Coherent quantum feedback

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In the conventional picture of quantum feedback control, sensors perform measurements on the system, a classical controller processes the results of the measurements, and actuators supply semiclassical potentials to alter the behavior of the quantum system. In this picture, the sensors tend to destroy coherence in the process of making measurements, and although the controller can use the actuators to act coherently on the quantum system, it is processing and feeding back classical information. This paper proposes an alternative method for quantum feedback control, in which the sensors, controller, and actuators are quantum systems that interact coherently with the system to be controlled. In this picture, the controller gets, processes, and feeds back *quantum* information. Controllers that operate using such quantum feedback loops can perform tasks such as entanglement transfer that are not possible using classical feedback. Necessary and sufficient conditions are presented for Hamiltonian quantum systems to be controllable and observable using both classical and quantum feedback.

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I. INTRODUCTION

Quantum control theory has a long history [1–12]. Experiments in elementary particles, atoms, solid-state systems, and optics involve the systematic measurement and manipulation of quantum systems. Quantum control theory has contributed significantly to the understanding of fundamental aspects of quantum mechanics, including the quantum Zeno effect [13,14], nondemolition measurements [15,16], and stochastic quantization [17]. Classical concepts of geometric control provide a basis for many quantum results [10–12]. For example, the field of nuclear magnetic resonance is largely concerned with the geometric control of collections of interacting nuclear spins [18–20]. Particularly significant are experimental applications of optimal control theory to quantum systems using NMR and optical techniques [6]. As quantum technologies have matured [5], a host of practical applications of quantum control have been realized in molecular dynamics [6], quantum optics [7,21–24], and quantum computation [21,22,25–28]. Efforts to protect quantum information from noise and decoherence [29–31] have led to proposals for quantum error correction [31] and entanglement purification [32], which can be thought of in terms of quantum feedback control. The idea of using quantum information in control situations has been proposed in the context of quantum ''smart matter'' [33] and in the ''all-optical'' feedback schemes in [7]. Recent experimental demonstrations of quantum teleportation [34] are examples of the application of feedback control to quantum communications.

The rapid development of quantum technologies together with the proliferation of results on quantum information and computation suggest that quantum control theory might profitably be reexamined from the perspective of quantum information. This paper presents results on the role of quantum information in quantum control. Control is largely about information [35]: for example, a feedback controller gets information about the system it is to control, processes that information, and feeds the information back into the system to change its behavior in a desired way. In the conventional picture of quantum feedback control, a feedback controller is a classical system that processes classical information obtained by making measurements on the quantum system to be controlled. In contrast, this paper proposes the idea of a quantum controller, a device that obtains *quantum* information by interacting with the quantum system to be controlled, processes that information using quantum logic, and feeds the information coherently back into the system. As will be shown, a controller that processes quantum information can perform tasks that a controller that processes classical information cannot. This paper analyzes the operation of such quantum controllers, proposes applications, and presents simple, experimentally accessible examples.

The analysis will proceed as follows: Experimental examples of various types of quantum control will be taken from quantum optics, atomic physics, and nuclear magnetic resonance. Then general theoretical results will be derived that apply to all quantum systems. Proofs of the theoretical results will be presented in the Appendix so as not to disrupt the exposition.

II. CONTROL THEORY

Control theory, quantum or classical, addresses a fundamental problem [36]: systems do not always behave the way one wants them to behave. Engines run too fast or too slowly; rooms are too hot or too cold; atoms can decay or nuclear spins dephase more rapidly than one desires. To improve a system's behavior, control theory adjoins to the system a second system, called a ''controller,'' which interacts with the original system in a way that improves its behavior. A governor can be added to an engine to regulate its speed; a thermostat can be added to a room to maintain a desired temperature; pulses of electromagnetic radiation can be applied to an atom or spin to decouple it from its surroundings and slow its decay or dephasing [37]. Together, the system

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and controller form a joint dynamical system. If the controller is well designed, this joint ''system-controller'' system behaves better (according to some appropriate metric) than the original system on its own.

Controllers are categorized according to the form of their interaction with the system to be controlled. If the interaction is one way, so that the controller acts on the system without obtaining any information about its state, then the controller is called ''open loop.'' In ''closed-loop'' control, by contrast, the controller acts on the basis of information that it obtains about the state of the system. A particularly important form of closed-loop control is feedback control, in which the controller obtains information about the system (i.e., the system acts on the controller via sensors), processes it, and feeds it back by acting on the system via actuators. Though more complicated than open-loop control, closed-loop control is typically more accurate as well: the acquisition of information about the system allows greater flexibility in control strategy.

A. Quantum control

Quantum control is the branch of control theory that applies to systems whose behavior is governed by the laws of quantum mechanics. Of course, quantum mechanics governs the behavior of all physical systems—cars and air conditioners as well as atoms and nuclear spins. However, quantum control is usually taken to apply specifically to systems such as atoms, spins, electrons, photons, Bose-Einstein condensates, etc., whose behavior does not admit an accurate classical description. Over the last few decades the rapid development of highly precise technologies for manipulating systems at the quantum scale has greatly expanded the repertoire of techniques available for quantum control. Stable, powerful lasers, and sophisticated cooling techniques, for example, have made the quantum regime much more accessible and controllable than before.

A particularly powerful method of quantum control is termed ''coherent control'' [6,7]. In coherent control, one manipulates the state of a quantum system by applying semiclassical potentials in a fashion that preserves quantum coherence. For example, to drive an atom coherently from its ground to its excited state, one shines on the atom a laser beam whose frequency is tuned to the energy difference between the states. The interaction between the beam and the induced dipole moment of the atom causes the atom to undergo Rabi oscillations, coherently driving the atom between its ground and excited states. Although the laser beam is itself a quantum system, composed of many photons in a coherent state, its effect on the atom (and the atom's effect on it) can be adequately modeled by treating the beam as an oscillating semiclassical potential [38,39]. This semiclassical model of the laser beam is accurate in regimes in which the effect of quantum fluctuations of the photons on the atom is small, and the effect of the atom on the quantum state of the photons (''back reaction'') is negligible [39].

Coherent methods can also be used in the context of feedback control. Here a measuring apparatus is used to extract classical information from the quantum system, and the potential applied to the system is a function of the information extracted. The measurement apparatus can be treated effectively classically, as in the Copenhagen interpretation of quantum mechanics [40]. That is, in coherent control and coherent control with feedback, the system is quantum mechanical while the controller is effectively classical.

There is no particular reason why the controller must be a fully classical system, however. Although it is often convenient to assume that the system to be controlled is quantum mechanical while the controller is classical, the division between ''quantum'' and ''classical'' need not take place at the system-controller boundary. In fact, recent advances in the theory and construction of quantum-information-processing devices such as quantum computers and quantum communication channels [21,22,25–28,31,32,34] suggest that certain control tasks can be accomplished only when the controller itself exhibits intrinsically quantum properties. This paper shows that there are indeed tasks that can only be accomplished by a controller capable of processing quantum information.

In particular, the accessibility of new technologies for quantum computation, quantum communications, and quantum-information processing allows the following innovation. In addition to using semiclassical potentials (quantum systems operating in an effectively classical regime) as control systems, use quantum systems themselves as part of the controller. For example, one can control the state of an atom using nonclassical (e.g., squeezed) light, in which both quantum fluctuations and the quantum back reaction are significant. Or one can construct a hybrid semiclassical/quantum controller by adjoining to the quantum system to be controlled a second quantum system, such as another atom, and then acting on the two quantum systems together using a semiclassical potential. As will be seen below, both of these methods are potentially more powerful than quantum control using semiclassical potentials alone. They are more powerful exactly because the incorporation of quantum systems in the controller allows the controller to exchange *quantum information* with the system to be controlled. We will call such devices quantum controllers. Quantum controllers are controllers of which some or all parts require an intrinsically quantum description.

B. Controllability and observability

We now present a more precise statement of the theoretical results to be derived in this paper. Two basic questions raised by control theory [36] are controllability—can a controller drive a system to a desired state?—and observability—can the controller's sensors completely determine the state of the system? This paper presents the following results in quantum-control theory.

