf{r},t)d^3r, \tag{A9}$$

can tentatively be decomposed as follows:

$$E_{\mathrm{em}}=\tfrac{1}{2}\,(E+E_A)+\Delta, \tag{A10}$$

where

$$E=\int_{-\infty}^{\infty}\boldsymbol{\Phi}^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}(\mathbf{r},t)d^3r, \tag{A11}$$

and

$$E_A=\int_{-\infty}^{\infty}\boldsymbol{\Phi}_A^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}_A(\mathbf{r},t)d^3r, \tag{A12}$$

are the energies of the photon (E) and antiphoton (E_A), respectively, and

$$\Delta=\tfrac{1}{2}\int_{-\infty}^{\infty}\boldsymbol{\Phi}_A^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}(\mathbf{r},t)d^3r+\text{c.c.} \tag{A13}$$

is the net energy associated with the photon and antiphoton interference. It readily appears from Eq. (A7) that

$$E=E_A\,. \tag{A14}$$

To determine Δ one starts from the relation

$$\int_{-\infty}^{\infty}\boldsymbol{\Phi}_A^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\Phi}(\mathbf{r},t)d^3r=2\int_{-\infty}^{\infty}\mathbf{f}_+^{(+)}(\mathbf{r},t)\cdot\mathbf{f}_-^{(+)}(\mathbf{r},t)d^3r, \tag{A15}$$

obtained by combining the Hermitian conjugate of Eq. (A1), and Eqs. (8) and (A3). By inserting the Fourier transforms

$$\mathbf{f}_{\pm}^{(+)}(\mathbf{r},t)=(2\pi)^{-3}\int_{-\infty}^{\infty}\mathbf{f}_{\pm}^{(+)}(\mathbf{q},t)e^{i\mathbf{q}\cdot\mathbf{r}}d^3q \tag{A16}$$

in Eq. (A15) and carrying out some trivial integrations, one next gets

$$\int_{-\infty}^{\infty}\mathbf{f}_+^{(+)}(\mathbf{r},t)\cdot\mathbf{f}_-^{(+)}(\mathbf{r},t)d^3r =(2\pi)^{-3}\int_{-\infty}^{\infty}\mathbf{f}_+^{(+)}(\mathbf{q},t)\cdot\mathbf{f}_-^{(+)}(-\mathbf{q},t)d^3q. \tag{A17}$$

Since $\mathbf{f}_+^{(+)}(\mathbf{q},t)=f_+^{(+)}(\mathbf{q},t)\hat{\epsilon}_+(\hat{\mathbf{q}})$ and $\mathbf{f}_-^{(+)}(-\mathbf{q},t)=f_-^{(+)}(-\mathbf{q},t)\hat{\boldsymbol{\epsilon}}_-(-\hat{\mathbf{q}})$, the integrand in Eq. (A17) is proportional to the scalar product $\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{q}})\cdot\hat{\boldsymbol{\epsilon}}_-(-\hat{\mathbf{q}})$. If one writes the helicity unit vectors in the usual forms $\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{q}})=(1/\sqrt{2})[\hat{\boldsymbol{\epsilon}}_1(\hat{\mathbf{q}})+i\hat{\boldsymbol{\epsilon}}_2(\hat{\mathbf{q}})]$ and $\hat{\boldsymbol{\epsilon}}_-(-\hat{\mathbf{q}})=(1/\sqrt{2})[\hat{\boldsymbol{\epsilon}}_1(-\hat{\mathbf{q}})-i\hat{\boldsymbol{\epsilon}}_2(-\hat{\mathbf{q}})]$, where the sets of unit vectors $(\hat{\boldsymbol{\epsilon}}_1(\hat{\mathbf{q}}),\hat{\boldsymbol{\epsilon}}_2(\hat{\mathbf{q}}),\hat{\mathbf{q}})$ and $(\hat{\boldsymbol{\epsilon}}_1(-\hat{\mathbf{q}}),\hat{\boldsymbol{\epsilon}}_2(-\hat{\mathbf{q}}),-\hat{\mathbf{q}})$ each form right-handed triads, it follows from elementary calculations of inner products that

