# Geometric phase for entangled spin pairs

Erik Sjöqvist*
*Department of Quantum Chemistry, Uppsala University, Box 518, Se-751 20 Uppsala, Sweden*
*and Atominstitut der Österreichischen Universitäten, Stadionallee 2, A-1020 Vienna, Austria*
(Received 19 November 1999; published 17 July 2000)

The entanglement dependence of the noncyclic geometric phase is analyzed. A pair of noninteracting spin-$\frac{1}{2}$ particles prepared in an arbitrarily entangled state and precessing in an external time-independent uniform magnetic field is considered. It is shown that the geometric phase reduces to a sum of one-particle geometric phases for product states and takes on the two values corresponding to the phase factors $\pm 1$ for maximally entangled states. If only one of the particles is affected by the magnetic field it is demonstrated that the influence of entanglement on the geometric phase may be interpreted as an effective reduction of the degree of polarization of the affected particle. The generalization to more than two precessing spin-$\frac{1}{2}$ particles, in the particular case where Schmidt decompositions exists, is briefly outlined. The geometric phase for a pair of spin-$\frac{1}{2}$ particles with a spin-spin interaction is calculated. In this model we show that the noncyclic geometric phase for a certain class of states may be interpreted solely in terms of the solid angle enclosed by the geodesically closed curve on a two-sphere parametrized by the evolving Schmidt coefficients. This suggests a geometric interpretation of Schmidt decompositions for spin-$\frac{1}{2}$ pairs analogous to that of the Poincaré sphere for a single spin $\frac{1}{2}$.

PACS number(s): 03.65.Bz

## I. INTRODUCTION

The concept of geometric phase, first introduced by Pancharatnam [1] in his study of interference between light waves in distinct states of polarization and rediscovered by Berry [2] for quantal systems undergoing cyclic adiabatic evolution, has been refined and applied during the past 15 years. Aharonov and Anandan [3] removed the need of adiabatic external parameters and pointed out that the geometric phase could be considered the anholonomy associated with the curvature of the projective Hilbert space. Samuel and Bhandari [4] extended the geometric phase to noncyclic and nonunitary evolutions by making use of the notion of geodesic closure in the projective Hilbert space. Subsequent work [5–8] were to show the redundancy of the geodesic closure, making the noncyclic geometric phase easier to calculate and conceptually more accessible. Recent applications of the noncyclic geometric phase can be found in molecular dynamics [9], linear response theory [10], and the theory of wave packet revivals [11].

Another important development having its roots in the quantal formalism is that of entanglement. Several conceptual consequences of this phenomenon were pointed out already in the 1930s [12,13], but it was not until the work by Bell [14] that entanglement became an experimental issue. Especially the recent development of parametric down-conversion techniques as a convenient tool for creating correlated multiphoton states, has made many of the curious implications of entanglement available in the laboratory (see, e.g., Ref. [15] for a review). Today we know that the importance of entanglement may extend to information theoretical applications such as quantum cryptography [16], quantum teleportation [17,18], and fast quantum computation [19] (a recent review of these achievements can be found in Ref. [20]).

In the present paper we analyze the entanglement dependence of the geometric phase in the case of SU(2) (spin) subsystems. Such an analysis seems pertinent in the context of the recent efforts [21–27] concerning phase effects for entangled states in two-photon interferometry. We analyze the noncyclic nonadiabatic two-particle geometric phase for two model systems both involving entangled spin-$\frac{1}{2}$ particles. In the first case noninteracting spin-$\frac{1}{2}$ prepared in an arbitrarily entangled Schmidt state undergo local SU(2) operations caused by an external time-independent uniform magnetic field. The second case concerns a spin-spin interaction model.