First, necessary and sufficient conditions are given for a finite-dimensional, Hamiltonian quantum system to be controllable and observable by a controller that makes measurements on the system, thereby generating classical information, and that feeds that information back to the system by acting on it coherently. Since the controller acts coherently but processes classical information, this form of control will be called coherent control with classical feedback. Previous work [1–7] on the problem of coherent control with classical feedback has focused on controlling a quantum system to a desired pure state $|\psi_d\rangle$: this paper supplements that work by showing how decohering processes such as measurement can be used to control the system to a desired mixed state described by a density matrix ρ_d. In addition to functioning as a sensor, a measurement apparatus can be used as a stochastic actuator to decohere a quantum system and to alter the purity of the system's state.

Second, the paper proposes a fundamentally different method for quantum feedback control in which the feedback loop preserves quantum coherence. Here the controller gains quantum information, which carries quantum phases, processes it using quantum logic to preserve those phases, and feeds the quantum information coherently back to the system. Since the controller is processing and feeding back quantum information, this method will be called coherent control with quantum feedback. Such coherent quantum feedback should be distinguished from the methods of coherent quantum control with classical feedback described in the previous paragraph: the term ''coherent control'' refers to a broad variety of highly successful techniques for controlling quantum systems in a fashion that respects quantum coherence [6,7]. Both of the methods discussed here use coherent control. When ''conventional'' coherent control is used with feedback, however, up until now the feedback loop has been taken to be classical and incapable of preserving quantum coherence. As will be seen below, a coherent controller that uses quantum feedback can accomplish tasks—in particular, the generation, transformation, and transfer of entanglement—that cannot be accomplished by coherent control with classical feedback. This paper provides necessary and sufficient conditions for finite-dimensional Hamiltonian quantum systems to be observable and controllable by controllers that use coherent quantum feedback. Both of these theoretical results are presented in the context of simple experiments, readily realizable using quantum optics and nuclear magnetic resonance, that highlight the difference between coherent control with classical feedback and coherent control with quantum feedback.

The methods used to derive these results will be those of geometric control applied to quantum systems [1–12]: these group-theoretic methods allow the easy mathematical treatment of Hamiltonian systems. For the sake of mathematical simplicity, the systems here will be taken to be finite dimensional. As all quantum systems with finite energy confined to a finite region of space are effectively finite dimensional, this is not a great restriction. The methods developed here could also be extended (at the cost of significant increase in mathematical complexity) to infinite-dimensional systems as in Ref. [1]. A greater restriction in the use of geometric control methods is the application to closed Hamiltonian systems. In particular, the more general case of quantum control is that of an open (i.e., non-Hamiltonian) finite-dimensional system coupled to an effectively infinite-dimensional environment, as in the case of control of chemical systems in a thermal bath. The mathematical treatment of such systems requires a semigroup [41,42] approach to quantum control. Semigroup methods and open quantum systems will be treated in further work [43].

III. FEEDBACK CONTROL OF QUANTUM SYSTEMS

A. Coherent control with classical feedback

The conventional method for controlling a quantum system using feedback is to make a measurement on the system to determine its state and then to apply a semiclassical potential whose value is conditioned on the result of the measurement to guide the system coherently to a desired state. As such a controller processes and feeds back classical information, this method may be termed coherent control with classical feedback. In this method of control, although the system itself must be described by the laws of quantum mechanics, the controller can be described classically. As an example of coherent control with classical feedback, consider Monroe *et al.*'s control of the $^2S_{1/2}$ hyperfine states $|F=2,m_F=2\rangle \equiv |\downarrow\rangle$, $|F=1,m_F=1\rangle \equiv |\uparrow\rangle$, of a single $^9\text{Be}^+$ ion in an ion trap [22]. Suppose that the ion is originally in an unknown superposition $|\psi\rangle = \alpha|\downarrow\rangle + \beta|\uparrow\rangle$, and the goal of the classical feedback loop is to put the ion in the state $|\uparrow\rangle$. The control loop begins by measuring the state of the ion by driving the cycling $|\downarrow\rangle \rightarrow {}^2P_{3/2}|F=3,m_F=3\rangle$ transition with $\sigma+$-polarized light and detecting the resulting ion fluorescence: fluorescence indicates that the ion is in the state $|\downarrow\rangle$. If the ion is found to be in the state $|\downarrow\rangle$, the controller (a classical digital computer) instructs the actuators (lasers) to effect a π pulse by driving a Raman transition through the virtual $^2P_{1/2}$ level to flip the atom into the $|\uparrow\rangle$ state. The net effect of the feedback loop is to put the ion in the state $|\uparrow\rangle$. The feedback loop is classical in the sense that the measurement provides a classical bit of information, and a classical controller decides on the basis of that bit whether or not to supply a Raman pulse to drive a coherent quantum transition in the ion.

From the perspective of control theory, coherent quantum control with classical feedback, though effective, has several drawbacks. First of all, measuring a quantum system almost inevitably disturbs it: even a nondemolition measurement that leaves the system in the state in which it was measured still typically alters the state of the system prior to the measurement [15,16]. The ion of the previous example is originally in an unknown coherent superposition of the ground and excited states. After fluorescence determines whether the ion is in its ground state or excited state, the initial quantum coherence between those states is irrevocably lost. Secondly, coherent control with classical feedback is stochastic: as a result of the measurement the system jumps to one state or another probabilistically. Although the ability to apply coherent operations conditioned on the results of measurements allows the controller to compensate for the probabilistic nature of their results, the introduction of stochastic effects significantly complicates the control process. A thorough and revealing analysis of stochastic effects in coherent control with classical feedback in the context of cavity quantum electrodynamics can be found in Ref. [7].

While the conventional view of quantum feedback and feedforward control looks at classical controllers interacting with quantum systems through semiclassical sensors and actuators, there is no reason why sensors, controllers, and actuators should not themselves be quantum systems. A conventional digital or operational-amplifier controller does not preserve quantum coherence. In contrast, recent developments in quantum computing [21,22,25–27] suggest the possibility of controllers constructed of quantum logic devices that preserve quantum coherence throughout the feedback loop. As will be shown below, controllers that use a quantum feedback loop can perform a number of tasks that controllers that use a classical feedback loop cannot. For example, they can use coherent feedback to guide a quantum system from an unknown initial state to a desired final state without destroying the initial state. In addition, a controller can use a quantum feedback loop to drive a quantum system to a target state that is entangled with another quantum system. Entanglement is a nonlocal quantum phenomenon that cannot be created by controllers using classical feedback loops.

B. Coherent control with quantum feedback

The following examples show how coherent quantum feedback control can be realized using optical or nuclear magnetic resonance techniques, and serve to highlight the difference between coherent control with classical feedback and coherent control with quantum feedback. The examples are selected for their simplicity and experimental accessibility. In each example, the system to be controlled is a simple quantum system such as an ion or nuclear spins. The quantum controller consists of other simple quantum systems such as ions, phonons, and nuclear spins that can be made to interact with the system to be controlled and that can perform simple quantum-information processing via techniques developed for quantum computing. Note that the simplicity of the quantum controllers and their physical proximity to the system to be controlled does not disqualify them as control devices: after all, classical controllers such as a governer for an engine or an operational-amplifier feedback controller for an electric circuit are simple devices that are physically integrated with the systems that they are designed to control. Indeed, the realization that coherent quantum control was experimentally possible came from the realization that an effective classical controller could be constructed from a single operational amplifier. As will be seen, an effective quantum controller can be constructed from a single ion or single nuclear spin.

First, we discuss how an ion in a trap can be subjected to coherent quantum feedback. Then we turn to an example of quantum feedback using nuclear spins.

