Functional approach to quantum decoherence and the classical final limit

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For a wide set of quantum systems it is demonstrated that the quantum regime can be considered as the transient phase, while the final classical statistical regime is a permanent state. A basis where exact matrix decoherence appears for these final states is found. The relation with the decoherence of histories formalism is studied. A set of final intrinsically consistent histories is found.

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I. INTRODUCTION

Following the idea that the interplay of observables and states is the fundamental ingredient of quantum mechanics[1] we have developed a paper [2] where we have studied the *relation* of the *state vectors* ρ of a close isolated quantum system (that belong to a convex set of states $\mathcal{S}$), to the *observables* O within this closed system (that belong to a space of observables $\mathcal{O}$). We consider that the essence of this relation is the mean value of an observable O in a state ρ, which is given by the equation:

$$\langle O\rangle_\rho = \mathrm{Tr}(\rho O) = (\rho|O). \tag{1}$$

In fact at the statistical level what we actually measure in an ensemble of identical states, are these kinds of averages, since we cannot either measure directly the state ρ or measure it with an infinite precision [3]. Moreover, these averages can be considered, as in the r.h.s. of Eq. (1), the result of a linear functional $(\rho| \in \mathcal{S}$ acting on a vector $|O) \in \mathcal{O}$, and therefore we can say that $\mathcal{S} \subset \mathcal{O}'$, being $\mathcal{O}'$ the dual of space $\mathcal{O}$. While for the usual states (mixed or pure) we can use $\mathrm{Tr}(\rho O)$, there are *generalized states* that can be defined as the functional $(\rho|O)$ as explained in earlier papers [2]. Many results were obtained using this formalism (see, e.g., [2], [4], [5]).

In this paper we will use the formalism of that in Ref. [2] to study the so-called ''classical limit problem,'' namely, the *statistical quantum mechanics → classical mechanics* limit that appears in some quantum systems when observed using certain spaces of observables $\mathcal{O}$. For conceptual reasons we will divide the problem in two different processes (which may or may not happen simultaneously).

(a) Statistical process. Namely the limit *statistical quantum mechanics → statistical classical mechanics,* where the phenomenon of *decoherence* combined with the disappearance of the uncertainty relations in the limit $\hbar \to 0$, originates the classical final stationary state. Almost all of the paper will be devoted to this problem. We will see how when $t \to \infty$ the quantum system reaches a classical final stationary state $\rho_*(q,p)$, where the statistical dynamics is trivial, since $\rho_*(q,p)$ is time independent, but the systems of the ensemble move according to the nontrivial laws of classical dynamics. In general we will have an unlocalized statistical classical state of many identical systems moving in phase space.

(b) Localization process. It is the evolution *statistical classical mechanics →classical mechanics.* In some special cases the evolution privileges a single space-time trajectory, in such a way that all trajectories (endowed with a nonnegligible positive probability) concentrate around it.[2] In this case we will have *correlations* and *localization.* Then we have the statistical classical state of all the systems practically moving along the same trajectory in such a way that we may consider that we are dealing with a *single classical system*. We will discuss this process in Sec. IV and Appendix B.

The usual technique to solve these problems is coarse graining. But in our method we will consider not only the coarse-graining average but *all* possible averages made using the observables of space $\mathcal{O}$, thus we are generalizing the coarse-graining idea.[3] In fact, among the observables of $\mathcal{O}$ there are some that, from the density matrix ρ, take into account only some component ρ_r, the so-called relevant part of ρ, and completely neglect the complementary component ρ_i, the so-called irrelevant part of ρ, i.e., these observables only measure (macroscopic) properties of what it is considered as the ''system'' (contained in ρ_r) and neglect or average the (microscopic) properties of the ''environment'' (contained in ρ_i). But we will consider not only this kind of observable but all observables in $\mathcal{O}$. Therefore the interplay-

[1]According to W. Zurek, ''The only sensible subject of consideration aimed at the interpretation of quantum theory . . . is the *relation* between the universal *state vector* and the state memory (records) of somewhat special system—such as *observers* —which are, for necessity, perceiving the Universe from within. It is the inability to appreciate the consequences of this rather simple but fundamental observation that has led to such desperate measures as the search of an alternative quantum physics [1].''

[2]In some cases this phenomenon does not happen for all the systems but only for a subsystem.

[3]At least the ''coarse graining'' as Zurek describes.

ing of observables and states will take the role of the coarse graining in this paper (see also the end of Sec. IIA). With this strategy we can not only obtain all the old results, but also we will find some new ones.

We will use this method to study the process (a) and to prove that certain quantum systems evolve from a statistical quantum state to the statistical classical final stationary state. In the same framework we will study the process (b) obtaining the classical motion of a single system.

The paper is organized as follows. In Sec. II we will see, using the Riemann-Lebesgue theorem, that transition (a) takes place in close systems endowed with a continuous spectrum and with just one bound state (as in the classical mixing systems). More general cases will be considered in Sec. II C. The main characteristics of the quantum laws are (1) the non-boolean nature of the way to find the probability of two exclusive events (this probability is the square modulus of the sum of their amplitudes and not the sum of the probabilities); (2) the uncertainty relations.

In the evolution from quantum mechanics to classical statistical mechanics the first characteristic disappears (and the Boolean method of adding probabilities is established) by the process of *decoherence* and the uncertainty relations can be neglected in the limit $\hbar \to 0$. Then we can use the laws of *classical statistical mechanics*. At this stage four remarks are in order.

(i) Using our language the generalized idea of decoherence can be introduced in the following way. At the quantum level the average (1) reads

$$\langle O \rangle_\rho^{(q)} = \sum_{\omega,\omega'} \rho_{\omega\omega'} O_{\omega'\omega} \tag{2}$$

where $\rho_{\omega\omega'}$ and $O_{\omega\omega'}$ are the components in some basis of the operators ρ and O, respectively. Equation (2) can be considered as the average of some quantities $O_{\omega\omega'}$ weighted by some generalized correlations $\rho_{\omega\omega'}$ (since the $\rho_{\omega\omega}$ are probabilities but the $\rho_{\omega\omega'}$, with $\omega \neq \omega'$, are quantum correlations). On the other hand, at the classical level we also have some quantities O_ω that correspond to a set $\{\omega\}$ of the exhaustive and exclusive alternatives, each one with a (Boolean) probability p_ω of measure ω for the observable O. The corresponding classical weighted average is

$$\langle O \rangle_\rho^{(cl)} = \sum_\omega p_\omega O_\omega \tag{3}$$

where $\Sigma_\omega p_\omega = 1$. The transition from the quantum phase to the classical one is therefore

$$\sum_{\omega,\omega'} \rho_{\omega\omega'} O_{\omega'\omega} \to \sum_\omega p_\omega O_\omega \tag{4}$$

at least for some O, which belong to a preferred subspace of $\mathcal{O}$ (i.e. to a subspace expanded by a complete set of commuting observables (CSCO) that we will define below; the eigenbasis of this set will be the so-called *final pointer basis*). If in (4) we take $\rho_{\omega\omega} = p_\omega$ and $O_{\omega\omega} = O_\omega$, the matrix $\rho_{\omega\omega'}$ must become diagonal in the final pointer basis. This is the essence of the transition (a), since the above relation will be valid for all observables of the CSCO and we will have

$$\langle O \rangle_\rho^{(q)} \to \langle O \rangle_\rho^{(cl)}. \tag{5}$$

If this transition takes place, Boolean logic is established in the statistical classical system, if we perform the measurement with the observables of the preferred CSCO. In the usual parlance we will then say that the density matrices that contain quantum interference terms become diagonal, in such away that these interferences are suppressed. Then the quantum way to find probabilities of exclusive and exhaustive alternatives, i.e., adding the corresponding amplitudes and computing the norm, becomes the classical Boolean way, just adding the probabilities.