$$\hat{\boldsymbol{\epsilon}}_+(\hat{\mathbf{q}})\cdot\hat{\boldsymbol{\epsilon}}_-(-\hat{\mathbf{q}})=0, \tag{A18}$$

independent of the *chosen angle* between $\hat{\boldsymbol{\epsilon}}_1(\hat{\mathbf{q}})$ and $\boldsymbol{\epsilon}_1(-\hat{\mathbf{q}})$. The result in Eq. (A18) hence implies that the right-hand side of Eq. (A17) is zero. Altogether the net energy associated with the photon ↔ antiphoton interference therefore is zero, i.e.,

$$\Delta=0. \tag{A19}$$

A combination of Eqs. (A10), (A14), and (A19) thus shows that

$$E_{\mathrm{em}}=E, \tag{A20}$$

as postulated in Eq. (10).

APPENDIX B: CALCULATION OF THE MAGNETIC-FIELD PROPAGATOR

1. Photon perspective

In order to derive the expression for the magnetic-field propagator $\overleftrightarrow{m}(\mathbf{R},\tau)$, which is cited in Eq. (54), starting from Eq. (52), we introduce the Huygens scalar Green function

$$d(R,\tau)=-\frac{1}{4\pi R}\,\delta\!\left(\frac{R}{c_0}-\tau\right), \tag{B1}$$

and note that the isotropic propagator in Eq. (50) is just $\overleftrightarrow{d}(R,\tau)=d(R,\tau)\overleftrightarrow{U}$. The curl of the transverse electric field $\mathbf{e}_T^{(+)}(\mathbf{r},t)$ is obtained, beginning with the result

$$\begin{aligned}\boldsymbol{\nabla}\times\left[\overleftrightarrow{d}(R,\tau)\cdot\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r}',t')}{\partial t'}\right]&=[\nabla d(R,\tau)]\times\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r}',t')}{\partial t'}\\&=\left[\frac{\partial d(R,\tau)}{\partial R}\right]\hat{\mathbf{R}}\times\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r}',t')}{\partial t'},\end{aligned} \tag{B2}$$

easily gotten by remembering that the nabla operator $\boldsymbol{\nabla}$ operates in $\mathbf{r}$ space. Utilizing also that

$$\frac{\partial d(R,\tau)}{\partial R}=\frac{1}{4\pi}\left[\frac{1}{R^2}\,\delta\!\left(\frac{R}{c_0}-\tau\right)-\frac{1}{R}\,\frac{\partial\delta\!\left(\frac{R}{c_0}-\tau\right)}{\partial R}\right],\quad \text{(B3)}$$

we find from Eq. (51)

$$\nabla\times\mathbf{e}_T^{(+)}(\mathbf{r},t'')=\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\left[\frac{1}{R^2}\,\delta\!\left(\frac{R}{c_0}-t''+t'\right)-\frac{1}{R}\,\frac{\partial\delta\!\left(\frac{R}{c_0}-t''+t'\right)}{\partial R}\right]\hat{\mathbf{R}}\times\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r}',t')}{\partial t'}\,d^3r'\,dt',\quad \text{(B4)}$$

since the nabla operation and the $\mathbf{r}'$ integration can be interchanged directly. For the sake of the subsequent calculation the time variable t has been renamed t''. To determine the magnetic field via Eq. (52) [with t' replaced by t''], one makes use of the formulas

$$\int_{-\infty}^{t}\delta\!\left(\frac{R}{c_0}-t''+t'\right)dt''=\theta\!\left(t-t'-\frac{R}{c_0}\right),\quad \text{(B5)}$$

and

$$\int_{-\infty}^{t}\frac{\partial\delta\!\left(\frac{R}{c_0}-t''+t'\right)}{\partial R}\,dt''=-\frac{1}{c_0}\,\delta\!\left(\frac{R}{c_0}-t+t'\right),\quad \text{(B6)}$$

and obtains consequently

$$\mathbf{b}^{(+)}(\mathbf{r},t)=-\frac{\mu_0}{4\pi c_0}\int_{-\infty}^{\infty}\left[\frac{c_0}{R^2}\,\theta\!\left(t-t'-\frac{R}{c_0}\right)+\frac{1}{R}\,\delta\!\left(\frac{R}{c_0}-t+t'\right)\right]\hat{\mathbf{R}}\times\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}(\mathbf{r}',t')}{\partial t'}\,d^3r'\,dt'.\quad \text{(B7)}$$