C. An ion-trap example

First, examine the problem of controlling the state of the ion in the ion trap using a quantum controller. (Recall that a quantum controller is a device that exchanges quantum information with the system to be controlled; consequently, at least part of a quantum controller requires a quantum description.) A simple method for creating a quantum controller is to add a second ion to the trap. The ions can be made to interact by their common interaction with their center-of-mass vibrational mode. As before, assume that the ion whose state we desire to control is in the unknown state $|\psi\rangle$, while the vibrational mode has been cooled to its ground state $|0\rangle_m$, and the second, ''controller'' ion has been prepared in the desired ''target'' state $|\phi\rangle_c=\gamma|\uparrow\rangle_c+\delta|\downarrow\rangle_c$. Just as in classical feedback control, by adjoining an additional system to act as the controller, we have now created a joint system-controller system, initially in the state

$$|\psi\rangle\otimes|0\rangle_m\otimes|\phi\rangle_c. \tag{1}$$

Now our job is to show that the joint system can possess superior properties to the system on its own.

The ion can now be controlled to its ground state using coherent quantum feedback as follows. First, focus light on the system ion and drive a spin-selected π pulse on the red sideband as in [26]: this pulse takes $|\uparrow\rangle\otimes|0\rangle_m$ to $-i|\downarrow\rangle\otimes|1\rangle_m$ and vice versa, while leaving $|\downarrow\rangle\otimes|0\rangle$ unchanged. The joint system-controller system is now in the state

$$|\downarrow\rangle\otimes(-i\alpha|1\rangle_m+\beta|0\rangle_m)\otimes|\phi\rangle_c=|\downarrow\rangle\otimes|\psi'\rangle_m\otimes|\phi\rangle_c. \tag{2}$$

We see that the quantum information contained in the original unknown state $|\psi\rangle$ of the system has been transferred coherently to the state of the vibrational mode (''coherent sensing''), albeit in a slightly altered form ($\alpha\rightarrow-i\alpha$).

Second, apply the same procedure to the control ion. It is straightforward to verify that the joint system-controller system is now in the state

$$\begin{aligned}|\downarrow\rangle\otimes(-i\gamma|1\rangle_m+\delta|0\rangle_m)\otimes(\alpha|1\rangle_c+\beta|0\rangle_c)\\=|\downarrow\rangle\otimes|\phi'\rangle_m\otimes|\psi\rangle_c.\end{aligned} \tag{3}$$

That is, this pulse exchanges the quantum information in the mode with the quantum information in the control ion, once again slightly altering it in the process. Note that the system ion is unaffected by this pulse: this step can be thought of as a form of coherent quantum-information processing within the quantum controller.

Third, repeat the first step. The resulting state is

$$|\phi\rangle\otimes|0\rangle_m\otimes|\psi\rangle_c. \tag{4}$$

This step coherently implants the target state into the system ion (''coherent actuation''), feeding back the quantum information processed in the second step.

The three steps, coherent sensing, coherent quantum-information processing within the quantum controller, and coherent quantum actuation, complete one cycle of a coherent quantum feedback loop that obtains quantum information about the system, processes it, and feeds it back. The net effect of this coherent quantum feedback loop is to exchange the initial unknown state of the system ion with the target state initially stored in the controller ion.

D. Discussion

There are a number of salient differences between coherent control with classical feedback and coherent control with quantum feedback. As noted above, coherent control with classical feedback is typically stochastic (an element of chance is introduced by quantum measurement) and destructive (the initial, unknown state of the system is irrevocably destroyed). Coherent control with quantum feedback, by contrast, is deterministic (each step in the quantum feedback loop above is completely reversible) and nondestructive (the initial unknown state of the system can be restored by repeating the feedback loop a second time). In addition, as will now be seen, coherent control with quantum feedback can be used to accomplish tasks that coherent control with classical feedback cannot.

E. A spin example

Let us now turn to a second example of coherent quantum feedback, this time using nuclear magnetic resonance. While the ion trap example above is technically feasible—ion-trap quantum computers loaded with several ions now exist—it is still a difficult experiment. In particular, the problem of focusing a laser on one ion but not the other is a hard one (using two species of ion would allow frequency addressing rather than spatial addressing). NMR, by contrast, has shown itself to be a flexible and experimentally accessible paradigm for quantum-information processing [44,45]. NMR quantum computations on three or more quantum bits involving tens or hundreds of steps are now commonplace.

Let us first rephrase our ion example above in the context of spins. Consider the problem of taking a quantum spin that is originally in the state $|\psi\rangle=\alpha|\uparrow\rangle+\beta|\downarrow\rangle$, where α and β are unknown, and putting it in the state $|\downarrow\rangle$. In coherent control with classical feedback, the controller begins by making a nondemolition measurement of the state of the spin (using, say, a Stern-Gerlach apparatus), giving $|\uparrow\rangle$ with probability $|\alpha|^2$ and $|\downarrow\rangle$ with probability $|\beta|^2$. The control algorithm is as follows: If the result of the measurement is $|\downarrow\rangle$, do nothing, while if the result of the measurement is $|\uparrow\rangle$, put the spin in a static magnetic field B and apply an electromagnetic pulse with frequency $\omega=2\mu B/\hbar$ to flip the spin (here μ is the spin's magnetic dipole moment). The spin is now in the state $|\downarrow\rangle$ as desired.

As before, coherent control with classical feedback requires measurement: a measurement apparatus is necessary to generate the classical information that the controller needs in order to perform feedback in the first place. But the fact that the feedback process is initiated by a measurement makes coherent control with classical feedback stochastic and irreversible: although the measurement reveals the state of the spin along some axis, it destroys the original coherent superposition.

To contrast coherent control by classical feedback with coherent control by quantum feedback, consider a quantum controller consisting of a second spin, initially in the state $|\downarrow\rangle_c$, that interacts with the first through the usual scalar interaction term $\gamma\sigma_z\sigma_z^c$ [19,20] so that the Hamiltonian for the two spins is $(\hbar/2)(\omega\sigma_z+\omega_c\sigma_z^c+\gamma\sigma_z\sigma_z^c)$, where $\omega_c=2\mu_c B/\hbar\neq\omega$ is the resonant frequency of the controller spin. Just as in the case of the ion-trap quantum feedback loop, in which the controller ion interacted with the system ion via their common center-of-mass mode to obtain, process, and feed back quantum information, here the controller spin will use its scalar interaction with the system spin to enact a quantum feedback loop.

The quantum feedback loop operates by enhancing the spin-spin interaction using conventional double-resonance techniques [19,20]. For example, applying a π pulse with frequency $\omega_c+\gamma$ coherently flips the controller spin if and only if the system spin is in the state $|\uparrow\rangle$ (in practice, instead of a single ''superselective'' pulse, a series of ''semiselective'' pulses are used to perform such a conditional spin-flipping operation). In the parlance of quantum computation this operation is called a controlled-NOT or CNOT. The two spins are now in the state $\alpha|\uparrow\rangle|\uparrow\rangle_c+\beta|\downarrow\rangle|\downarrow\rangle_c$. Clearly, the controller spin has become correlated with the system spin in the sense that measuring the state of the controller spin would reveal the state of the system spin. The controlled-NOT operation has caused the controller spin to obtain quantum information about the system spin.

In addition to inducing quantum correlation between the system and controller spins, the CNOT operation has disturbed the state of the system spin: initially in the pure state $|\psi\rangle$, the spin is now in the mixed state described by a density matrix $\rho'=\alpha\bar{\alpha}|\uparrow\rangle\langle\uparrow|+\beta\bar{\beta}|\downarrow\rangle\langle\downarrow|$. The controller spin is in an identical mixed state. No irreversible measurement has taken place, however. The disturbance can be removed and the correlation undone by applying a second pulse with the same frequency to flip the second spin back again, returning both spins to their initial states. With a quantum feedback loop, in contrast to coherent control with classical feedback, the disturbance introduced by the sensors is reversible and can be undone by the actuators.