(ii) In this paper decoherence is essentially studied in systems with continuous spectrum. The case of the discrete spectrum, and the causes of decoherence in this case, are discussed in Sec. II C.

(iii) In the case of the continuous spectrum the essence of the method is the following: If $\omega \in \mathbb{R}^+$ are the eigenvalues of H and we call $\nu = \omega - \omega'$, the $\rho_{\omega\omega'}$ of Eq. (2) is a function $\rho(\nu, \dots)$. Then the time limit of its evolution is given by the Riemann-Lebesgue theorem, which prescribes that

$$\lim_{t\to\infty} \int_{-a}^{a} e^{-i\nu t} \rho(\nu, \dots) d\nu = 0 \tag{6}$$

if $\rho(\nu, \dots)$ is integrable. All the diagonal terms ($\nu = 0$) and all the off-diagonal terms ($\nu \neq 0$) vanish. Therefore this theorem cannot be used as a computation method in the case of continuous spectrum. Nevertheless when we consider the problem within a cube of size L, we define $\rho_{\omega\omega'}$ there, and when we make $L \to \infty$, it can be shown that a s*ingular structure* appears for $\rho(\nu, \dots)$ and the corresponding singular diagonal term remains as it should. The method introduced in paper [2] is precisely designed to rigorously deal with these singular structures. It has yielded good results in papers [2], [5], [4].

(iv) Before the classical stationary state limit is reached usually the system goes through a ''classical phase'' where the state can be considered as classical but it is not yet in its final classical stationary state. But our method can only be used when $t \to \infty$. It only allows to find the ''statistical classical final limit.'' So, we essentially study this final stationary state but we believe that our method can be generalized to cover the classical phase before the final stationary state, so we will discuss these matters in Sec. V. Moreover, we believe that the understanding of the final limit will enhance the chances to understand the much more difficult problem of the classical phase, in the clearest and concise way.

In Sec. III we reach to the principal aim of the formalism of transition (a), which is to create a bridge between quantum and classical mechanics, precisely between quantum mechanics and classical *statistical* mechanics at equilibrium. We know that the uncertainty relations disappear, when $\hbar \to 0$ (more precisely when the characteristic dimension of the system makes $\hbar$ a negligible quantity). Then, let us consider a system where the quantum state is defined by a density matrix ρ, and a set of classical trajectories in phase space labeled by some constants $x, l_1, \dots, l_N, a_1, \dots, a_N$, where

x corresponds to the energy, $l_1, \ldots, l_N$ to other dynamical momentum variables, and, $a_1, \ldots, a_N$ to configuration variables. The aim of the theory is (1) to transform the matrix ρ into a classical density function in phase space $\rho(q,p)$ when $\hbar \to 0$, (2) to decompose $\rho(q,p)$ as

$$\rho(q,p) = \sum_{x,l_1,\ldots,l_N,a_1,\ldots,a_N} p_{x,l_1,\ldots,l_N,a_1,\ldots,a_N} \times \rho_{x,l_1,\ldots,l_N,a_1,\ldots,a_N}(q,p), \tag{7}$$

where q and p are the position and momentum coordinates and the classical densities $\rho_{x,l_1,\ldots,l_N,a_1,\ldots,a_N}(q,p)$ would correspond to each classical trajectory[4] (in the classical sense that it is peaked in the trajectory and thus it rapidly vanishes when going from the near vicinity of the trajectory to the far zones of the phase space) and $p_{x,l_1,\ldots,l_N,a_1,\ldots,a_N}$ is the probability of each trajectory.

We will obtain (when $\hbar \to 0$) these results as follows: (1) $\rho(q,p)$ will be the Wigner function corresponding to the matrix ρ; (2) $\rho_{x,l_1,\ldots,l_N,a_1,\ldots,a_N}(q,p)$ will be the Wigner functions of the wave packets going along the classical trajectories labeled by the constant of the motion $x, l_1, \ldots, l_N$, and passing by the initial point of coordinates $a_1, \ldots, a_N$.

We will see that all this happens after a convenient decoherence time and we will obtain the last expansion [cf. Eq. (42)] and therefore what we consider the best bridge between classical and quantum statistical concepts (see paper [6] for a very similar conclusion).

We will devote Sec. IV to discuss transition (b), namely the localization process. Eventually in some cases this process takes place and *correlations* appear and we reach a single classical state if the localization process is efficient enough. Then we can use the laws of *classical mechanics*. This phenomenon happens if the dynamic of the system and the initial conditions are such that some canonically conjugated variables correlate (see Appendix B). We will see how this fact can be incorporated in our formalism. We will draw our main conclusions and comments in Sec. V.

Appendix A is devoted to compare our results with those in the literature. In Appendix B we deal with correlations and localization. Finally, in Appendix C we translate the results into the language of usual decoherence of histories.

II. DECOHERENCE

A. Decoherence in the energy

Let us consider an isolated quantum system with $N+1$ dynamical variables and a Hamiltonian endowed with a continuous spectrum and just one bounded state. So the discrete part of the spectrum of H has only one value ω_0 and the continuous spectrum is, let say, $0 \leqslant \omega < \infty$ (how the discrete spectrum behaves in the continuous limit can be seen in the papers of [7], [8]). Eventually we will give the collective name x to both ω_0 and ω. Let us assume that it is possible to diagonalize the Hamiltonian H, together with N observables O_i $(i=1,\ldots,N)$. The operators $(H, O_1, \ldots, O_N)$ form a *complete set of commuting observables* (CSCO). For simplicity we also assume a discrete spectrum for the N observables O_i. Therefore we write

$$H = \omega_0 \sum_m |\omega_0, m\rangle\langle\omega_0, m| + \int_0^\infty \omega \sum_m |\omega, m\rangle\langle\omega, m| d\omega, \tag{8}$$

where $\omega_0 < 0$ is the energy of the ground state, and $m \doteq \{m_1, \ldots, m_N\}$ labels a set of discrete indexes, which are the eigenvalues of the observables $O_1, \ldots, O_N$. $\{|\omega_0, m\rangle, |\omega, m\rangle\}$ is a basis of generalized eigenvectors of the CSCO:

$$H|\omega_0, m\rangle = \omega_0 |\omega_0, m\rangle, \quad H|\omega, m\rangle = \omega |\omega, m\rangle,$$

$$O_i|\omega_0, m\rangle = m_i |\omega_0, m\rangle, \quad O_i|\omega, m\rangle = m_i |\omega, m\rangle.$$