Since

$$\hat{\mathbf{R}}\times\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}}{\partial t'}=\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\hat{\mathbf{R}}+\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{\Theta}}+\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Phi}})\cdot\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}}{\partial t'}\quad \text{(B8)}$$

because the dyadic sum $\hat{\mathbf{R}}\hat{\mathbf{R}}+\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{\Theta}}+\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Phi}}$ is just the unit tensor, we finally may write

$$\hat{\mathbf{R}}\times\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}}{\partial t'}=(\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Theta}}-\hat{\boldsymbol{\Theta}}\hat{\boldsymbol{\Phi}})\cdot\frac{\partial\boldsymbol{\mathcal{J}}_T^{(+)}}{\partial t'},\quad \text{(B9)}$$

remembering that $\hat{\mathbf{R}}$, $\hat{\boldsymbol{\Theta}}$, and $\hat{\boldsymbol{\Phi}}$ in this cyclic order form a right-hand triad. By inserting Eq. (B9) into Eq. (B7), it appears that the analytical signal belonging to the magnetic field can be written in the form given in Eq. (53), with a magnetic-field propagator as cited in Eq. (54). Q.E.D.

2. Electron perspective

To determine the curl of the retarded transverse electric field, $\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t)$, given in Eq. (67), let us consider the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, and let us assume that $\mathbf{A}$ and $\mathbf{B}$ depend on the position vector $\mathbf{r}$ whereas $\mathbf{C}$ is independent of $\mathbf{r}$. The following tensor identity then holds:

$$\nabla\times[\mathbf{A}(\mathbf{r})\mathbf{B}(\mathbf{r})\cdot\mathbf{C}]=\{[\nabla\times\mathbf{A}(\mathbf{r})]\mathbf{B}(\mathbf{r})-\mathbf{A}(\mathbf{r})\times\nabla\mathbf{B}(\mathbf{r})\}\cdot\mathbf{C}.\quad \text{(B10)}$$

Since the step function $\theta[(R/c_0)-\tau]$ appearing in Eq. (68) ensures that no R^{-3} singularity is present in $\overleftrightarrow{D}_T(\mathbf{R},\tau)$ for $\tau>0$, the integral over $\mathbf{r}'$ in Eq. (67) converges absolutely, and the $\nabla\times$ and $\int_{-\infty}^{\infty}$ operations can be interchanged in the determination of $\nabla\times\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t)$. Thus, we have

$$\nabla\times\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t)=\mu_0\int_{-\infty}^{\infty}\nabla\times\left[\overleftrightarrow{D}_T(\mathbf{R},\tau)\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}\right]d^3r'\,dt'.\quad \text{(B11)}$$

Since $\overleftrightarrow{D}_T(\mathbf{R},\tau)$ consists of a sum of dyadic terms, see Eq. (68), the identity in Eq. (B10) helps us to calculate $\nabla\times[\overleftrightarrow{D}_T\cdot(\partial\boldsymbol{\mathcal{J}}^{(+)}/\partial t')]$.

It appears from Eq. (68) that the transverse electromagnetic propagator contains two dyadic terms of the form $\alpha(R)\hat{\mathbf{R}}\hat{\mathbf{R}}$, where $\alpha(R)=d(R,\tau)$, and $\alpha(R)=3c_0^2\tau\theta(\tau)\theta[(R/c_0)-\tau]/(4\pi R^3)$, respectively. By setting $\mathbf{A}(\mathbf{r})=\alpha(R)\hat{\mathbf{R}}$, $\mathbf{B}(\mathbf{r})=\hat{\mathbf{R}}$, and $\mathbf{C}=\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')/\partial t'$, one obtains for these terms by means of Eq. (B10)

$$\nabla\times\left[\alpha(R)\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}\right]=(\{\nabla\times[\alpha(R)\hat{\mathbf{R}}]\}\hat{\mathbf{R}}-\alpha(R)\hat{\mathbf{R}}\times\nabla\hat{\mathbf{R}})\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}.\quad \text{(B12)}$$