The state $\alpha|\uparrow\rangle|\uparrow\rangle_c+\beta|\downarrow\rangle|\downarrow\rangle_c$ exhibits a peculiarly quantum form of correlation called entanglement. Entangled states are known to exhibit strange, apparently nonlocal quantum effects, the best known of which is the Einstein-Podolsky-Rosen (EPR) effect [46–48]. Creating and controlling entangled states is a crucial part of new quantum technologies such as quantum cryptography, quantum computation, and teleportation [8,9,25–28,32,33]. The interaction between the controller spin and the system spin has entangled system with controller. The key point here is that entanglement cannot be created without an exchange of quantum information. A classical controller cannot be entangled with the quantum system it is controlling. Quantum feedback loops typically create entanglement between system and controller at some stage in their operation.

A second coherent interaction between the two spins now controls the spin coherently to the state $|\downarrow\rangle$: simply apply to the system in state $\alpha|\uparrow\rangle|\uparrow\rangle_c+\beta|\downarrow\rangle|\downarrow\rangle_c$ a pulse with frequency $\omega+\gamma$ to flip the first spin if and only if the second spin is up. The state of the two spins is now $|\downarrow\rangle(\alpha|\uparrow\rangle_c+\beta|\downarrow\rangle_c)$. That is, not only has coherent quantum feedback put the first spin in the state $|\downarrow\rangle$, it has coherently put the second spin in the initial state of the first spin. No stochastic operation has taken place, and the initial state of the controlled spin has not been destroyed: rather, it has been coherently transferred to the state of the controller.

F. Comparison

In both the spin and the ion-trap cases, adjoining a second quantum system as part of the controller allows one to control the spin in ways that are not possible using a fully classical controller. In contrast to coherent control with classical feedback, coherent control with quantum feedback is neither stochastic nor destructive.

Although both the spin and ion-trap quantum feedback loops accomplish the same task, an exchange of the unknown system state with the known target state of the controller, they operate in slightly different ways. Most noticeably, in the ion-trap quantum feedback loop there is a clear directionality to the transmission of quantum information. The quantum information from the system ion is transferred first to the center-of-mass mode and then to the control ion; when the quantum information in the center-of-mass mode is transferred to the control ion, the information on the control ion is transferred to the center-of-mass mode and from there is transferred to the system ion. Here, information moves around the loop in one direction.

In the spin example, by contrast, although at first glance the first controlled-NOT operation looks like a classic ''sensing'' operation (the sensor changes in response to the state of the system), closer inspection reveals that it actually induces a two-way flow of quantum information, resulting in a symmetric, entangled state for the two spins. Similarly, the second controlled-NOT operation looks at first glance like a classic ''actuation'' operation (act on the system conditioned on the state of the controller), it also involves a two-way flow of quantum information that disentangles the state of the two spins and exchanges the initial controller state with the initial system state. This effect highlights another feature of quantum feedback loops: where in quantum control with classical feedback sensing and actuation are two distinct steps, in quantum control with quantum feedback sensing and actuation are often indistinguishable. A quantum sensor *is* a quantum actuator and vice versa. Only in certain well-defined situations, as in the ion-trap quantum feedback loop, is it possible to identify a unidirectional flow of quantum information around the loop. In a typical quantum feedback loop, quantum information flows both ways.

G. Entanglement transfer

The previous examples of quantum feedback loops were designed to show simply how quantum feedback differs from classical feedback, and how the ability of a controller to exchange quantum information with the controlled system allows it to perform feedback control of quantum systems in a way that is neither stochastic nor destructive. Quantum feedback loops can accomplish other tasks that are not possible classically. Before going on to the theoretical description of quantum feedback, let us look briefly at one such task, entanglement transfer.

For the sake of compactness, we describe entanglement transfer only in the case of spin systems. An ion-trap version of entanglement transfer could easily be accomplished by adding a third ion to the trap. Here, the goal of the control process is to put the system spin in an entangled state $(1/\sqrt{2})(|\uparrow\rangle|\uparrow\rangle_a+|\downarrow\rangle|\downarrow\rangle_a)$ where $|\uparrow\rangle_a$ and $|\downarrow\rangle_a$ are states of a third spin (the ''ancilla''). As noted above, such states can readily be produced by making the system spin interact directly with the ancilla spin. Suppose, however, that we are not allowed to make the two spins interact directly. It is a well-known fact that if two quantum systems are not entangled initially, they cannot become entangled through the exchange of classical information alone [40]. That is, no classical feedback loop that exchanges information between the system and ancilla can entangle them.

By contrast, because of its ability to transfer quantum information, a quantum feedback loop that mediates between the two spins can readily induce entanglement between them. To accomplish the entanglement transfer, prepare the ancilla spin in the state $(1/\sqrt{2})(|0\rangle_a+|1\rangle_a)$, then entangle the ancilla and controller spins by performing a controlled-NOT on the controller spin with the ancilla spin as control. The three spins are now in the state

$$(\alpha|\uparrow\rangle+\beta|\downarrow\rangle)(1/\sqrt{2})(|\uparrow\rangle_c|\uparrow\rangle_a+|\downarrow\rangle_c|\downarrow\rangle_a). \qquad (5)$$

Now perform the quantum feedback procedure given above, supplementing it by applying a third π pulse with frequency $\omega_c+\gamma$ to flip the controller spin if and only if the system spin is in the state $|\uparrow\rangle$. It is easily verified that the final state is

$$(1/\sqrt{2})(|\uparrow\rangle|\uparrow\rangle''+|\downarrow\rangle|\downarrow\rangle'')(\alpha|\uparrow\rangle'+\beta|\downarrow\rangle'). \qquad (6)$$

That is, the quantum feedback loop accomplishes the goal of producing the desired entanglement between the system spin and the ancilla despite the fact that the system and ancilla spins never interact directly. By contrast, as noted above, a coherent controller that operates by classical feedback cannot drive the system to such an entangled target state without acting on the third spin directly.

IV. THEORETICAL CHARACTERIZATION OF QUANTUM FEEDBACK

Control of quantum systems can be accomplished by either classical or quantum controllers. A classical controller is one whose operation can be described classically: it obtains classical information about a quantum system by measurement, processes that information using a classical technique (e.g., classical digital or analog computation), and feeds the processed information back to the quantum system via semiclassical potentials. In contrast, a quantum controller is one whose operation cannot be described classically: at least part of its functioning involves obtaining, processing, and feeding back quantum information. As demonstrated in the experimental examples above, quantum controllers can accomplish tasks such as entanglement transfer that classical controllers cannot.

We now turn to the theoretical characterization of quantum feedback. As noted above, the central questions that control theory asks are whether a system is controllable by a particular control method—can it be driven to a desired state?—and whether it is observable—can the method determine the underlying state of the system? In what follows, we derive necessary and sufficient conditions for quantum systems to be controllable and observable by a variety of methods: open-loop coherent control, closed-loop coherent control with classical feedback, and closed-loop coherent control with quantum feedback. The first two of these methods have been well studied in the quantum-control-theory literature. We present results on them merely for the sake of completeness and to lay down a mathematical framework for the derivation of our results on coherent quantum feedback.

A. Open-loop coherent control

First, we review well-known results in open-loop coherent control. Open-loop controllers act without obtaining knowledge about the underlying state of the system. More precisely, the controller is provided with some information about the system's initial state, but obtains no further information during the control process. In the quantum case, an open-loop coherent controller acts by applying time-dependent potentials $\Sigma_i \gamma_i(t) H_i$ to the system. Controllability is the problem of taking a quantum system from some initial state to a desired final state. A quantum system is open-loop controllable if the potentials can be modulated by varying $\gamma_i(t)$ so as to take the system from an arbitrary known initial state $|\psi\rangle$ to a desired final state $|\psi_d\rangle$. This form of controllability is called open loop because the initial state of the system is assumed to be known, and no measurement is made on the system. The problem of coherent open-loop controllability of finite-dimensional Hamiltonian quantum systems has long been known to possess an elegant geometric solution [1,2,6,10–12].

Result 1. Coherent controllability: open-loop case. A quantum system with Hamiltonian H is open-loop controllable by a coherent controller if and only if the algebra $\mathcal{A}$ generated from $\{H, H_i\}$ by commutation is the full algebra of Hermitian operators for the system.