The most general observable that we are going to consider in our model reads:

$$\begin{aligned} O = & \sum_{mm'} O(\omega_0)_{mm'} |\omega_0, m\rangle\langle\omega_0, m'| \\ & + \sum_{mm'} \int_0^\infty d\omega O(\omega)_{mm'} |\omega, m\rangle\langle\omega, m'| \\ & + \sum_{mm'} \int_0^\infty d\omega O(\omega, \omega_0)_{mm'} |\omega, m\rangle\langle\omega_0, m'| \\ & + \sum_{mm'} \int_0^\infty d\omega' O(\omega_0, \omega')_{mm'} |\omega_0, m\rangle\langle\omega', m'| \\ & + \sum_{mm'} \int_0^\infty \int_0^\infty d\omega d\omega' O(\omega, \omega')_{mm'} |\omega, m\rangle\langle\omega', m'|, \end{aligned} \tag{9}$$

where $O(\omega)_{mm'}$, $O(\omega, \omega_0)_{mm'}$, $O(\omega_0, \omega)_{mm'}$, and $O(\omega, \omega')_{mm'}$ are ordinary functions of the real variables ω and ω' (these functions must have some mathematical properties in order to develop the theory; these properties are listed in [2]). Namely, the most general observables have a singular component (the second term of the r.h.s. of the last equation) and a regular part (all the other terms). If the singular term would be missing the Hamiltonian (8) would not belong to the space of the chosen observables [2]. We will say that these observables belong to a space $\mathcal{O}$. This space has the *basis* $\{|\omega_0, mm'), |\omega, mm'), |\omega\omega_0, mm'), |\omega_0\omega', mm'), |\omega\omega', mm')\}$:

$$|\omega_0, mm') \doteq |\omega_0, m\rangle\langle\omega_0, m'|, \quad |\omega, mm') \doteq |\omega, m\rangle\langle\omega, m'|,$$

$$|\omega\omega_0, mm') \doteq |\omega, m\rangle\langle\omega_0, m'|,$$

[4] The dimension of the phase space considered is $2(N+1)$. Then there are $(N+1)$ momenta and $(N+1)$ coordinates. So $N+1$ is the number of parameters necessary to label the momenta of the classical space-time trajectories, and N the number necessary to label the origins of the trajectories.

$$|\omega_0\omega',mm'\rangle\doteq|\omega_0,m\rangle\langle\omega',m'|,$$

$$|\omega\omega',mm'\rangle\doteq|\omega,m\rangle\langle\omega',m'|. \tag{10}$$

The quantum states ρ are measured by the observables just defined, computing the mean values of these observables in the quantum states, i.e., in the usual notation: $\langle O\rangle_\rho = \mathrm{Tr}(\rho^\dagger O)$ [3]. These mean values, generalized as in paper [2], can be considered as linear functionals ρ, mapping the vectors O on the real numbers, that we can call $(\rho|O)$ [9]. In fact, this is a generalization of the usual mean value definition. Then $\rho\in\mathcal{S}\subset\mathcal{O}'$, where $\mathcal{S}$ is a convenient convex set contained in $\mathcal{O}'$, the space of linear functionals over $\mathcal{O}$ [10], [11]. The basis of $\mathcal{O}'$ (that can also be considered as the *co-basis* of $\mathcal{O}$) is $\{(\omega_0,mm'|,\ (\omega,mm'|,\ (\omega\omega_0,mm'|,\ (\omega_0\omega',mm'|,\ (\omega\omega',mm'|\}$ defined as functionals by the equations:

$$\begin{aligned}
(\omega_0,mm'|\omega_0,nn') &= \delta_{mn}\delta_{m'n'},\\
(\omega,mm'|\eta,nn') &= \delta(\omega-\eta)\delta_{mn}\delta_{m'n'},\\
(\omega\omega_0,mm'|\eta\omega_0,nn') &= \delta(\omega-\eta)\delta_{mn}\delta_{m'n'},\\
(\omega_0\omega',mm'|\omega_0\eta',nn') &= \delta(\omega'-\eta')\delta_{mn}\delta_{m'n'},\\
(\omega\omega',mm'|\eta\eta',nn') &= \delta(\omega-\eta)\delta(\omega'-\eta')\delta_{mn}\delta_{m'n'}
\end{aligned} \tag{11}$$

and all other $(.|.)$ are zero. In particular we have

$$(\omega_0,mm'|O)=O(\omega_0)_{mm'}=\langle\omega_0,m|O|\omega_0,m'\rangle \tag{12}$$

for any $O\in\mathcal{O}$. But $(\omega,mm'|O)=O(\omega)_{mm'}$ is not equal to $\langle\omega,m|O|\omega,m'\rangle$, which is not even defined if O is given by Eq. (9). Therefore $(\omega,mm'|$ can only be considered as a functional, being a typical generalized state. Then, a generic quantum state reads

$$\begin{aligned}
\rho=&\sum_{mm'}\overline{\rho(\omega_0)}_{mm'}(\omega_0,mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega\overline{\rho(\omega)}_{mm'}(\omega,mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega\overline{\rho(\omega,\omega_0)}_{mm'}(\omega\omega_0,mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega'\overline{\rho(\omega_0,\omega')}_{mm'}(\omega_0\omega',mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega\int_0^\infty d\omega'\overline{\rho(\omega,\omega')}_{mm'}(\omega\omega',mm'|,
\end{aligned} \tag{13}$$

where $\overline{\rho(\omega_0)}_{mm}$ and $\overline{\rho(\omega)}_{mm}$ are real and non-negative, $\overline{\rho(\omega,\omega_0)}_{mm'}=\rho(\omega_0,\omega)_{m'm}$, and $\overline{\rho(\omega,\omega')}_{mm'}=\rho(\omega',\omega)_{m'm}$. Moreover, $\rho(\omega_0)_{mm'}$ and $\rho(\omega)_{mm'}$ satisfy the total probability condition

$$(\rho|I)=\sum_m\rho(\omega_0)_{mm}+\sum_m\int_0^\infty d\omega\rho(\omega)_{mm}=1, \tag{14}$$

where $I=\Sigma_m|\omega_0,m\rangle\langle\omega_0,m|+\int_0^\infty d\omega\Sigma_m|\omega,m\rangle\langle\omega,m|$ is the identity operator in $\mathcal{O}$. Equation (14) is the extension to state functionals of the usual condition $\mathrm{Tr}\,\rho^\dagger=1$, used when ρ is a density operator.

The time evolution of the quantum state ρ reads

$$\begin{aligned}
\rho(t)=&\sum_{mm'}\overline{\rho(\omega_0)}_{mm'}(\omega_0,mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega\overline{\rho(\omega)}_{mm'}(\omega,mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega\overline{\rho(\omega,\omega_0)}_{mm'}e^{i(\omega-\omega_0)t}(\omega\omega_0,mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega'\overline{\rho(\omega_0,\omega')}_{mm'}e^{i(\omega_0-\omega')t}(\omega_0\omega',mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega\int_0^\infty d\omega'\overline{\rho(\omega,\omega')}_{mm'}\\
&\times e^{i(\omega-\omega')t}(\omega\omega',mm'|
\end{aligned} \tag{15}$$