## II. NONCYCLIC GEOMETRIC PHASE

In this section we review briefly the kinematic approach to the noncyclic geometric phase for unitary evolutions [6]. Consider the Hilbert space trajectory $\tilde{\Gamma}: t \in [0,\tau] \rightarrow |\Psi(t)\rangle$ with $\Psi(t)$ normalized and the "end points" $\Psi(0)$ and $\Psi(\tau)$ nonorthogonal. By noting that the dynamical phase change $\delta\eta(t)$ at $t$ along $\tilde{\Gamma}$ is naturally given by $\delta\eta(t) = \arg\langle\Psi(t)|\Psi(t+\delta t)\rangle \approx -i\langle\Psi(t)|\dot{\Psi}(t)\rangle\delta t$, we may define the geometric phase associated with the projective Hilbert space image $\Gamma$ of $\tilde{\Gamma}$ by removing the accumulation of these dynamical phases from the total phase, i.e.,

$$\Phi_G[\Gamma] = \arg\langle\Psi(0)|\Psi(\tau)\rangle + i\int_0^\tau dt\langle\Psi(t)|\dot{\Psi}(t)\rangle. \quad (1)$$

The geometric phase $\Phi_G[\Gamma]$ is real-valued, reparametrization invariant, and projective geometric [5,6]. It depends only on the curve $\Gamma$ in the projective Hilbert space. It can be demonstrated that $\Phi_G[\Gamma]$ reduces to the Aharonov-Anandan

*Electronic address: eriks@kvac.uu.se

 

formula [3] for cyclic evolutions and to the expressions in Refs. [2,8] for adiabatic states.

As an example consider a spin-$\frac{1}{2}$ precessing in a time-independent uniform magnetic field pointing in the $z$ direction. Assuming that the initial spin state $|\mathbf{n}(0)\rangle$ makes an angle $\theta$ with the $z$ axis the state at any later time is given by

$$|\mathbf{n}(t)\rangle = e^{-i\varphi(t)/2}\cos\frac{\theta}{2}|+z\rangle + e^{i\varphi(t)/2}\sin\frac{\theta}{2}|-z\rangle. \tag{2}$$

Here $\varphi(t)=\varphi(0)+\omega t$, where $\varphi(0)$ is the initial angle with the $x$ axis and $\omega$ is the Larmor frequency being proportional to the magnetic field strength. The curve $\Gamma$ in the projective Hilbert space is isomorphic to the curve $C_{\mathbf{n}}: t\in[0,\tau] \to \mathbf{n}(t)=[\sin\theta\cos\varphi(t),\sin\theta\sin\varphi(t),\cos\theta]$ on the Poincaré sphere with antipodal points corresponding to orthogonal states, i.e., $\langle -\mathbf{n}|\mathbf{n}\rangle=0$. Inserting Eq. (2) into Eq. (1) we obtain the noncyclic geometric phase acquired during the evolution as (see, e.g., Ref. [28])

$$\Phi_G[C_{\mathbf{n}}] = -\arctan\left(\cos\theta\tan\frac{\omega\tau}{2}\right) + \frac{\omega\tau}{2}\cos\theta$$
$$= -\frac{1}{2}\Omega[C_{\mathbf{n}}^{g-c}], \tag{3}$$

where $\Omega[C_{\mathbf{n}}^{g-c}]$ is the solid angle enclosed by the curve $C_{\mathbf{n}}^{g-c}$ that consists of $C_{\mathbf{n}}$ and the shortest geodesic on the Poincaré sphere connecting the end points $\mathbf{n}(0)$ and $\mathbf{n}(\tau)\neq -\mathbf{n}(0)$. Note that for a given curve $C_{\mathbf{n}}$, $\Phi_G[C_{\mathbf{n}}]$ is independent of the strength of the Hamiltonian as it depends on the precession angle $\omega\tau$, but not on $\omega$ itself.

## III. GEOMETRIC PHASE FOR ENTANGLED SU(2) STATES

### A. Description of entanglement

According to Schmidt's theorem [29,30] the state of any spin-$\frac{1}{2}$ pair (or indeed any pair of two-state systems) may be decomposed as

$$|\Psi\rangle = e^{-i\beta/2}\cos\frac{\alpha}{2}|\mathbf{n}\rangle_1|\mathbf{m}\rangle_2 + e^{i\beta/2}\sin\frac{\alpha}{2}|-\mathbf{n}\rangle_1|-\mathbf{m}\rangle_2, \tag{4}$$