Since

$$\nabla\hat{\mathbf{R}}=\frac{1}{R}(\overleftrightarrow{U}-\hat{\mathbf{R}}\hat{\mathbf{R}}),\quad \text{(B13)}$$

as one may show by an explicit calculation in Cartesian coordinates, for instance, and

$$\nabla\times[\alpha(R)\hat{\mathbf{R}}]=\mathbf{0},\quad \text{(B14)}$$

Eq. (B12) is reduced to

$$\nabla\times\left[\alpha(R)\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}\right]=-\frac{\alpha(R)}{R}\hat{\mathbf{R}}\times\overleftrightarrow{U}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}.\qquad(B15)$$

The two remaining terms in the dyadic expression for $\overleftrightarrow{D}_T(\mathbf{R},\tau)$ have the form $\beta(R)\overleftrightarrow{U}$, with $\beta(R)=d(R,\tau)$ and $\beta(R)=\alpha(R)/3$. For these terms one gets

$$\nabla\times\left[\beta(R)\overleftrightarrow{U}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}\right]=\frac{\partial\beta(R)}{\partial R}\hat{\mathbf{R}}\times\overleftrightarrow{U}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'},\qquad(B16)$$

cf. Eq. (B2).

By means of Eqs. (68), (B15), and (B16) it appears that the far-field contribution to the integral in Eq. (B11) equals

$$\nabla\times\left[d(R,\tau)(\overleftrightarrow{U}-\hat{\mathbf{R}}\hat{\mathbf{R}})\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}\right]=\left[\frac{\partial d(R,\tau)}{\partial R}+\frac{d(R,\tau)}{R}\right]\hat{\mathbf{R}}\times\overleftrightarrow{U}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'},\qquad(B17)$$

and the near-field contribution is given by

$$\nabla\times\left[\frac{c_0^2\tau}{4\pi R^3}\theta(\tau)\theta\left(\frac{R}{c_0}-\tau\right)(\overleftrightarrow{U}-3\hat{\mathbf{R}}\hat{\mathbf{R}})\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}\right]=\frac{c_0^2\tau}{4\pi}\theta(\tau)\left[\frac{\partial}{\partial R}\left(\frac{\theta\left(\frac{R}{c_0}-\tau\right)}{R^3}\right)+\frac{3\theta\left(\frac{R}{c_0}-\tau\right)}{R^4}\right]\hat{\mathbf{R}}\times\overleftrightarrow{U}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}.\qquad(B18)$$

Since Eq. (B3) can be rewritten in the form

$$\frac{\partial d(R,\tau)}{\partial R}+\frac{d(R,\tau)}{R}=-\frac{1}{4\pi R}\frac{\partial\delta\left(\frac{R}{c_0}-\tau\right)}{\partial R},\qquad(B19)$$

and a direct calculation gives

$$\frac{\partial}{\partial R}\left(\frac{\theta\left(\frac{R}{c_0}-\tau\right)}{R^3}\right)+\frac{3\theta\left(\frac{R}{c_0}-\tau\right)}{R^4}=\frac{1}{c_0R^3}\delta\left(\frac{R}{c_0}-\tau\right),\qquad(B20)$$

we finally have

$$\nabla\times\mathbf{e}_{T,R}^{(+)}(\mathbf{r},t'')=\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\left[-\frac{1}{R}\frac{\partial\delta\left(\frac{R}{c_0}-t''+t'\right)}{\partial R}+\frac{c_0(t''-t')}{R^3}\theta(t''-t')\delta\left(\frac{R}{c_0}-t''+t'\right)\right]\hat{\mathbf{R}}\times\overleftrightarrow{U}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}d^3r'dt'.\qquad(B21)$$

To obtain the retarded magnetic field, $\mathbf{b}_R^{(+)}(\mathbf{r},t)$, from Eq. (69), with t' replaced by t'', and with the expression in Eq. (B21) inserted, one makes use of Eq. (B6) and the formula

$$\int_{-\infty}^{t}(t''-t')\theta(t''-t')\delta\left(\frac{R}{c_0}-t''+t'\right)dt''=\frac{R}{c_0}\theta\left(t-t'-\frac{R}{c_0}\right).\qquad(B22)$$