The mean value of an observable O in a quantum state ρ reads

$$\begin{aligned}
\langle O\rangle_{\rho(t)}=&(\rho(t)|O)\\
=&\sum_{mm'}\overline{\rho(\omega_0)_{mm'}}O(\omega_0)_{mm'}\\
&+\sum_{mm'}\int_0^\infty d\omega\overline{\rho(\omega,\omega')}_{mm'}O(\omega)_{mm'}\\
&+\sum_{mm'}\int_0^\infty d\omega\overline{\rho(\omega,\omega_0)}_{mm'}e^{i(\omega-\omega_0)t}O(\omega,\omega_0)_{mm'}\\
&+\sum_{mm'}\int_0^\infty d\omega'\overline{\rho(\omega_0,\omega')}_{mm'}e^{i(\omega_0-\omega')t}\\
&\times O(\omega_0,\omega')_{mm'}+\sum_{mm'}\int_0^\infty d\omega\int_0^\infty d\omega'\overline{\rho(\omega,\omega')}_{mm'}\\
&\times e^{i(\omega-\omega')t}O(\omega,\omega')_{mm'}.
\end{aligned} \tag{16}$$

Using the Riemann-Lebesgue theorem we obtain the weak limit, for all $O\in\mathcal{O}$

$$\lim_{t\to\infty}\langle O\rangle_{\rho(t)}=\langle O\rangle_{\rho_*}, \tag{17}$$

where we have introduced the diagonal asymptotic or final stationary state functional

$$\begin{aligned}
\rho_*=&\sum_{mm'}\overline{\rho(\omega_0)}_{mm'}(\omega_0,mm'|\\
&+\sum_{mm'}\int_0^\infty d\omega\overline{\rho(\omega)}_{mm'}(\omega,mm'|.
\end{aligned} \tag{18}$$

Therefore, in a weak sense we have

$$W\lim_{t\to\infty}\rho(t)=\rho_*. \tag{19}$$

Thus, any quantum state weakly goes to a linear combination of the energy diagonal states $(\omega_0,mm'|$ and $(\omega,mm'|$ (the energy off-diagonal states $(\omega\omega_0,mm'|$, $(\omega_0\omega',mm'|$, and $(\omega\omega',mm'|$ are not present in ρ_*). This is the case if we observe and measure the system evolution with *any possible observable of space* $\mathcal{O}$. Then, from the observational (or generalized coarse-graining) point of view, we have decoherence of the energy levels when $t\to\infty$, even that, from the strong limit (fine-graining) point of view the off-diagonal terms never vanish, they just oscillate, since we cannot directly use the Riemann-Lebesgue theorem in the operator equation (15).

Some observations are in order.

(i) The real existence of the two singular parts of O and ρ is assured by the physics of the problem. The singular part of the observables is just a necessary generalization of the singular part of the Hamiltonian, which has a singular part $|\omega)$ [Eq. (8)]. The states must also be singular objects since, intuitively, we realize that a continuous by continuous matrix will decohere in a matrix with some kind of singularity in the diagonal. The method is precisely designed to deal with this object.

(ii) From Eq. (17) we can again see that what we are doing is just a generalized version of coarse graining, where a projector on the ''relevant'' part of the system is defined. The ''relevant'' part of the states, $(\rho|$ is in our case $(\rho|O)$ for all $O\in\mathcal{O}$, i.e., the ''projection'' of ρ on the class of observables of the form given in Eq. (9). An ''irrelevant'' projection would be a $(\rho|O')$, where $O'\notin\mathcal{O}$.

B. Decoherence in the other momentum dynamical variables

Having established the decoherence in the energy levels we must consider the decoherence in the other dynamical variables O_i, of the CSCO where we are working. We will call these variables ''momentum variables.'' For the sake of simplicity we will consider, as in the previous section, that the spectra of these dynamical variables are discrete. As the expression of ρ_* given in Eq. (18) involves only the time independent components of $\rho(t)$, it is impossible that a different decoherence process would take place to eliminate the off-diagonal terms in the remaining N dynamical variables. Therefore, the only thing to do is to find if there is a basis where the off-diagonal components of $\rho(\omega_0)_{mm'}$ and $\rho(\omega)_{mm'}$ vanish at any time before the final state is reached. This basis in fact exists, it is constant in time, and it will be called the *final pointer basis*.

Let us consider the following change of basis

$$|\omega_0,r\rangle=\sum_m U(\omega_0)_{mr}|\omega_0,m\rangle,\quad |\omega,r\rangle=\sum_m U(\omega)_{mr}|\omega,m\rangle, \tag{20}$$

where r and m are short notations for $\underline{r}\doteq\{r_1,\ldots,r_N\}$ and $m\doteq\{m_1,\ldots,m_N\}$, and $[U(x)^{-1}]_{mr}=\overline{U(x)}_{rm}$ (x denotes either $\omega_0<0$ or $\omega\in\mathbb{R}^+$).

The new basis $\{|\omega_0,r\rangle,|\omega,r\rangle\}$ verifies the generalized orthogonality conditions

$$\langle\omega_0,r|\omega_0,r'\rangle=\delta_{rr'},\quad \langle\omega,r|\omega',r'\rangle=\delta(\omega-\omega')\delta_{rr'},$$

$$\langle\omega_0,r|\omega,r'\rangle=\langle\omega,r|\omega_0,r'\rangle=0.$$

It is easy to obtain the components of the states $\rho\in\mathcal{S}$ in the new basis

$$\rho(\omega_0)_{rr'}=\sum_{mm'}[U(\omega_0)^{-1}]_{rm}\rho(\omega_0)_{mm'}[U(\omega_0)]_{m'r'},$$

$$\rho(\omega)_{rr'}=\sum_{mm'}[U(\omega)^{-1}]_{rm}\rho(\omega)_{mm'}[U(\omega)]_{m'r'},$$

$$\rho(\omega,\omega')_{rr'}=\sum_{mm'}[U(\omega)^{-1}]_{rm}\rho(\omega,\omega')_{mm'}[U(\omega')]_{m'r'},$$

$$\rho(\omega_0,\omega')_{rr'}=\sum_{mm'}[U(\omega_0)^{-1}]_{rm}\times\rho(\omega_0,\omega')_{mm'}[U(\omega')]_{m'r'},$$

$$\rho(\omega,\omega_0)_{rr'}=\sum_{mm'}[U(\omega)^{-1}]_{rm}\rho(\omega,\omega_0)_{mm'}[U(\omega_0)]_{m'r'},$$

As $\overline{\rho(\omega_0)}_{mm'}=\rho(\omega_0)_{m'm}$ and $\overline{\rho(\omega)}_{mm'}=\rho(\omega)_{m'm}$, it is possible to choose $U(\omega_0)$ and $U(\omega)$ in such a way that the off-diagonal parts of $\rho(\omega_0)_{rr'}$ and $\rho(\omega)_{rr'}$ would vanish, i.e.,

$$\rho(\omega_0)_{rr'}=\rho_r(\omega_0)\delta_{rr'},\quad \rho(\omega)_{rr'}=\rho_r(\omega)\delta_{rr'}.$$

Therefore, there is a final pointer basis for the observables given by $\{|\omega_0,rr'),\ |\omega,rr'),\ |\omega\omega_0,rr'),\ |\omega_0\omega',rr'),\ |\omega\omega',rr')\}$ and defined as in Eq. (10). The corresponding final pointer basis for the states $\{(\omega_0,rr'|,\ (\omega,rr'|,\ (\omega\omega_0,rr'|,\ (\omega_0\omega',rr'|,\ (\omega\omega',rr'|\}$ diagonalizes the time independent part of $\rho(t)$ and therefore it diagonalizes the final state ρ_*,

$$\rho_*=W\lim_{t\to\infty}\rho(t)=\sum_r\rho_r(\omega_0)(\omega_0,rr|+\sum_r\int_0^\infty d\omega\rho_r(\omega)(\omega,rr|. \tag{21}$$