$\mathbf{n}$ and $\mathbf{m}$ being two points on the Poincaré sphere and the subscripts denote spin 1 and 2, respectively. With more than two particles Schmidt decompositions of this type generally do not exist [31]. In fact special conditions for the existence of Schmidt decompositions in tripartite [32] and multipartite [33] pure states have been found recently. The "angle" $\alpha$ in Eq. (4) determines the degree of entanglement in the state [34]: $\alpha=0$ or $\alpha=\pi$ correspond to product states and maximal entanglement is obtained for $\alpha=\pi/2$. In analogy with the Poincaré sphere interpretation of a single spin-$\frac{1}{2}$, Eq. (4) suggests that $\alpha$ and $\beta$ parametrize a two-sphere, which we shall call the "Schmidt sphere." A point $\mathbf{e}$ on the Schmidt sphere is given by $\mathbf{e}=(\sin\alpha\cos\beta,\sin\alpha\sin\beta,\cos\alpha)$, so that antipodal points correspond to orthogonal states, i.e., $\langle -\mathbf{e}|\mathbf{e}\rangle=0$. The projective Hilbert space for the spin-pair is the three-dimensional complex projective space $\mathrm{CP}^3$ [35]. Existence of Schmidt decompositions of the form Eq. (4) makes it possible to represent points in $\mathrm{CP}^3$ uniquely by the vectors $(\mathbf{n},\mathbf{m},\mathbf{e})$.

### B. Spin precession

Consider two noninteracting entangled spin-$\frac{1}{2}$ particles that undergo spin precession in an external time-independent uniform magnetic field in the $z$ direction. The Hamiltonian operator is

$$H=\omega_1 S_{1,z}+\omega_2 S_{2,z}, \tag{5}$$

where $\omega_1$ and $\omega_2$ are the Larmor frequencies, and $S_{1,z}$ and $S_{2,z}$ are the corresponding $z$ components of the spin operators associated with the two particles. Thus the time evolution operator takes a product form, which implies that the Schmidt parameters $\alpha$ and $\beta$ are constant and the state evolves according to

$$|\Psi(0)\rangle = e^{-i\beta/2}\cos\frac{\alpha}{2}|\mathbf{n}(0)\rangle_1|\mathbf{m}(0)\rangle_2$$
$$+ e^{i\beta/2}\sin\frac{\alpha}{2}|-\mathbf{n}(0)\rangle_1|-\mathbf{m}(0)\rangle_2$$
$$\to |\Psi(t)\rangle = e^{-i\beta/2}\cos\frac{\alpha}{2}|\mathbf{n}(t)\rangle_1|\mathbf{m}(t)\rangle_2$$
$$+ e^{i\beta/2}\sin\frac{\alpha}{2}|-\mathbf{n}(t)\rangle_1|-\mathbf{m}(t)\rangle_2 \tag{6}$$

with $|\pm\mathbf{n}(t)\rangle$ and $|\pm\mathbf{m}(t)\rangle$ given by Eq. (2). The geometric phase for this state reads

$$\Phi_G[\Gamma] = -\arctan\left(\frac{\cos\alpha[\cos\theta_1\tan(\omega_1\tau/2)+\cos\theta_2\tan(\omega_2\tau/2)]}{1-(\cos\theta_1\cos\theta_2+\sin\alpha\cos\beta\sin\theta_1\sin\theta_2)\tan(\omega_1\tau/2)\tan(\omega_2\tau/2)}\right) + \cos\alpha\left(\frac{\omega_1\tau}{2}\cos\theta_1+\frac{\omega_2\tau}{2}\cos\theta_2\right), \tag{7}$$

where $\Gamma$ is the path in the projective Hilbert space $\mathrm{CP}^3$. The quantities $(\theta_1,\omega_1\tau)$ and $(\theta_2,\omega_2\tau)$ are the spherical polar angles of $\mathbf{n}$ and $\mathbf{m}$, respectively. They parametrize the Poincaré sphere pertaining to the two spins. In the general case the geometric phase and the four spin paths $C_{+\mathbf{n}}$, $C_{-\mathbf{n}}$, $C_{+\mathbf{m}}$, and $C_{-\mathbf{m}}$ on the Poincaré sphere with the concomitant solid angles are unrelated. $\Phi_G[\Gamma]$ depends nontrivially on the constant relative phase $\beta$ in the Schmidt decomposition, i.e., the geometric phase may be different for distinct states that have the same degree of entanglement and follow the same one-particle paths $C_{\pm\mathbf{n}}$ and $C_{\pm\mathbf{m}}$. The dynamical phase for each spin is reduced by the entanglement and they add separately to $\Phi_G[\Gamma]$ as the spins do not interact. Thus the inseparability of $\Phi_G[\Gamma]$ in Eq. (7) can entirely be traced back to the inseparability of the total phase $\arg\langle\Psi(0)|\Psi(\tau)\rangle$.