The spin in the example above is open-loop controllable by a coherent controller since NMR methods allow it to be taken from any given state to any desired state: the algebra generated by the Hamiltonian corresponding to the static field $B\sigma_z$ and the applied Hamiltonian, $B_x \sigma_x \sin \omega t$ can easily be seen to generate the full algebra of SU(2) by commutation. Result 1 is a quantum analog of the geometric theory of classical nonholonomic control [10–12]. A familiar example of a classical nonholonomic control problem is parallel parking: a car cannot be driven sideways directly, but can still be parked by edging first in one direction and then in another. In the quantum case, the algebra $\mathcal{A}$ determines what set of states can be reached by edging the quantum system first in one direction, then in another, a method that can be called ''parking Schrödinger's car.''

B. Closed-loop quantum control: The role of measurement

Now let us turn to closed-loop quantum control. Central to any discussion of closed-loop quantum control with classical feedback (''traditional'' quantum feedback control) is the role of measurement. As is well known, measurement plays an important and often problematic role in quantum mechanics [40]. Fundamental difficulties arise in attempting to describe how quantum-mechanical systems interact with systems that behave in classical ways (the ''measurement problem''). As noted above, quantum measurements are stochastic and destructive, while the underlying dynamics of quantum mechanics is deterministic and reversible. As a result, the treatment of measurement in quantum control is often the most technically difficult part of the control process [7]. In particular, even in the case of quantum control with classical feedback, measurement is not only a sensing process, but a stochastic actuation process as well.

The stochastic nature of measurement in quantum mechanics is useful, as well as problematic. For example, note that the definition of controllability given above for open-loop coherent control is specific to pure initial and final states. This is because open-loop coherent control takes pure states to pure states. More generally, if the system's initial state is described by a known density matrix ρ, then Hamiltonian time evolution of the sort described preserves the eigenvalues of ρ. If a system is open-loop controllable as described above, then a known initial ρ can be taken to any ρ_d with the same eigenvalues.

To extend this controllability result to unknown initial states ρ and to arbitrary final states ρ_d, either we must use open-system techniques such as thermal relaxation, or we must introduce closed-loop control. Control of open, non-Hamiltonian quantum systems will be discussed in further work [43]. Here we examine feedback control of quantum systems. Suppose that the controller can make measurements on S [for the sake of simplicity, assume that these measurements are projective von Neumann measurements; the more general case of positive operator valued measures [40] will be considered elsewhere] corresponding to a finite set of Hermitian observables $\{\mathcal{M}_j\}$ and then apply potentials $\Sigma_i \gamma_i(m_j, t) H_i$ that depend on the results m_j of the measurements. Note that, unlike the classical case in which measurements can be assumed to be noninvasive in principle, a quantum measurement typically has a stochastic, coherence-destroying effect on the system measured. A measuring apparatus for a quantum system is not only a sensor, but a stochastic actuator as well. A quantum system S is closed-loop controllable if and only if a closed-loop controller can take S from an arbitrary unknown initial state ρ to any desired final state ρ_d. We then have the following result.

Result 2. Coherent controllability: closed-loop case. A quantum system with Hamiltonian H is closed-loop controllable to an arbitrary mixed state ρ_d by a coherent controller with classical feedback if and only if (i) at least one of the $\mathcal{M}_j \neq I$ (that is, the controller can make some nontrivial measurement on the system) and (ii) the algebra generated by $\{H, H_i\}$ is the full algebra of Hermitian operators for the system.

For example, the spin above is clearly closed-loop controllable by classical feedback using the techniques described. The proof of this result is given in the Appendix. The ''if'' part follows because even when one can make a nondemolition measurement of only a single bit of information, the open-loop controllability of the system allows that bit to correspond to projections onto arbitrary subspaces; repeated measurements then allow the value of any operator to be determined and the system to be guided to a desired pure state. To construct a desired mixture ρ_d, the sensors can now be used as stochastic actuators to destroy the system's coherence in a controlled fashion. The ''only if'' part follows because, if the system is not open-loop controllable, then the set of states that can be reached conditioned on the results of measurements is of lower dimension than the Hilbert space of the system.

C. Quantum observability

The close relationship between open- and closed-loop controllability for quantum systems has implications for the related notion of observability. The classical definition of observability must be somewhat altered for quantum systems since the irreversible disturbance introduced by measurement implies that no procedure can reveal an arbitrary unknown initial state of a quantum system. Accordingly, a quantum system will be called observable by a coherent controller by classical feedback if the proper sequence of controls and measurements can be used to observe any desired feature of the initial state of the system. Specifically, the system is observable if the controller can make a measurement that reveals the projection of the original state along any desired set of orthogonal axes in Hilbert space. Result 2 immediately implies the following.

Result 3. Observability by classical feedback. A Hamiltonian quantum system is observable by a coherent controller with classical feedback if and only if it is closed-loop controllable (proof in the Appendix.)

In the example above, NMR techniques, together with the ability to measure the component of spin along the z axis, clearly allow one to measure the spin along any axis. In addition, if one can manipulate the spin so as to measure it along any axis, then one can also manipulate it sufficiently to control its state to any desired state, conditioned on the result of the measurement.

D. Coherent control with quantum feedback

Now turn to coherent control with quantum feedback. Here our controller possesses a quantum subsystem that can be made to interact with the quantum system to be controlled. This interaction allows the system to exchange quantum information with the controller. The exchange of quantum information is not possible when the controller is classical. In addition, we may be able to apply coherent control as above, applying quantum potentials to the system and the quantum subsystem of the controller together.

A quantum system will be said to be controllable by fully quantum feedback if there is some initial state for the controller (possibly entangled with the state of another quantum system), a sequence of interactions with the controller and a sequence of applied semiclassical potentials that takes the system from some initial state ρ to a desired final state ρ_d which can also be entangled with another quantum system.

More precisely, to allow the exchange of quantum information between system and controller—quantum feedback—some of the applied potentials that make the system interact with a quantum controller are coherent interactions of the form $\Sigma_i \gamma_i(t) H^i_{\rm SC}$ where $H^i_{\rm SC}$ are Hermitian operators that couple the system to the controller and $\gamma_i(t)$ is a coupling constant that can be turned on and off to make the system and controller interact. (Alternatively, the $H^i_{\rm SC}$ can be ''on'' all the time, and suitable ''bang-bang'' controls applied to the system and controller to effectively turn the couplings on and off [37]; the mathematical exposition is similar for both cases and so only the time-dependent interactions will be treated here.) For an interaction to allow the exchange of quantum information $H^i_{\rm SC}$ cannot equal either $H^i_{\rm S}\otimes I_{\rm C}$ or $I_{\rm S}\otimes H^i_{\rm C}$, where I is the identity operator: otherwise the interaction reduces to coherent control by the application of semiclassical potentials as above. As noted above, for a quantum controller, there is no fundamental distinction between sensors and actuators: an interaction that can function as an actuator can also function as a sensor, and vice versa. (Of course, some interactions are more useful for sensing functions and some are more useful for actuation.)

Assume that the quantum part of the controller has a Hilbert space of large dimension, and that it itself is controllable by coherent open-loop control as in Sec. IV A above. Let $\{O_i = {\rm tr}_{\rm C} H^i_{\rm SC}\rho_{\rm C}\}$ be the set of Hermitian operators that can act on the system given different states $\rho_{\rm C}$ for the controller. We then have the following result (proof in the Appendix).

Result 4(a). Quantum controllability (a). A quantum system with Hamiltonian H is controllable by fully quantum feedback if and only if the algebra $\mathcal{A}$ generated from $\{H, O_i\}$ by commutation is the full algebra of Hermitian operators for the system.

More generally, we have the following.

Result 4(b). Quantum controllability (b). A quantum system is controllable by fully quantum feedback if and only if the system together with the quantum part of the controller are controllable by coherent control.