Now we can define the *final exact pointer observables* [1]:

$$P_i=\sum_r P_r^i(\omega_0)|\omega_0,r\rangle\langle\omega_0,r|+\int_0^\infty d\omega\sum_r P_r^i(\omega)|\omega,r\rangle\langle\omega,r|. \tag{22}$$

As H and P_i are diagonal in the basis $\{|\omega_0,r\rangle, |\omega,r\rangle\}$, the set $\{H,P_i,\ldots P_N\}$ is precisely the complete set of commuting observables (CSCO) related to this basis, where ρ_* is diagonal in the corresponding cobasis for states. For simplicity we define the operators P_i such that $P_r^i(\omega_0)=P_r^i(\omega)=r_i$, thus

$$P_i|\omega_0,r\rangle=r_i|\omega_0,r\rangle, \quad P_i|\omega,r\rangle=r_i|\omega,r\rangle. \tag{23}$$

Therefore $\{|\omega_0,r\rangle, |\omega,r\rangle\}$ is the final observers' pointer basis where there is a perfect decoherence in the corresponding state cobasis. Moreover the generalized states $(\omega_0,rr|$ and $(\omega,rr|$ are constants of the motion, and therefore these exact pointer observables have a constant statistical entropy and will be ''at the top of the list'' of Zurek's ''predictability sieve'' [1]. The final pointer basis is therefore defined by the dynamics of the model and by the quantum state considered.

Therefore (i) decoherence in the energy is produced by the time evolution when $t\to\infty$; (ii) decoherence in the other dynamical variables can be seen if we choose an adequate basis, namely the final pointer basis.

Essentially we have given a partial answer, for this kind of model, to the fundamental question of Gell-Mann and Hartle [12] (precisely only an answer in the case when $t\to\infty$): For each H and each initial state ρ there is only one final pointer basis and therefore only one ''quasiclassical domain or realm'' [13].[5]

Our main result is Eq. (21). *When $t\to\infty$ then $\rho(t)\to\rho_*$ and in this state the dynamical variables $H,P_1,\ldots,P_N$ are well defined. Therefore the eventual conjugated variables to these momentum variables (namely, configuration variables, if they exist) are completely undefined.*

In fact, calling $\mathbb{L}_i$ the generator of the displacements along the eventual configuration variable conjugated to P_i, we have $(\mathbb{L}_i\rho_*|O)=(\rho_*|\mathbb{L}_i^\dagger O)=(\rho_*|[P_i,O])=0$ for all $O\in\mathcal{O}$ as it can be proved by direct computation using Eqs. (9), (11), (18), and (22). Then $\mathbb{L}_i\rho_*=0$, and ρ_* is homogeneous in these configuration variables.

C. Decoherence characteristic decaying time, the permanent quantum states case, and the role of the environment

From the preceding section it may seem that the process of decoherence must be found in all the physical systems. It is not so and there are two reasons.

(i) Characteristic decaying times can be computed using analytic continuation technics, as in paper [10]. For example, in particular models we can find the characteristic times for the system (e.g., an oscillator) and the field (e.g., the environments or bath) as below Eq. (56) of the last quoted paper. If the maximal characteristic time γ^{-1} is very large, even if theoretically the decoherence process will always take place, it will be so slow that the system will behave as a quantum one for a very long time. Then there will be no measurable decoherence.

(ii) It may also happen that more than one of the γ would be zero. Then, Hamiltonian H has more than one bound state, let us say n (or even part of its spectrum is discrete). Then the first term of the r.h.s. of Eq. (16) must be changed to

$$\sum_{ij}\rho_{ji}O_{ij}e^{i(\omega_i-\omega_j)t}=\sum_i\rho_{ii}O_{ii}+\sum_{i\neq j}\rho_{ji}O_{ij}e^{i(\omega_i-\omega_j)t}, \tag{24}$$

where $i,j=1,\ldots n$, and as the second term of the r.h.s. does not vanish when $t\to\infty$, decoherence does not take place. This is the case of a theoretical atom, not coupled to the electromagnetic field, where the electrons will remain forever in their exited states, and they will never decay. Then the atom never goes to a decohered state. But if the atom is coupled to an electromagnetic field (that usually it is called the ''environment,'' as in Appendix B) there will be only one bound state, the second term of the r.h.s. of Eq. (24) will be absent, and decoherence will occur. In fact, in many examples the role of the ''environment'' is just to introduce a continuous spectrum to be coupled in such a way that only one bound state remains and the decoherence is complete. In other cases fluctuations (or imperfections) of continuous nature take the role of the continuous spectrum and produce the average and make the off-diagonal term disappear. This is the case of the spin recombination experiment ([3], p. 180) that takes place in a single crystal interferometer.

(iii) More generally, using only observables from a subset $\Omega\in\mathcal{O}$ we may only involve some components of the state functional, e.g., those constructed with the eigenvectors of H that eventually expand the space Ω. Then if we only consider the observables of Ω it may be that the components of the state related with these observables become decohered, because their decoherence times are small, while the other components remain undecohered, because they have a larger decoherence time. Then we will have a system that is partially decohered and partially not decohered (which in fact is the case of the universe where there are both classical and quantum phenomena).

III. THE CLASSICAL STATISTICAL LIMIT

A. Expansion in sets of classical motions

In this section we will use the Wigner integrals that introduce an isomorphism between quantum observables O and states ρ and their classical analogs $O^W(q,p)$ and $\rho^W(q,p)$ [15]:

$$O^W(q,p)=\int d\lambda\left\langle q-\frac{\lambda}{2}\middle|O\middle|q+\frac{\lambda}{2}\right\rangle\exp\left(\frac{i\lambda p}{\hbar}\right),$$

$$\rho^W(q,p)=\left(\frac{1}{\pi\hbar}\right)^{N+1}\int d\lambda(\rho||q+\lambda\rangle\langle q-\lambda|)\exp\left(\frac{2i\lambda p}{\hbar}\right). \tag{25}$$

[5]But of course this unique consistent set depends of the chosen space of observable $\mathcal{O}$ (see more in Appendix C).

It is possible to prove that $\int dq\,dp\,\rho^W(q,p)=(\rho|I)=1$, but ρ^W is not in general non-negative. It is also possible to deduce that

$$(\rho^W|O^W)=\int dq\,dp\,\rho^W(q,p)O^W(q,p)=(\rho|O), \quad (26)$$

and therefore to the mean value in the classical Liouville space it corresponds to the mean value in the quantum Liouville space. Moreover, calling L the classical Liouville operator, and $\mathbb{L}$ the quantum Liouville–Von Neumann operator, we have

$$L[\rho^W(q,p)]=[\mathbb{L}\rho]^W(q,p)+O(\hbar), \quad (27)$$

where $L\rho^W(q,p)=i\{H^W(q,p),\rho^W(q,p)\}_{PB}$ and

$$(\mathbb{L}\rho|O)=(\rho|[H,O]). \quad (28)$$

Finally, if $O=O_1O_2$, where O_1 and O_2 are two quantum observables, we have

$$O^W(q,p)=O_1^W(q,p)O_2^W(q,p)+O(\hbar). \quad (29)$$

We will prove that the distribution function $\rho_*^W(q,p)$ that corresponds to the state functional ρ_* via the Wigner integral is a non-negative function of the classical constants of the motion, in our case $H^W(q,p)$, $P_1^W(q,p),\ldots,P_N^W(q,p)$, obtained from the corresponding quantum operators H, $P_1,\ldots,P_N$.