For vanishing entanglement with the state at the north $(\alpha=0)$ or south $(\alpha=\pi)$ pole of the Schmidt sphere, we have

$$\begin{aligned}\Phi_G[\Gamma]&=\mp\arctan\left(\frac{\cos\theta_1\tan(\omega_1\tau/2)+\cos\theta_2\tan(\omega_2\tau/2)}{1-\cos\theta_1\cos\theta_2\tan(\omega_1\tau/2)\tan(\omega_2\tau/2)}\right)\\&\quad\pm\left(\frac{\omega_1\tau}{2}\cos\theta_1+\frac{\omega_2\tau}{2}\cos\theta_2\right)\\&=\Phi_G[C_{\pm\mathbf{n}}]+\Phi_G[C_{\pm\mathbf{m}}],\end{aligned}\tag{8}$$

where we have used the identity $\arctan[(x+y)/(1-xy)]=\arctan x+\arctan y$ and the fact that $-\Phi_G[C_{\mathbf{n}}]=\Phi_G[C_{-\mathbf{n}}]$. As expected the two-particle geometric phase for product states may be analyzed in terms of the geodesically closed solid angle for each spin. This is true for any multiparticle system: the geometric phase for any product state equals the sum of geometric phases acquired by each subsystem. Equation (8) implies that the geometric phase for two identical particles being prepared in the same spin polarization state, is doubled compared to that of the one-particle case, while for opposite polarization the geometric phase vanishes. These effects were predicted by Klyshko [25] and observed in a two-photon experiment by Brendel *et al.* [23] in the special case of cyclic evolution along geodesic segments on the Poincaré sphere.

In the case of maximal spin entanglement $\alpha=\pi/2$, corresponding to states lying on the equator of the Schmidt sphere, the dynamical phase vanishes and $\langle\Psi(0)|\Psi(\tau)\rangle=\cos(\omega_1\tau/2)\cos(\omega_2\tau/2)-(\cos\theta_1\cos\theta_2+\cos\beta\times\sin\theta_1\sin\theta_2)\sin(\omega_1\tau/2)\sin(\omega_2\tau/2)$ is real valued. It follows that

$$\Phi_G[\Gamma]=\begin{cases}0 & \text{if } \langle\Psi(0)|\Psi(\tau)\rangle>0,\\ \text{undefined} & \text{if } \langle\Psi(0)|\Psi(\tau)\rangle=0,\\ \pi & \text{if } \langle\Psi(0)|\Psi(\tau)\rangle<0.\end{cases}\tag{9}$$

This result is reminiscent of the sign-change property of the geometric phase for real-valued wave functions [8], such as in the molecular Aharonov-Bohm effect [37]. In Eq. (9) we see that the dependence of $\Phi_G[\Gamma]$ on the one-particle quantities $\theta_1,\theta_2,\omega_1\tau,\omega_2\tau$ is indirect as $\Phi_G[\Gamma]$ depends only on the sign of $\langle\Psi(0)|\Psi(\tau)\rangle$. Note that as the dynamical phase vanishes the geometric and total phases are identical. This makes it possible to test $\Phi_G[\Gamma]$ directly in the case of maximal spin entanglement by observing the total phase [28,36].