Results 4(a,b) follow directly from the theory of open quantum systems taken together with the control concepts introduced above [27,40–42] [note that we are assuming that the quantum part of the controller has a dimension large (at least N^2) compared with the dimension (N) of the system's Hilbert space and that the controller is coherently controllable on its own]. Results 4(a) and 4(b) for coherent control using a quantum feedback loop correspond to results 1 and 2 for coherent control using a classical feedback loop. The equivalence between quantum sensors and quantum actuators implies that when a quantum controller acts on a quantum system it almost invariably gets information about the system, and vice versa. As an example of results (4), the two-spin quantum controller in the example above is clearly capable of controlling the other spin to any desired state, entangled or not.

Just as in the case of coherent control with classical feedback, care must be taken in defining observability for coherent control with quantum feedback: the controller is not a classical device that makes measurements on the system, but a quantum system in its own right that becomes correlated with the system. No irreversible measurement ever takes place. A quantum system will be said to be observable by a quantum controller if the initial state of the system, together with all its entanglements with any other quantum systems, can be transferred to an analogous state of the controller. The controller can then use this transferred state as the target state to which to control some other quantum system. This fundamentally quantum definition of observability is the natural converse to the quantum definition of controllability in result 4. Given results 1–4, the following result should come as no surprise.

Result 5. Quantum observability. A Hamiltonian quantum system is observable by a quantum controller if and only if it is controllable by the controller.

Proofs of results 4 and 5 are given in the Appendix. As the example of the three spins shows, an interaction with a quantum controller that puts a Hamiltonian system in a desired state necessarily transfers the initial state or the system, together with its entanglements, to an analogous final state of the controller. As noted above, control of a quantum Hamiltonian system using a classical feedback loop cannot in general determine the initial state of the quantum system. Since the controller is classical, the quantum state of the system certainly cannot be transferred to an analogous classical state of the controller. (It is interesting to note, however, that when a classical Hamiltonian controller controls a *classical* Hamiltonian system using a classical feedback loop, the original state of the system is necessarily transferred to the controller.)

V. CONCLUSION

This paper explored the properties of coherent control using both classical and quantum feedback, and gave necessary and sufficient conditions for controllability and observability of Hamiltonian quantum systems in a variety of settings. Conventional coherent control of quantum systems by classical feedback involves the acquisition and processing of classical information. A quantum feedback controller, by contrast, acquires and processes quantum information. Quantum information, measured in quantum bits or ‘‘qubits,’’ carries quantum phase information as well as classical information. A controller that feeds back quantum information can perform tasks, such as entanglement transfer, that controllers that feed back classical information cannot perform. The potential experimental realizations of quantum controllers discussed here were based on nuclear magnetic resonance; this paper's results could also be realized using quantum logic devices such as ion traps [22], high-Q cavities in quantum optics [7,21,23], and quantum dots [49]. The ‘‘all-optical’’ control proposed in Ref. [7] is a specific example of coherent quantum feedback control in a quantum optical setting.

Although the difficulty of constructing quantum controllers is likely to limit their application initially, such controllers could play a key role in the development of quantum technologies such as quantum computation and quantum communications. The work of Ramakrishna and Rabitz [6] has pointed out the close relationship between the open-loop geometric quantum control methods described above and the construction of quantum logic gates detailed in Refs. [50], [51]. Indeed, the recently reported experimental results in quantum teleportation [34] represent applications of quantum feedback control (in these experiments, entangled states are combined with a classical feedback loop to transfer quantum information). Quantum controllers could have application to a variety of problems, including problems with classical analogs such as trajectory control, and problems with no classical analog such as preventing decoherence. As the theory of quantum error correction shows [31], strategies for disturbance rejection are harder to devise for quantum systems than for classical. However, in the same way that polarized light allows one to observe effects that are not accessible with unpolarized light, entangled states generated and manipulated by quantum controllers might be used to allow more efficient observation and control of a variety of systems. A particularly important open question is the extent to which the controllability and observability results reported here for Hamiltonian quantum systems can be extended to open quantum systems.

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APPENDIX: PROOFS OF RESULTS 1–5

1. Coherent control: Open-loop case

A quantum system with Hamiltonian H is open-loop controllable by a coherent controller if and only if the algebra $\mathcal{A}$ generated from $\{H,H_i\}$ by commutation is the full algebra of Hermitian operators for the system.

This quantum-control result follows from well-known results in classical geometric control theory [1–4,10–12]: a simple demonstration is given by Ramakrishna *et al.* [6]. (Note that the many-particle version of this result is a fundamental result in quantum computation [50,51]: since almost any H_i together with H generates the full algebra of Hermitian operators, almost any quantum logic gate is universal.) The proof is straightforward. The system is open-loop controllable if and only if one can generate any unitary operator $U\in U(N)$ where N is the dimension of the Hilbert space $\mathcal{H}$ of the system. By assumption, one can apply a time-dependent Hamiltonian of the form $H(t)=H+\Sigma_i\gamma_i(t)H_i$, where the $\gamma_i(t)$ can be picked by the controller. That is, one can construct any unitary time evolution of the form $U=\mathcal{T}\exp[-i\int_0^t H(t')dt']$ where $\mathcal{T}$ is the time-ordering operator. Expanding the exponential in a power series yields the usual expression

$$U=1-i\int_0^t dt_1 H(t_1)-\int_0^t dt_1\int_0^{t_1} dt_2\, H(t_1)H(t_2)+\cdots. \tag{A1}$$

Substituting the explicit expression for $H(t)$ into Eq. (A1) and then normal-ordering terms in the order $H,H_1,H_2,\ldots$

shows that one has sufficient leeway in the choice of the moments $\langle \gamma_1^{m_1}\gamma_2^{m_2}\cdots\rangle$ to construct any U of the form e^{-iHt}, where $H\in\mathcal{A}$.

An interesting related question is that of trajectory tracking: can one make the open-loop system follow any desired trajectory in Hilbert space? The answer to this question can be given by creating a quantum version of Sussman's theorem [11,12] for the small-time local controllability of classical nonholonomic systems. A sufficient condition for small-time local controllability of a quantum system is that one be able to apply not merely H_i, but $-H_i$ as well: in addition, one must be able to cancel out all ''bad brackets,'' expressions of commutators of the H_i in which each H_i appears with even multiplicity (i.e., 0, 2, 4, etc., times). That is, for the system to be small-time locally controllable, the bad brackets must be linear combinations of good brackets of lower order. If in addition the system has drift from its natural Hamiltonian H, one must be able to cancel out H by applying the H_i as well.

2. Coherent controllability: Closed-loop case

A quantum system with Hamiltonian H is closed-loop controllable by a coherent controller using a classical feedback loop if and only if (i) at least one of the $\mathcal{M}_j\neq I$—that is, the controller can make some nontrivial measurement on the system—and (ii) the algebra generated by $\{H,H_i\}$ is the full algebra of Hermitian operators for the system.

As noted in the paper, the ''if'' part of this result holds because the open-loop controllability of the system allows one to make any nondemolition measurement, even of just a single bit of information, function as a nondemolition measurement that discriminates between members of an arbitrary basis $\{|e_j\rangle\}$ of pure states for the Hilbert space of the system. To see that such a nondemolition measurement can be performed, let $\{|e_j\rangle\}$ be a basis with respect to which one of the measurement operators M is diagonal: $M=\Sigma_k m_k P_k$, where P_k is a projection operator onto the kth eigenspace $\mathcal{H}_k$ of M, $P_k=\Sigma_{|e_k\rangle\in\mathcal{H}_k}|e_k\rangle\langle e_k|$. If each of the $\mathcal{H}_k$ is one dimensional, then M already discriminates between the $\{|e_j\rangle\}$ perfectly, and one can implement the control strategy described in the next paragraph. If some of the $\mathcal{H}_k$ are multidimensional, M can still be made to discriminate between the $\{|e_j\rangle\}$ perfectly by measuring M, using the open-loop controllability of the system to apply U_+ where $U_+|e_j\rangle=|e_{j+1}\rangle$ if $j\neq N=\dim\mathcal{H}$, and $U_+|e_N\rangle=|e_1\rangle$, and then measuring M again. Since U_+ cyclically permutes the $|e_j\rangle$, after at most N measurements, the sequence of results $m_{k_1}m_{k_2}\cdots m_{k_N}$ obtained completely determines which $|e_j\rangle$ the system is in, and the net result is to perform a nondemolition measurement that discriminates between the $\{|e_j\rangle\}$. To make a nondemolition measurement corresponding to another basis $\{|e_j'\rangle\}$, simply use the open-loop controllability of the system to apply a $U_{e'e}$ that maps $|e_j'\rangle\rightarrow U_{e'e}|e_j'\rangle=|e_j\rangle$, then make a nondemolition measurement to discriminate between the $\{|e_j\rangle\}$ as above, and then apply $U_{e'e}^{-1}$ to map $|e_j\rangle$ back to $|e_j'\rangle$.