Hence, one gets

$$\mathbf{b}_R^{(+)}(\mathbf{r},t)=\frac{\mu_0}{c_0}\int_{-\infty}^{\infty}\left[-\frac{1}{4\pi R}\delta\left(\frac{R}{c_0}-\tau\right)-\frac{c_0}{4\pi R^2}\theta\left(\tau-\frac{R}{c_0}\right)\right]\hat{\mathbf{R}}\times\overleftrightarrow{U}\cdot\frac{\partial\boldsymbol{\mathcal{J}}^{(+)}(\mathbf{r}',t')}{\partial t'}d^3r'dt',\qquad(B23)$$

and by a comparison with Eq. (54) [with Eq. (55) inserted], we immediately obtain the propagator result cited in Eq. (70).

APPENDIX C: CALCULATION OF $\mathcal{J}_T(Q)$ AND $\mathcal{J}_L(Q)$ FOR THE HYDROGEN $1S\leftrightarrow 2P_Z$ TRANSITION

To determine the Fourier transform $\boldsymbol{\mathcal{J}}(\mathbf{q})$ of the effective transition current density, $\boldsymbol{\mathcal{J}}(\mathbf{r})$, given in polar coordinates in Eq. (92), it is convenient first to write it in Cartesian coordinates. Thus

$$\boldsymbol{\mathcal{J}}(\mathbf{r})=B\left(\frac{b}{3}z\hat{\mathbf{r}}+\hat{\mathbf{z}}\right)e^{-br},\qquad(C1)$$

where $\hat{\mathbf{r}}=(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})/r$ with $r=(x^2+y^2+z^2)^{1/2}$. Next we perform a rotation of the Cartesian coordinate system so that the new axes are given by the unit vectors $\hat{\boldsymbol{\theta}}_q$, $\hat{\boldsymbol{\varphi}}_q$, and $\hat{\mathbf{q}}$. These vectors are just the local unit vectors belonging to a spherical coordinate representation of the wave vector $\mathbf{q}$. [The polar axis is assumed to coincide with the $\hat{\mathbf{z}}$ axis.] In the new coordinate system we denote the arbitrary vector by

$$\mathbf{r}_0=x_0\hat{\boldsymbol{\theta}}_q+y_0\hat{\boldsymbol{\varphi}}_q+z_0\hat{\mathbf{q}},\qquad(C2)$$

and since the transformation from the old to new coordinates implies that $r \Rightarrow r_0 = (x_0^2 + y_0^2 + z_0^2)^{1/2}$, $\hat{\mathbf{r}} \Rightarrow \hat{\mathbf{r}}_0 = \mathbf{r}_0 / r_0$, $z \Rightarrow z_0 \cos\theta_q - x_0 \sin\theta_q$, and $\hat{\mathbf{z}} \Rightarrow \hat{\mathbf{q}} \cos\theta_q - \hat{\boldsymbol{\theta}}_q \sin\theta_q$, the transition current density in Eq. (C1) goes into

$$\mathscr{J}(\mathbf{r}_0) = B\left[\frac{b}{3}(z_0\cos\theta_q - x_0\sin\theta_q)\hat{\mathbf{r}}_0 + (\hat{\mathbf{q}}\cos\theta_q - \hat{\boldsymbol{\theta}}_q\sin\theta_q)\right]e^{-br_0}. \tag{C3}$$

The Fourier transform hence is given by

$$\mathscr{J}(\mathbf{q}) = B\int_{-\infty}^{\infty} e^{-iqz_0}\exp\left(-\frac{3r_0}{2a_0}\right)\times\left[\frac{1}{2a_0}(z_0\cos\theta_q - x_0\sin\theta_q)\hat{\mathbf{r}}_0 + \hat{\mathbf{q}}\cos\theta_q - \hat{\boldsymbol{\theta}}_q\sin\theta_q\right]d^3r_0, \tag{C4}$$