Suppose now that only one of the particles (1 say) is affected by the external magnetic field. We shall see that the second particle still may influence the two-particle geometric phase through entanglement. The state now evolves according to

$$\begin{aligned}|\Psi(0)\rangle&=e^{-i\beta/2}\cos\frac{\alpha}{2}|\mathbf{n}(0)\rangle_1|\mathbf{m}(0)\rangle_2\\&\quad+e^{i\beta/2}\sin\frac{\alpha}{2}|-\mathbf{n}(0)\rangle_1|-\mathbf{m}(0)\rangle_2\\&\to|\Psi(t)\rangle=e^{-i\beta/2}\cos\frac{\alpha}{2}|\mathbf{n}(t)\rangle_1|\mathbf{m}(0)\rangle_2\\&\quad+e^{i\beta/2}\sin\frac{\alpha}{2}|-\mathbf{n}(t)\rangle_1|-\mathbf{m}(0)\rangle_2.\end{aligned}\tag{10}$$

The geometric phase for this state can be found by inserting $\omega_2=0$ into Eq. (7) yielding

$$\begin{aligned}\Phi_G[\Gamma]&=-\arctan\left(\cos\alpha\cos\theta_1\tan\frac{\omega_1\tau}{2}\right)\\&\quad+\frac{\omega_1\tau}{2}\cos\alpha\cos\theta_1,\end{aligned}\tag{11}$$

which differs from the one-particle geometric phase $\Phi_G[C_{\mathbf{n}}]$ in Sec. II, except for vanishing entanglement. This effect is perhaps most striking when the spin basis of the affected spin in the Schmidt decomposition is parallel (antiparallel) to the magnetic field, i.e., $\theta_1=0$ $(\theta_1=\pi)$. In this case we obtain

$$\Phi_G[\Gamma]=\mp\arctan\left(\cos\alpha\tan\frac{\omega_1\tau}{2}\right)\pm\frac{\omega_1\tau}{2}\cos\alpha,\tag{12}$$

where the upper (lower) sign corresponds to $\theta_1=0$ $(\theta_1=\pi)$. By comparing this result with Eq. (3) we see that $\alpha$ formally plays the role of the constant polar angle of the affected spin.

The two-particle geometric phase in Eq. (11) may be interpreted in terms of the improper mixture $\frac{1}{2}(1+\cos\alpha\,\mathbf{n}\cdot\boldsymbol{\sigma}_1)$ obtained by tracing over the unaffected system. This interpretation makes $\Phi_G[\Gamma]$ equivalent (up to a sign) to the one-particle phase $\Phi_G[C_{\mathbf{n}};P]=-\arctan[P\cos\theta\tan(\omega\tau/2)]+(\omega\tau/2)P\cos\theta$ [28] with the degree of polarization $P$ and the degree of entanglement $|\cos\alpha|$ identified. However, it should be kept in mind that $\Phi_G[\Gamma]$ in Eq. (11) differs conceptually from the corresponding one-particle phase $\Phi_G[C_{\mathbf{n}};P]$ as the former cannot be tested on one of the particles alone; it pertains to the whole system and must therefore be observed in coincidence.

Kwiat and Chiao [21] and Grayson *et al.* [22] observed the geometric phase using two-photon coincidence technique by letting one of the photons make a geodesic cycle on the Poincaré sphere. The observed phase in these experiments corresponds to the cyclic case of Eq. (11) for disentangled polarizations $(\alpha=0)$.

The nonclassical dependence of the geometric phase on the exact fixed location on the Schmidt sphere could be checked nonlocally as the spins do not interact and entanglement may persist over large spatial distances. This property may be used to obtain an adiabatic geometric phase of the entangled spin pair. The idea is to use the fact that Schmidt decompositions of spin pairs single out two spin directions $\mathbf{n}$ and $\mathbf{m}$ that each can be transported locally by two spatially separated slowly varying magnetic fields: spin 1 is located in a spatial region with a magnetic field $\mathbf{B}_1=B_1\mathbf{n}$ and spin 2 in a spatial region where $\mathbf{B}_2=B_2\mathbf{m}$. By slowly changing the magnetic field pair, the spin bases $|\pm\mathbf{n}\rangle$ and $|\pm\mathbf{m}\rangle$ follow adiabatically. Keeping the angles of $\mathbf{B}_1$ and $\mathbf{B}_2$ with respect to the $z$ axis fixed, the general expression (7) still holds and the geometric phase becomes an entanglement dependent function of the path taken by the directions of the magnetic field pair $(\mathbf{B}_1,\mathbf{B}_2)$ that are localized at two spatially separated regions. In this way the geometric phase can be controlled as an arbitrary path of the spin pair could be generated.