As long as one can perform such a nondemolition measurement, a simple control strategy suffices to put the system in a desired pure state. First, make such a measurement, get the result $|e_j\rangle$ for some j; second, use the open-loop controllability of the system [guaranteed by condition (ii)] to construct a U that takes $|e_j\rangle\rightarrow U|e_j\rangle=|\psi_d\rangle$. If the desired final state is a mixed state ρ_d, first write ρ_d in diagonal form: $\rho_d=\Sigma_j p_j|\chi_j\rangle\langle\chi_j|$. Then prepare the system in a pure state $|\psi\rangle$ such that $|\langle\psi|\chi_j\rangle|^2=p_j$. Finally, make a nondemolition measurement corresponding to $\{|\chi_j\rangle\}$. The result is the state ρ_d. This proves the ''if'' part of result 2.

To prove the ''only if'' part of result 2, we show that if either condition (i) or condition (ii) is false, then the system is not controllable. If (i) is false, then the system is trivially uncontrollable, as the controller can make no measurement on the system at all. Suppose then that (i) is true but (ii) is false, i.e., the algebra $\mathcal{A}$ generated from $\{H,H_i\}$ by commutation is not the full algebra of $N\times N$ Hermitian matrices for the Lie group $U(N)$, but rather some subalgebra corresponding to a subgroup $\tilde{U}\subset U(N)$. The controller then can drive the system to any pure state of the form $V|m_k^j\rangle$, where $V\in\tilde{U}$ and $|m_k^j\rangle$ is the kth eigenstate of M_j, and to no other states. The system is controllable if the set of states $\{V|m_k^j\rangle\}$ is in fact the set of all pure states for the system. However, as we now show, the system can only be driven to a manifold of states of dimension strictly less than the dimension of the manifold of all states. Accordingly, if (i) is true but (ii) is false, the system is not controllable by a coherent controller using classical feedback.

In particular, if (ii) is false, the set of reachable states cannot include all pure states. The set of all normalized pure states constitutes a $(2N-1)$-dimensional manifold over the real numbers (i.e., the surface of an N-dimensional sphere of radius 1 over the complex numbers). But if $\tilde{U}$ is a strict subset of $U(N)$, then the set of points generated by $V|e_j\rangle$, $V\in\tilde{U}$ and $|e_j\rangle$ a member of an orthonormal basis as above, is a set with dimension strictly lower than $2N-1$; otherwise, the algebra of $\tilde{U}$ would contain operators of the form $|e_j\rangle\langle e_k|+|e_k\rangle\langle e_j|$, $i|e_j\rangle\langle e_k|-i|e_k\rangle\langle e_j|$ for arbitrary k. But the algebra generated by these operators by commutation is the full algebra of $U(N)$, in contradiction to the assumption that $\tilde{U}$ was a strict subgroup of $U(N)$. The set of states of the form $V|m_k^j\rangle$ for a finite number of m_k^j therefore constitutes a manifold of states with strictly lower dimension than the manifold of all pure states, and the system is not controllable. This proves the ''only if'' part of result 2.

3. Observability by a coherent controller using classical feedback

A Hamiltonian quantum system is observable by a coherent controller using classical feedback if and only if it is closed-loop controllable.

The ''if'' part of result 3 follows immediately from the proof of the ''if'' part of result 2: part of the proof of result 2 showed how conrollability could be used to construct a nondemolition measurement that discriminates between the members of any desired basis. The ''only if'' part is proved as follows. From the proof of result 2, it is clear that, if the system is not closed-loop controllable, the set of measurements that can be made consists of measurements that can be built up of repeated nondemolition measurements corresponding to operators of the form $VM_jV^\dagger$ for some $V\in\tilde{U}$ and for some M_j. However, the proof of the ''only if'' part of result 2 above shows immediately that such measurements can discriminate only between sets of orthogonal states that can be mapped to the eigenstates of the M_j by some $V\in\tilde{U}$. But by the same argument as in result 2, the set of states that can be so mapped to any given $|m_k^j\rangle$ constitutes a manifold of states with strictly lower dimension than the manifold of all pure states. As a result, almost all states (a set of measure 1) are members of bases between whose members the controller cannot distinguish. So if the system is not closed-loop controllable, then it is not observable.

4. Quantum controllability

A quantum system with Hamiltonian H is controllable by a coherent controller by fully quantum feedback if and only if the algebra $\mathcal{A}$ generated from $\{H,O_i\}$ by commutation is the full algebra of Hermitian operators for the system.

Result 4 can be proved by applying result 1 for the open-loop controllability of Hamiltonian systems to the joint system consisting of system and controller taken together. We simply construct the algebra of available operations and see what transformations it allows the controller to perform.

In the absence of the applied interactions $\Sigma_i\gamma_i(t)H^i_{\rm SC}$ the system and controller evolve according to a Hamiltonian $H\otimes I'\oplus I\otimes H'$, where H' is the Hamiltonian for the controller and by assumption can be chosen at will. By result 1, the set of joint time evolutions for the system and controller taken together is given by $U=e^{-iAt}$ where $A\in$ the algebra generated by $\{H\otimes I'\oplus I\otimes H',H^i_{\rm SC}\}$ via commutation. Consequently, as long as one of the $H^i_{\rm SC}$ interactions is nontrivial in the sense described in the text, by varying H' and the initial state $\rho_{\rm C}$ of the controller and by judiciously varying the $\gamma_i(t)$, we can obtain any operator in the algebra $\mathcal{A}\otimes\mathcal{A}'$, where $\mathcal{A}$ is the algebra generated from $\{H,O_i\}$ by commutation, and where $\mathcal{A}'$ is the full algebra for the controller. (Recall that $\{O_i=\mathrm{tr}_{\rm C}H^i_{\rm SC}\rho_{\rm C}\}$ is the set of Hermitian operators that can act on the system given different states $\rho_{\rm C}$ for the controller.)

The ''if'' part of result 4 now follows because if $\mathcal{A}$ is the full algebra of Hermitian operators for the system, then by result 1 it is possible to arrange any desired joint unitary evolution for system and controller. In particular, it is possible to generate a unitary evolution that exchanges the state of the system with the state of the register in the controller that holds the desired state ρ_d, which may be entangled with some other system: i.e., the operation that takes $\rho\otimes\rho_d\rightarrow\rho_d\otimes\rho$ is clearly unitary, and so can be generated by the proper schedule of interactions between system and controller. This proves the ''if'' part of result 4.

The ''only if'' part of result 4 follows because, if $\mathcal{A}$ is not the full algebra for the system, then the set of transformations for the system and controller together does not allow an arbitrary transformation of the state of the system. In particular, no schedule of interactions can apply the transformation $e^{-i\bar{A}t}$ to the initial state of the system, where $\bar{A}\notin\mathcal{A}$. This proves the ''only if'' part of result 4.

5. Quantum observability

A Hamiltonian quantum system is observable by a coherent controller using quantum feedback if and only if it is controllable by the controller.