From Eq. (21) we have

$$\rho_*=W\lim_{t\to\infty}\rho(t)=\sum_r \rho_r(\omega_0)(\omega_0,rr| + \sum_r \int_0^\infty d\omega\rho_r(\omega)(\omega,rr|, \quad (30)$$

so we must compute

$$\rho_{\omega r}^W(q,p)\doteq\left(\frac{1}{\pi\hbar}\right)^{N+1}\int(\omega,rr||q+\lambda\rangle\langle q-\lambda|)e^{2ip\lambda}d\lambda. \quad (31)$$

We know from [2] Sec. II C [or we can directly prove from Eqs. (21)–(23)] that

$$(\omega_0,rr|H^n)=\omega_0^n,\quad (\omega,rr|H^n)=\omega^n,\quad (\omega_0,rr|P_i^n)=r_i^n,$$
$$(\omega,rr|P_i^n)=r_i^n, \quad (32)$$

for $i=1,\ldots,N$ and $n=0,1,2,\ldots$. Using the relation (29) between quantum and classical products of observables and relation (26) between quantum and classical mean values, in the limit $\hbar\to0$ (we will consider that we always take this limit when we refer to classical equations below) we deduce that the characteristic property of the distribution $\rho_{\omega r}^W(q,p)$, that corresponds to the state functional $(\omega,rr|$, is

$$\int \rho_{\omega r}^W(q,p)[H^W(q,p)]^n dqdp=\omega^n,$$
$$\int \rho_{\omega r}^W(q,p)[P_i^W(q,p)]^n dqdp=r_i^n, \quad (33)$$

for any natural number n. Thus $\rho_{\omega r}^W(q,p)$ must be the functional:[6]

$$\rho_{\omega r}^W(q,p)=c\,\delta[H^W(q,p)-\omega]\delta[P_1^W(q,p)-r_1]\cdots\times\delta[P_N^W(q,p)-r_N]. \quad (34)$$

For the distribution $\rho_{\omega_0 r}^W(q,p)$, which corresponds to the state functional $(\omega_0,rr|$, we obtain

$$\rho_{\omega_0 r}^W(q,p)=c\,\delta[H^W(q,p)-\omega_0]\delta[P_1^W(q,p)-r_1]\cdots\times\delta[P_N^W(q,p)-r_N]. \quad (35)$$

Therefore, going back to Eq. (30) and since the Wigner relation is linear, we have

$$\rho_*^W(q,p)=\sum_r \rho_r(\omega_0)\rho_{\omega_0 r}^W(q,p) + \sum_r \int_0^\infty d\omega\rho_r(\omega)\rho_{\omega r}^W(q,p). \quad (36)$$

Also we obtain $\rho_*^W(q,p)\geqslant0$, because $\rho_r(\omega_0)$ and $\rho_r(\omega)$ are non-negative.

Therefore, the classical state $\rho_*^W(q,p)$ is a linear combination of the generalized classical states $\rho_{xr}^W(q,p)$ (where x is either ω_0 or ω), having well-defined values $x, r_1,\ldots,r_N$ of the classical observables $H^W(q,p)$, $P_1^W(q,p),\ldots,P_N^W(q,p)$ and the corresponding classical canonically conjugated variables completely undefined since the $\rho_{xr}^W(q,p)$ are not functions of these variables. *So we reach, in the classical case, the same conclusion as in the quantum case (see end of Sec. II B).* But now all the classical canonically conjugated variables $a_0,a_1,\ldots,a_N$ do exist since they can be found solving the corresponding Poisson brackets differential equations.

As the momenta $H^W,P_1^W,\ldots,P_N^W$, or any function of these momenta, which we will call generically Π, are also constant of the motion, then we have $d/dt\Pi=-\partial H/\partial\alpha=0$, where α is the classically conjugated variable to Π. So H is just a function of Π and

$$\frac{d}{dt}\alpha=\frac{\partial H(\Pi)}{\partial\Pi}=\varpi(\Pi)=\text{const.} \quad (37)$$

So

$$\alpha_j(t)=\varpi_j(\Pi)t+\alpha_j(0),\quad j=0,1,\ldots,N. \quad (38)$$

---

[6]We must also take into account that, as ρ_*, $(x,rr)\mathbb{L}_i=0$ (see the end of Sec. II B). Then classically $\mathbb{L}_i^W\rho_{xr}^W=0$ [as in Eq. (27)]. So $\rho_{xr}^W(q,p)$ cannot be a function of the canonical conjugated variables to $H^W,P_1^W\ldots,P_N^W$. Then Eqs. (34) and (35) are just multiplied by a normalization constant $c=V^{-1}$, where $V=(2\pi)^{N+1}$ is the volume of the tori in the bounded case (see below). The nonintegrable case and the case $V\to\infty$ will be considered elsewhere. We have omitted the $O(\hbar)$ of Eqs. (27) and (29). If we reintroduce these $O(\hbar)$ we will see that Eqs. (34) and (35) are only valid in the limit $\hbar\to0$. If $\hbar$ is only very small the δ are just functions strongly peaked at the zero value of their variables.

Thus (going back to the old coordinates) in the set of classical motions contained in the densities (34) and (35) the momenta $H, P_1, \ldots, P_N$, are completely defined and the origin of the corresponding motions, that we will respectively call $a_0(0)$, $a_1(0), \ldots$ and $a_N(0)$, are completely undefined, in such a way that the motions represented in the last equation homogeneously fill the surface, where H^W, $P_1^W, \ldots,$ and P_N^W have constant values, which now turns out to be a usual torus of phase space.[7] This is the interpretation that we give to the densities (34) and (35), which are just functions of the variables H^W, $P_1^W, \ldots, P_N^W$, but they are not of the classical conjugated variables $a_0, a_1, \ldots, a_N$.

Then, Eq. (36) can be considered as the expansion of $\rho_*^W(q,p)$ in the sets of classical motions contained in $\rho_{xr}^W(q,p)$, each one with a probability $\rho_r(x)$ $(x=\omega_0,\omega)$.

Summing up (i) we have shown that the quantum state functional $\rho(t)$ evolves to a final diagonal state ρ_*; (ii) this quantum state ρ_* has $\rho_*^W(q,p)$ as its corresponding classical density; (iii) this classical density can be decomposed in sets of classical motions where H^W, $P_1^W, \ldots, P_N^W$ remain constant. The origin of these motions $a_0(0), a_1(0), \ldots, a_N(0)$ are homogeneously distributed; (iv) From Eqs. (34)–(36) we obtained that $\rho_*^W(q,p) = f[H^W(q,p), P_1^W(q,p), \ldots, P_N^W(q,p)] \geqslant 0$.