We end this section by a brief outline of the generalization to spin precession in a time-independent uniform magnetic field for more than two noninteracting spin-$\frac{1}{2}$ particles in an entangled Schmidt state. With $N$ spins such a state may be written as

$$|\Psi\rangle=\cos\frac{\alpha}{2}e^{-i\beta/2}\prod_{j=1}^{N}|\mathbf{n}_j\rangle+\sin\frac{\alpha}{2}e^{i\beta/2}\prod_{j=1}^{N}|-\mathbf{n}_j\rangle. \tag{13}$$

The evolution of this state is described by the set of mappings $\{\mathbf{n}_j(0)\}\rightarrow\{\mathbf{n}_j(\tau)\}$ on the Poincaré sphere. Introducing the spin angles $\{\theta_j\}$ with respect to the magnetic field in the $z$ direction and the Larmor frequencies $\{\omega_j\}$, it is useful to introduce the entanglement independent quantities

$$\zeta=\prod_{j=1}^{N}\left(\cos\frac{\omega_j\tau}{2}-i\cos\theta_j\sin\frac{\omega_j\tau}{2}\right) \tag{14}$$

and

$$\gamma=\prod_{j=1}^{N}\sin\theta_j\sin\frac{\omega_j\tau}{2}. \tag{15}$$

The geometric phase for the evolving multiparticle state becomes

$$\Phi_G[\Gamma]=\arctan\left(\frac{\cos\alpha\,\mathrm{Im}\,\zeta}{\mathrm{Re}\,\zeta+(-1)^K\gamma\sin\alpha\cos\beta}\right)+\cos\alpha\sum_{j=1}^{2K}\frac{\omega_j\tau}{2}\cos\theta_j \tag{16}$$

for $N=2K$, $K$ being a positive integer, and

$$\Phi_G[\Gamma]=\arctan\left(\frac{\cos\alpha\,\mathrm{Im}\,\zeta}{\mathrm{Re}\,\zeta+(-1)^{K+1}\gamma\sin\alpha\sin\beta}\right)+\cos\alpha\sum_{j=1}^{2K+1}\frac{\omega_j\tau}{2}\cos\theta_j \tag{17}$$

for $N=2K+1$, $K$ again being a positive integer. Here $\Gamma$ is a path in the Schmidt subspace of the full projective Hilbert space $\mathrm{CP}^{2^N-1}$ of the $N$-particle system. Vanishing entanglement yields

$$\Phi_G[\Gamma]=\pm\arg\zeta\pm\sum_{j=1}^{N}\frac{\omega_j\tau}{2}\cos\theta_j=\sum_{j=1}^{N}\Phi_G[C_{\pm\mathbf{n}_j}] \tag{18}$$

that naturally generalizes Eq. (8). The geometric phase for maximal entanglement is given by Eq. (9), where now $\langle\Psi(0)|\Psi(\tau)\rangle=\mathrm{Re}\,\zeta+(-1)^K\gamma\cos\beta$ for even $N$ and $\langle\Psi(0)|\Psi(\tau)\rangle=\mathrm{Re}\,\zeta+(-1)^{K+1}\gamma\sin\beta$ for odd $N$. In the case where one of the particles ($N$ say) does not interact with the external magnetic field, it follows that $\gamma=0$ and the geometric phase reads

$$\Phi_G[\Gamma]=\arctan(\cos\alpha\arctan[\arg\zeta])+\cos\alpha\sum_{j=1}^{N-1}\frac{\omega_j\tau}{2}\cos\theta_j. \tag{19}$$

Thus the noninteracting particle influences the geometric phase through entanglement. In the case where the entanglement vanishes the geometric phase reduces to Eq. (18). This expresses the fact that states in which entanglement may persist for the remaining $N-1$ particles cannot be reached by Schmidt states of the form Eq. (13). States of the former type require a different treatment beyond that of the Schmidt analysis in this work.