The ''if'' part of result 5 follows directly from the proof of the ''if'' part of result 4: the same schedule of interactions between controller and system that transfers the desired state from the controller to the system *ipso facto* transfers the state of the system, together with all its entanglements, to the controller. The ''only if'' part of result 5 follows from the ''only if'' part of result 4. The set of transformations that can be effected by the quantum controller consists of unitary operators for system and controller that lie in the Lie group corresponding to the algebra $\mathcal{A}\otimes\mathcal{A}'$. If the algebra $\mathcal{A}$ is not the full algebra for the system, then such operations do not allow the controller to distinguish between two states $|\psi\rangle$ and $|\psi'\rangle=\bar{V}|\psi\rangle$, where $\bar{V}=e^{-i\bar{A}t}\in U(N)/\tilde{U}$. As a result, if the algebra $\mathcal{A}$ is not the full algebra for the system, then the system is not observable by the quantum controller. This proves result 5.

[1] G. M. Huang, T. J. Tarn, and J. W. Clark, J. Math. Phys. **24**, 2608 (1983).

[2] *Information Complexity and Control in Quantum Physics*, Proceedings of the 4th International Seminar on Mathematical Theory of Dynamical Systems and Microphysics, Udine, 1985, edited by A. Blaquiere, S. Diner, and G. Lochak (Springer-Verlag, New York, 1987). This volume contains multiple references to work in quantum-control theory before 1985.

[3] A. Blaquiere, *Modeling and Control of Systems in Engineering, Quantum Mechanics, Economics and Biosciences*, Proceedings of the Bellman Continuum Workshop, Sophia Antipolis, 1988 (Springer-Verlag, New York, 1989).

[4] A. G. Butkovskiy and Yu. I. Samoilenko, *Control of Quantum-Mechanical Processes and Systems* (Kluwer, Dordrecht, 1990).

[5] *Quantum Control and Measurement*, Proceedings of the ISQM Satellite Workshop ARL, Hitachi, 1992, edited by H. Ezawa and Y. Murayama (North-Holland, Amsterdam, 1993).

[6] A. Peirce, M. Dahleh, and H. Rabitz, Phys. Rev. A **37**, 4950 (1988); **42**, 1065 (1990); R. S. Judson and H. Rabitz, Phys. Rev. Lett. **68**, 1500 (1992); W. S. Warren, H. Rabitz, and M. Dahleh, Science **259**, 1581 (1993); V. Ramakrishna, M. V. Salapaka, M. Dahleh, H. Rabitz, and A. Peirce, Phys. Rev. A **51**, 960 (1995); V. Ramakrishna and H. Rabitz, *ibid.* **54**, 1715 (1996).

[7] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. **70**, 548

(1993); Phys. Rev. A **49**, 1350 (1994); **49**, 4110 (1994); H. M. Wiseman, *ibid.* **49**, 2133 (1994).
[8] M. Keller and G. Mahler, J. Mod. Opt. **41**, 2537 (1994).
[9] J. Schlienz and G. Mahler, Phys. Rev. A **52**, 4396 (1995).
[10] R. W. Brockett, SIAM J. Control **10**, 265 (1972).
[11] H. Sussmann and V. Jurdjevic, J. Diff. Eqns. **12**, 95 (1972); V. Jurdjevic and H. Sussmann, *ibid.* **12**, 313 (1972).
[12] *Differential Geometric Control Theory*, edited by R. W. Brockett, R. S. Millman, and H. J. Sussman (Birkhäuser, Boston, 1983); *Nonholonomic Motion Planning*, edited by Z. Li and J. F. Canney (Kluwer, Boston, 1993).
[13] C. B. Chiu, E. C. G. Sudarshan, and B. Misra, Phys. Rev. D **16**, 520 (1977).
[14] A. Peres, Am. J. Phys. **48**, 931 (1980).
[15] V. B. Braginsky, Y. I. Yorontsov, and K. S. Thorne, Science **209**, 547 (1980).
[16] C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmerman, Rev. Mod. Phys. **52**, 341 (1980).
[17] S. Mitter, in *Information Complexity and Control in Quantum Physics* (Ref. [2]), p. 151.
[18] C. P. Slichter, *Principles of Magnetic Resonance*, 3rd ed. (Springer-Verlag, New York, 1990).
[19] O. R. Ernst, G. Bodenhausen, and A. Wokaun, *Principles of Nuclear Magnetic Resonance in One and Two Dimensions* (Oxford University Press, Oxford, 1987).
[20] L. Elmsley and A. Pines, in *Lectures on Pulsed NMR*, 2nd ed., Proceedings of the International School of Physics ''Enrico Fermi,'' Course CXXIII, edited by B. Maraviglia (Società Italiana di Fisica, Varenna, 1994).
[21] Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi, and H. J. Kimble, Phys. Rev. Lett. **75**, 4710 (1995); H. Mabuchi and H. J. Kimble, Opt. Lett. **19**, 749 (1993); A. S. Parkins, P. Marte, P. Zoller, and H. J. Kimble, Phys. Rev. Lett. **71**, 3095 (1993).
[22] C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).
[23] M. Brune *et al.*, Phys. Rev. Lett. **72**, 3339 (1984); P. Domokos, J. M. Raimond, M. Brune, and S. Haroche, Phys. Rev. A **52**, 3554 (1995).
[24] H. Walther, in *Control of Quantum-Mechanical Processes and Systems* (Ref. [4]), p. 113.
[25] For a review of quantum computing, see D. Divincenzo, Science **270**, 255 (1995).
[26] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[27] S. Lloyd, Science **273**, 1073 (1996).
[28] C. H. Bennett, Phys. Today **48** (10), 24 (1995).
[29] R. Landauer, Nature (London) **335**, 779 (1988).
[30] R. Landauer, Phys. Lett. A **217**, 188 (1996).
[31] P. W. Shor, Phys. Rev. A **52**, R2493 (1995); A. R. Calderbank and P. W. Shor, *ibid.* **54**, 1098 (1996); A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996); R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, *ibid.* **77**, 198 (1996).
[32] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[33] T. Hogg and G. Chase, in *Physcomp96*, Proceedings of the Fourth Workshop on Physics and Computation, Boston University, 1996, edited by T. Toffoli, M. Biafore, and J. Leao (New England Complex Systems Institute, Boston, 1996), pp. 147–152.
[34] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993); D. Bouwmeester *et al.*, Nature (London) **390**, 575 (1997); S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998); A. Furusawa *et al.*, Science **282**, 706 (1998).
[35] H. Touchette and S. Lloyd, Phys. Rev. Lett. **84**, 1156 (2000).
[36] D. G. Luenberger, *Introduction to Dynamic Systems* (Wiley, New York, 1979).
[37] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998); L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999); L. Viola, S. Lloyd, and E. Knill, *ibid.* **83**, 4888 (1999).
[38] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973); L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).
[39] A. Anderson, Phys. Rev. Lett. **75**, 3377 (1995).
[40] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1995).
[41] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Springer-Verlag, Berlin, 1987).
[42] E. B. Davies, *Quantum Theory of Open Systems* (Academic, London, 1976).
[43] S. Lloyd and L. Viola (unpublished).
[44] D. G. Cory, A. F. Fahmy, and T. F. Havel, in *PhysComp96* (Ref. [33]), pp. 87–91; S. Somaroo *et al.*, Phys. Rev. Lett. **82**, 5381 (1999); C. H. Tseng *et al.*, Phys. Rev. A **61**, 012302 (2000).
[45] N. A. Gershenfeld and I. L. Chuang, Science **275**, 350 (1997); I. L. Chuang *et al.*, Nature (London) **393**, 143 (1998); I. L. Chuang, N. Gershenfeld, and M. Kubinec, Phys. Rev. Lett. **80**, 3408 (1998); L. M. K. Vandersypen *et al.*, Appl. Phys. Lett. **76**, 646 (2000).
[46] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[47] D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1951).
[48] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[49] C. B. Murray, C. R. Kagan, and M. G. Bawendi, Science **270**, 1335 (1995).
[50] D. Deutsch, A. Barenco, and A. Ekert, Proc. R. Soc. London, Ser. A **449**, 669 (1995).
[51] S. Lloyd, Phys. Rev. Lett. **75**, 346 (1995).