since $\mathbf{q}\cdot\mathbf{r} \Rightarrow qz_0$. The integration over the $\mathbf{r}_0$ space is adequately carried out in spherical coordinates for which the polar axis coincides with the $\hat{\mathbf{q}}$ axis. The polar and azimuth angles we denote by α and β, respectively. By inserting $(x_0, y_0, z_0) = r_0(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$, $\hat{\mathbf{r}}_0 = \sin\alpha\cos\beta\hat{\boldsymbol{\theta}}_q + \sin\alpha\sin\beta\hat{\boldsymbol{\varphi}}_q + \cos\alpha\hat{\mathbf{q}}$, and $d^3r_0 = r_0^2\sin\alpha d\beta d\alpha dr_0$, in Eq. (C4), one obtains after having carried out the trivial integrations over β

$$\mathscr{J}(\mathbf{q}) = -\pi B\hat{\boldsymbol{\theta}}_q\sin\theta_q\int_0^\infty\int_0^\pi e^{-iqr_0\cos\alpha}e^{-3r_0/(2a_0)}\times\left(\frac{r_0}{2a_0}\sin^2\alpha + 2\right)r_0^2\sin\alpha d\alpha dr_0 + \pi B\hat{\mathbf{q}}\cos\theta_q\int_0^\infty\int_0^\pi e^{-iqr_0\cos\alpha}e^{-3r_0/(2a_0)}\times\left(\frac{r_0}{a_0}\cos^2\alpha + 2\right)r_0^2\sin\alpha d\alpha dr_0. \tag{C5}$$

Since the longitudinal and transverse parts of $\mathscr{J}(\mathbf{q})$ may be obtained from, respectively,

$$\mathscr{J}_L(\mathbf{q}) = \hat{\mathbf{q}}\hat{\mathbf{q}}\cdot\mathscr{J}(\mathbf{q}) \tag{C6}$$

and

$$\mathscr{J}_T(\mathbf{q}) = (\hat{\boldsymbol{\theta}}_q\hat{\boldsymbol{\theta}}_q + \hat{\boldsymbol{\varphi}}_q\hat{\boldsymbol{\varphi}}_q)\cdot\mathscr{J}(\mathbf{q}), \tag{C7}$$

a comparison with Eq. (C5) indicates that

$$\mathscr{J}(\mathbf{q}) = \hat{\mathbf{q}}\mathscr{J}_L(\mathbf{q}) + \hat{\boldsymbol{\theta}}_q\mathscr{J}_T(\mathbf{q}), \tag{C8}$$

where

$$\mathscr{J}_L(\mathbf{q}) = \pi B\cos\theta_q\int_0^\infty\int_0^\pi e^{-iqr_0\cos\alpha}e^{-3r_0/(2a_0)}\times\left(\frac{r_0}{a_0}\cos^2\alpha + 2\right)r_0^2\sin\alpha d\alpha dr_0 \tag{C9}$$

and

$$\mathscr{J}_T(\mathbf{q}) = -\pi B\sin\theta_q\int_0^\infty\int_0^\pi e^{-iqr_0\cos\alpha}e^{-3r_0/(2a_0)}\times\left(\frac{r_0}{2a_0}\sin^2\alpha + 2\right)r_0^2\sin\alpha d\alpha dr_0. \tag{C10}$$

It appears from Eq. (C8) that the transverse part of the transition current density only has a component in the $\hat{\boldsymbol{\theta}}_q$ direction, as expected due to the fact that $\mathscr{J}(\mathbf{r})$ in Eq. (92) is independent of the azimuth angle $\hat{\boldsymbol{\varphi}}$. If one makes the substitution $u = \cos\alpha$ in Eqs. (C9) and (C10), the integrations over u are easily carried out, and after this has been done only elementary integrals over r_0 need to be performed. The final results for the transverse and longitudinal parts of the hydrogen $1s \leftrightarrow 2p_z$ transition current density in the wave-vector domain are presented in the main text [Eqs. (98) and (99)]. The corresponding expressions in direct space may be found in, e.g., Ref. [26].