### C. Spin-spin interaction

In this section we consider a closed (isolated) quantal system consisting of two spin-$\frac{1}{2}$ particles with a spin-spin interaction described by the Hamiltonian operator

$$H=(2\lambda/\hbar)\mathbf{S}_1\cdot\mathbf{S}_2, \tag{20}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin operators pertaining to the spin pair and $\lambda$ is the strength of the interaction. This model is of particular interest for the implementation of the geometric phase in quantum computation using nuclear magnetic resonance technique [38]. In a physically realistic situation $\lambda$ decreases with the spatial distance between the two particles, thus making the results of the analysis below only locally testable. The coupled spin dynamics is straightforwardly calculated by transforming to the singlet and triplet states $|S,M\rangle$. It can be seen that all nonstationary spin states are cyclic with cyclic time $\tau_c=\pi/\lambda$.

The degree of entanglement may change due to the interaction between the two spins; a product state may evolve into an entangled state and vice versa and it is impossible to make one of the subsystems evolve but not the other. For the same reason the relative phase in the Schmidt decomposition may change. To stress this mobility on the Schmidt sphere in the context of the geometric phase, it is convenient to consider a superposition of the type

$$|\Psi(0)\rangle=\cos(a/2)|1,0\rangle+\sin(a/2)|0,0\rangle. \tag{21}$$

The spin-spin interaction generates the time evolution $\cos(a/2)\to e^{-i\lambda t/2}\cos(a/2)$ and $\sin(a/2)\to e^{i3\lambda t/2}\sin(a/2)$, from which we obtain

$$|\Psi(t)\rangle=\frac{1}{\sqrt{2}}\Bigg[\left(e^{-i\lambda t}\cos\frac{a}{2}+e^{i\lambda t}\sin\frac{a}{2}\right)|+\rangle_1|-\rangle_2 + \left(e^{-i\lambda t}\cos\frac{a}{2}-e^{i\lambda t}\sin\frac{a}{2}\right)|-\rangle_1|+\rangle_2\Bigg] \quad (22)$$

up to an unimportant overall phase. The basis states $|\pm\rangle_{1,2}$ (expressed along $z$, say) in Eq. (22) are time independent and the geometric phase originates purely from the evolution on the Schmidt sphere. To see this explicitly we introduce the time-dependent Schmidt parameters $\alpha$ and $\beta$ by rewriting Eq. (22) on the symmetric form

$$|\Psi(t)\rangle=e^{-i\beta(t)/2}\cos\frac{\alpha(t)}{2}|+\rangle_1|-\rangle_2 + e^{i\beta(t)/2}\sin\frac{\alpha(t)}{2}|-\rangle_1|+\rangle_2 \quad (23)$$

again ignoring an overall phase. The Schmidt parameters are given by $\cos\alpha(t)=\sin a\cos 2\lambda t$ and $\tan\beta(t)=-\tan a\sin 2\lambda t$. The image curve $\Gamma$ in the projective Hilbert space is isomorphic to the curve $C_{\mathbf{e}}:t\in[0,\tau]\to\mathbf{e}(t)=[\sin\alpha(t)\cos\beta(t),\sin\alpha(t)\sin\beta(t),\cos\alpha(t)]$ on the Schmidt sphere. Inserting Eq. (23) into Eq. (1) we obtain the noncyclic two-particle geometric phase acquired during the evolution as

$$\Phi_G[C_{\mathbf{e}}]=-\arctan\left(\frac{\cos([\alpha(\tau)+\alpha(0)]/2)}{\cos([\alpha(\tau)-\alpha(0)]/2)}\tan\frac{\beta(\tau)}{2}\right)+\int_0^\tau dt\frac{\dot\beta(t)}{2}\cos\alpha(t) = -\frac{1}{2}\Omega[C_{\mathbf{e}}^{g-c}], \quad (24)$$

where $\alpha(0)=\pi/2-a$ and $\Omega[C_{\mathbf{e}}^{g-c}]$ is the solid angle enclosed by the curve $C_{\mathbf{e}}^{g-c}$ that consists of $C_{\mathbf{e}}$ and the shortest geodesic on the Schmidt sphere connecting the end points $\mathbf{e}(0)$ and $\mathbf{e}(\tau)\neq-\mathbf{e}(0)$. In the cyclic case the geometric phase reduces to

$$\Phi_G[C_{\mathbf{e}}]=-\frac{1}{2}\oint_{C_{\mathbf{e}}}d\beta(1-\cos\alpha)=-\frac{1}{2}\Omega[C_{\mathbf{e}}], \quad (25)$$