B. Expansion in terms of classical motions

We can now expand the densities given in Eqs. (34)–(36) in terms of classical motions. In fact, since

$$\int \prod_{i=0}^{N} \delta[a_i(q,p) - a_i(t)] \prod_{i=0}^{N} da_i(0) = 1, \tag{39}$$

where $a_j(t) = \varpi_j(P^W)t + a_j(0)$, we can write Eq. (36) as

$$\begin{aligned}\rho_*^W(q,p) = \int \sum_r \rho_r(\omega_0) \rho_{\omega_0 r}^W(q,p) \prod_{i=0}^{N} \delta[a_i(q,p) - a_i(t)] \\ \times \prod_{i=0}^{N} da_i(0) + \int \sum_r \int_0^\infty d\omega \rho_r(\omega) \rho_{\omega r}^W(q,p) \\ \times \prod_{i=0}^{N} \delta[a_i(q,p) - a_i(t)] \prod_{i=0}^{N} da_i(0).\end{aligned} \tag{40}$$

We define

$$\begin{aligned}\rho_{x,r,a(0)}^W(q,p,t) \doteq \delta[H^W(q,p) - x] \\ \times \delta[P_1^W(q,p) - r_1] \cdots \delta[P_N^W(q,p) - r_N] \\ \times \delta[a_0(q,p) - a_0(t)] \cdots \delta[a_N(q,p) \\ - a_N(t)],\end{aligned} \tag{41}$$

which corresponds to the classical distribution of a motion with momenta $x, r_1, \ldots, r_N$ and initial conditions $a_0(0), \ldots, a_N(0)$, and therefore to a single classical motion. So we can write Eq. (40) as

$$\begin{aligned}\rho_*^W(q,p) = \int \sum_r \rho_r(\omega_0) \rho_{\omega_0,r,a(0)}^W(q,p,t) \prod_{i=0}^{N} da_i(0) \\ + \int \sum_r \int_0^\infty d\omega \rho_r(\omega) \rho_{\omega,r,a(0)}^W(q,p,t) \prod_{i=0}^{N} da_i(0).\end{aligned} \tag{42}$$

We have proved Eq. (7) as stated in the Introduction.

The densities $\rho_{x,r,a(0)}^W(q,p,t)$ represent a point in phase space with momenta $H^W = x$, $P_1^W = r_1, \ldots, P_N^W = r_N$ and coordinates $a_j(t) = \varpi_j(P^W)t + a_j(0)$, i.e., they represent single classical trajectories.

Then we have obtained the final classical limit. When $t \to \infty$ the quantum state functional ρ becomes a diagonal state ρ_*. The corresponding classical distribution $\rho_*^W(q,p)$ can be expanded as a linear combination of density functions $\rho_{\omega_0,r,a(0)}^W(q,p,t)$ and $\rho_{\omega,r,a(0)}^W(q,p,t)$, representing classical trajectories, each one weighted by their corresponding probabilities $\rho_r(\omega_0)$ and $\rho_r(\omega)$. As the limit when $t \to \infty$ of our quantum model we have obtained a statistical classical mechanical model [3], and the *classical statistical realm* is obtained.

IV. CORRELATIONS AND LOCALIZATION

From many examples (e.g., [16]) we know that eventually correlations and the localization appear when $t \to \infty$, at least in some variables and in some quantum systems. For, example in Appendix B we give an example obtained using our method, where we can see that correlations appear in variables Q and P, when $t \to \infty$ [see Eq. (B18)]. As this state with correlations is a final state let us call it ρ_* and let us see how it can be incorporated in our formalism.

As ρ_* is a final stationary state it can be decomposed as in Eq. (21). From Eq. (11) we have

$$\begin{aligned}(\omega_0, rr | \omega_0, r'r') = \delta_{rr'}, \quad (\omega, rr | \omega, r'r') = \delta_{rr'}, \\ (\omega_0, rr | \omega, r'r') = 0.\end{aligned} \tag{43}$$

Thus from Eq. (43) we have

$$(\rho_* | \omega_0, rr) = \rho_r(\omega_0), \quad (\rho_* | \omega, rr) = \rho_r(\omega). \tag{44}$$

So, given ρ_*, endowed with correlations and computed by any method (including ours, see Appendix B) we can find the corresponding initial conditions $\rho_r(\omega_0), \rho_r(\omega)$ that yield, when $t \to \infty$, to this correlated state.[8] In general, all final de-

[7] If H^W, $P_1^W, \ldots, P_N^W$ are isolating constants of the motion, the tori are not broken [14]. In the nonintegrable case the tori are broken. This case will be considered elsewhere.

[8] The remaining initial conditions $\rho_{rr'}(\omega_0)$, $\rho_{rr'}(\omega)$, $\rho_{rr'}(\omega,\omega')$, $\rho_{rr'}(\omega_0,\omega')$, $\rho_{rr'}(\omega,\omega_0)$ are irrelevant since the corresponding terms disappear when $t \to \infty$.

cohered stationary states (but not any quantum state) can be decomposed in this way, in particular our correlated state.

We can repeat all these formulas in the classical perspective of Sec. III using the relation (26) between quantum and classical symbols, computing the initial conditions $\rho_r(\omega_0),\rho_r(\omega)$ using classical formulas:

$$(\rho^W_*(q,p)|\rho^W_{\omega_0 r}(q,p))=\rho_r(\omega_0),$$

$$(\rho^W_*(q,p)|\rho^W_{\omega r}(q,p))=\rho_r(\omega). \tag{45}$$

In this way the correlation and localization phenomena can be incorporated in our formalism. But it is difficult to use coordinates x,r,a to directly obtain the final state ρ_* since this state looks quite unfamiliar in these coordinates, but it turns out to be the minimal uncertainty wave packet if we study the problem in the usual coordinates q, p, as we prove in the Appendix B via an example.[9] Furthermore the correlation phenomenon only appears if the potential and the initial conditions are such that all the trajectories with nonnegligible probability are concentrated by the dynamics eventually yielding a ''maximally localized'' or ''minimal uncertainty'' wave packet (as in the example of Appendix B). It is difficult to see this fact in the abstract unfamiliar frame of the coordinates x,r,a, because the potentials are hidden by the diagonalization even if the initial conditions are obviously present [i.e., in the choice $\rho_r(\omega_0)$, $\rho_r(\omega)$]. Anyhow the phenomenon is there, since we obtain localization when $t\to\infty$. In this way we can consider the localized wave packet like a single classical system and the limit *classical statistical mechanics → classical mechanics* is obtained because the motion of the wave packet satisfies the classical equations (38) as in all the trajectories. Now the processes (a) and (b) are explained and the limit *statistical quantum mechanics→classical mechanics* is completed. The *classical realm* is present.

V. COMMENTS AND CONCLUSIONS

Some observations are in order.