[1] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms, Introduction to Quantum Electrodynamics* (Wiley Interscience, New York, 1989), and references therein.
[2] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions, Basic Processes and Applications* (Wiley Interscience, New York, 1992), and references therein.
[3] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995), and references therein.
[4] O. Keller, Phys. Rev. A **58**, 3407 (1998).
[5] L. D. Landau and R. Peierls, Z. Phys. **62**, 188 (1930).
[6] R. J. Cook, Phys. Rev. A **25**, 2164 (1982).
[7] R. J. Cook, Phys. Rev. A **26**, 2754 (1982).
[8] R. J. Cook, in *Coherence and Quantum Optics V*, edited by L. Mandel and E. Wolf (Plenum, New York, 1984), p. 539.
[9] T. Inagaki, Phys. Rev. A **49**, 2839 (1994).
[10] I. Bialynicki-Birula, Acta Phys. Pol. A **86**, 97 (1994).
[11] I. Bialynicki-Birula, in *Coherence and Quantum Optics VII*, edited by J. H. Eberly, L. Mandel, and E. Wolf (Plenum, New York, 1996), p. 313.
[12] I. Bialynicki-Birula, in *Progress in Optics XXXVI*, edited by E. Wolf (North-Holland, Amsterdam, 1996), p. 245.
[13] J. E. Sipe, Phys. Rev. A **52**, 1875 (1995).
[14] H. Weber, *Die partiellen Differential-Gleichungen der Mathematischen Physik nach Riemann's Vorlesungen* (Friedrich Vieweg und Sohn, Braunschweig, 1901), p. 348.

[15] L. Silberstein, Ann. Phys. (Leipzig) **22**, 579 (1907).
[16] L. Silberstein, Ann. Phys. (Leipzig) **24**, 783 (1907).
[17] L. Silberstein, *The Theory of Relativity* (MacMillan, London, 1914).
[18] J. R. Oppenheimer, Phys. Rev. **38**, 725 (1931).
[19] M. Hawton, Phys. Rev. A **59**, 954 (1999).
[20] M. Hawton, Phys. Rev. A **59**, 3223 (1999).
[21] E. A. Power and S. Zinau, Philos. Trans. R. Soc. London, Ser. A **251**, 427 (1959).
[22] E. A. Power, *Introductory Quantum Electrodynamics* (Longmans, London, 1964).
[23] R. G. Wooley, Mol. Phys. **22**, 1013 (1971).
[24] E. A. Power and T. Thirunamachandran, Phys. Rev. A **25**, 2473 (1982).
[25] W. P. Healy, Phys. Rev. A **22**, 2891 (1980).
[26] O. Keller, Phys. Rep. **268**, 85 (1996), and references therein.
[27] O. Keller, in *Progress in Optics XXXVII*, edited by E. Wolf (North-Holland, Amsterdam, 1997), p. 257, and references therein.
[28] O. Keller, J. Opt. Soc. Am. **16**, 835 (1999).
[29] R. Y. Chiao and A. M. Steinberg, in *Progress in Optics XXXVII*, edited by E. Wolf (North-Holland, Amsterdam, 1997), p. 345, and references therein.
[30] O. Keller, Phys. Rev. A **60**, 1652 (1999).
[31] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
[32] M. H. L. Pryce, Philos. Trans. R. Soc. London, Ser. A **195**, 62 (1948).
[33] R. Acharya and E. C. G. Sudarshan, J. Math. Phys. **1**, 532 (1960).
[34] J. M. Jauch and C. Piron, Helv. Phys. Acta **40**, 559 (1967).
[35] W. O. Amrein, Helv. Phys. Acta **42**, 149 (1969).
[36] E. R. Pike and S. Sarkar, in *Frontiers in Quantum Optics*, edited by E. R. Pike and S. Sarkar (Adam Hilger, Bristol, 1986), p. 282.
[37] C. Adlard, E. R. Pike, and S. Sarkar, Phys. Rev. Lett. **79**, 1583 (1997).
[38] I. Bialynicki-Birula, Phys. Rev. Lett. **80**, 5247 (1998).
[39] G. Molière, Ann. Phys. (Leipzig) **6**, 146 (1949).
[40] R. H. Good, Jr., Phys. Rev. **105**, 1914 (1957).
[41] M. Born and E. Wolf, *Principles of Optics, Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 7th ed. (Cambridge University Press, New York, 1999).
[42] J. A. Kong, *Electromagnetic Wave Theory* (Wiley Interscience, New York, 1986).