where now $\Omega[C_{\mathbf{e}}]$ is the solid angle enclosed by the loop $C_{\mathbf{e}}$ on the Schmidt sphere. The value of $\Phi_G[\Gamma]$ may be found by inserting $\alpha$ and $\beta$ into the expression for the path $C_{\mathbf{e}}$ yielding $C_{\mathbf{e}}:t\in[0,\tau]\to\mathbf{e}(t)=(\cos a,-\sin a\sin 2\lambda t,\sin a\cos 2\lambda t)$. The curve $C_{\mathbf{e}}$ is shown in Fig. 1. It follows immediately that the geometric phase is $-\arctan(\cos a\tan\lambda\tau)+\lambda\tau\cos a$, which in the cyclic case reduces to $-\pi(1-\cos a)$.

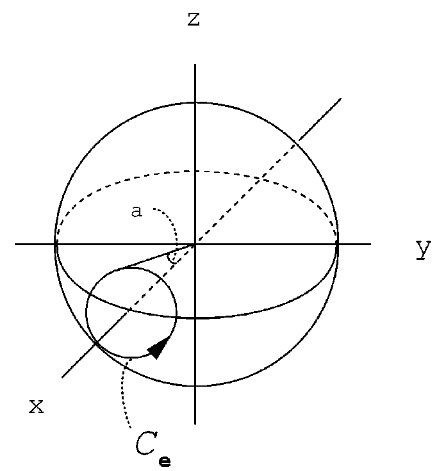


FIG. 1. Curve $C_{\mathbf{e}}$ on the Schmidt sphere. In the cyclic case this curve defines a cone with opening angle $a$ and enclosed solid angle $2\pi(1-\cos a)$.

By including the triplet states $|1,\pm1\rangle$, the one-particle bases in the Schmidt decomposition become time dependent. For such cases the state follows a path in $CP^3$ that can neither be projected solely onto the Poincaré nor solely onto the Schmidt sphere, which therefore both lose their role in the interpretation of the two-particle geometric phase.

The solid angle interpretation of the geometric phase as given by Eqs. (24) and (25) constitutes the main result of this section. It shows the significance of the Schmidt sphere and provides a clear demonstration of the importance of the evolving entanglement for the geometric phase in closed quantal systems with interacting parts.

## IV. CONCLUSIONS

The influence of entanglement on the noncyclic two-particle geometric phase has been studied for two different spin-$\frac{1}{2}$ models. The geometric phase for noninteracting spin-$\frac{1}{2}$ particles precessing in an external time-independent magnetic field and prepared in an entangled Schmidt state has been shown to exhibit a rich entanglement dependence. An experimental technique that could prepare arbitrarily entangled polarization states has been developed recently [39]. This opens up the possibility to test the full entanglement dependence of the two-particle phase in the laboratory using photons that undergo local SU(2) operations [27]. We have found a class of states where the geometric phase in a spin-spin interaction model can be interpreted solely in terms of the solid angle enclosed by the curve on the two-sphere parametrized by the evolving Schmidt coefficients. This suggests a geometric interpretation of Schmidt decompositions for spin-$\frac{1}{2}$ pairs analogous to that of the Poincaré sphere for a single spin $\frac{1}{2}$.

The present analysis could be extended to cases involving three or more SU(2) particles where Schmidt decompositions do not exist as well as to subsystems with higher-dimensional Hilbert spaces. For more than two particles general descriptions of entanglement, being applicable to any number of subsystems and not based on the Schmidt form, are available [40]. It would be interesting to apply these ideas in the calculation of the geometric phase. In the case of pairs of entangled higher-dimensional Hilbert spaces, it is clear that the Schmidt sphere has to be generalized, precisely as the Poincaré sphere has to be generalized for higher-dimensional single-particle Hilbert spaces [41–43].

## ACKNOWLEDGMENTS

I wish to thank Björn Hessmo for discussions on two-photon phases and Osvaldo Goscinski for comments on the manuscript. This work was supported by the European PECNO (''perfect crystal neutron optics'') TMR-network, Grant No. ERB-FMRX-CT96-0057, and the Swedish Natural Science Research Council (NFR).

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