A. Sketch of the classical limit

Using the final pointer basis obtained in Sec. II B, the time dependent Wigner function [namely, the diagonalized version of Eq. (15)] is

$$\begin{aligned}\rho^W(q,p,t)=&\sum_r \overline{\rho_r(\omega_0)}\rho^W_{\omega_0 r}(q,p)+\sum_r \int_0^\infty d\omega\,\overline{\rho_r(\omega)}\rho^W_{\omega r}(q,p)+\sum_{rr'}\int_0^\infty d\omega\,\overline{\rho(\omega,\omega_0)}_{rr'}e^{i(\omega-\omega_0)t}\rho^W_{\omega\omega_0 rr'}(q,p)\\&+\sum_{rr'}\int_0^\infty d\omega'\,\overline{\rho(\omega_0,\omega')}_{rr'}e^{i(\omega_0-\omega')t}\rho^W_{\omega_0\omega rr'}(q,p)+\sum_{rr'}\int_0^\infty d\omega\int_0^\infty d\omega'\,\overline{\rho(\omega,\omega')}_{rr'}e^{i(\omega-\omega')t}\rho^W_{\omega\omega' rr'}(q,p)\\=&\rho^W_*(q,p)+\Delta\rho(q,p,t),\end{aligned} \tag{46}$$

where the coefficients $\rho_r(\omega_0)$ and $\rho_r(\omega)$ are the probabilities of each ''classical'' final history, and $\Delta\rho(q,p,t)$ corresponds to ''quantum'' nondecohered histories.[10] It is clear that when $t\to\infty$ (really after a decoherence time γ^{-1}) the terms corresponding to these histories vanish according to Riemann-Lebesgue theorem (see paper [2]).

Now we know (i) that when $\hbar\to 0$, $\rho^W(q,p,t)$ satisfies the classical Liouville equation, namely the laws of classical mechanics; (ii) that $\rho^W_*(q,p)\geqslant 0$, but that the second term $\Delta\rho(q,p,t)$ is not positive definite, so $\rho^W(q,p,t)=\rho^W_*(q,p)+\Delta\rho(q,p,t)$ is not positive definite and therefore cannot be considered as a classical density. Nevertheless $\Delta\rho$ vanishes when $t\to\infty$, so that $\rho^W(q,p,t)$ is ''almost'' positive definite. Therefore with an adequate ''coarse graining,''[11] the averaged $\rho^W(q,p,t)$ may be positive definite, for $t\gg\gamma^{-1}$, and also would satisfy the classical Liouville equation, in its way towards classical equilibrium. This $\rho^W(q,p,t)$ would be a ''classical limit'' before equilibrium. We will follow this line of research elsewhere. For the moment it is clear that the nonfinal pointer basis has for limit the final pointer basis when $t\to\infty$. This fact may help us to find both the nonfinal pointer basis and the classical limit.

B. Local vs global equilibrium

In the last subsection we have considered the case where classicality is reached before classical equilibrium. In this subsection we will see this process in a different way. We can decompose the global system in a set of local systems. In these subsystems, classical local equilibrium can be reached after a time γ^{-1} with positive definite local equilibrium den-

[9]In fact, it is not possible to formulate a general theory in q, p coordinates because, in order to make the computations, we must know the relation of these coordinates with the energy and other momenta, and this is only defined in specific models.

[10]We do not write these histories in detail since they will disappear below.

[11]Or, in our language, observed by an observer space smaller than $\mathcal{O}$.

sities (and, among other things we will be able to define a classical local equilibrium entropy [17]). Since the classical local equilibrium densities of each subsystem are positive definite the classical density of the whole system will be positive definite. But the whole system is out of equilibrium and its evolution is defined by the classical Liouville equation. Now, as the system is already classical, since all its parts are classical, we can study its evolution towards equilibrium with a classical relaxation time τ, that we are supposing $\tau > \gamma^{-1}$, with the usual classical methods (and the global entropy of classical phenomenological thermodynamics will become maximal).

Of course, if the interaction is such that $\tau < \gamma^{-1}$ this process is not possible and we will directly reach the final equilibrium without a previous stage of local equilibrium. But the case $\tau > \gamma^{-1}$ is the usual one and it takes place if the global long-range interactions are unimportant with respect to the local short-range interactions. Therefore the system can be decomposed into a set of many weakly interacting subsystems that can be considered as quasi-isolated. The local interactions transform these quantum subsystems in classical subsystems in equilibrium, and the system can be described as a nonhomogeneous distribution of local subsystem with classical momenta x, $l_1, \ldots, l_N$. These momenta, and the subsystems density are different in each of them, producing a state of classical nonequilibrium that will reach equilibrium, due to the global long-range forces at a time $\tau > \gamma^{-1}$ (see [18]).

C. Final conclusion

Using the interplay of observables and states, that we have considered as functionals over the space of observables, we have found an *exact final* pointer basis and an *intrinsically* consistent set of final histories. So, given a Hamiltonian H and a state ρ we have found the exact final pointer basis $\{|x, r_1, \ldots, r_N\rangle\}$ and we have shown that ρ_*^W, the Wigner function of ρ_*, can be expanded in Wigner functions corresponding to the co-basis $(x, r_1, \ldots, r_N|$. To obtain this results or similar ones almost all the authors use coarse-graining methods based in projectors and try to obtain a limit. So they essentially use the weak limit of Eq. (17), namely,

$$\lim_{t\to\infty}(\rho(t)|O) = (\rho_*|O), \forall O \in \mathcal{O}.$$

But, at least in the classical case, we know that this weak limit exists if and only if the system is mixing [19]. And the system is mixing if it has a continuous spectrum ([2], [5], [4]) and the present paper can be considered as an extension of the theorem, which says that the mixing evolutions have a weak limit towards equilibrium [20] but now formulated in the quantum case. Thus the only way to deal with the problem (at least in the limit $t\to\infty$) in an exact way is to use a method, such as ours, specially adapted to deal with the singularities inherent to that continuous spectrum. If not we are limited to perform approximate calculations.

Nevertheless approximated methods are important and, in some cases, unavoidable to obtain the nonfinal pointer basis, but they can be better understood if they are compared with exact methods. We will continue our research following this subject.

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APPENDIX A: COMPARISON WITH THE LITERATURE

In this Appendix we would like to compare our method with those that can be found in the literature, where the models are studied using the variables Q and P.

Let us first see what is the shape of the diagonal states $(\omega_0|$, $(\omega|$ or ρ_* in Q and P. In order to determinate the diagonal states, like ρ_*, in the configuration and momentum basis (and also to find the correlations in these variables) we are forced to go to a particular model where the relation among H, Q, and P is defined. We consider a coupled system of an oscillator and a bath such that the Hamiltonian reads [21]:

$$H = \frac{1}{2}\Omega(p^2+q^2) + \frac{1}{2}\int \omega(p_\omega^2+q_\omega^2)d\omega + \lambda\int V(\omega)(qq_\omega + pp_\omega)d\omega, \tag{A1}$$

where the first term corresponds to the oscillator (with bound eigenstates $|\omega_i\rangle$ and ground state $|\omega_0\rangle$), the second term to a field, the ''bath'' (with eigenstates $|\omega_1, \omega_2, \ldots, \omega_n\rangle$), and the third is an interaction term (with a $p \leftrightarrow q$ symmetry), which leaves just one set of possible final states

$$\rho = (\omega_0| = \sum_n \int \prod d\omega_i d\omega_i' \rho_{0\omega_1,\omega_2,\ldots,\omega_n\omega_1',\omega_2',\ldots,\omega_n'} \times (\omega_0,\omega_1,\omega_2,\ldots,\omega_n\omega_1',\omega_2',\ldots,\omega_n'|, \tag{A2}$$

namely, the oscillator in the ground state and the bath in any state (see [21]). The $(\omega_0,\omega_1,\omega_2,\ldots,\omega_n\omega_1',\omega_2',\ldots,\omega_n'|$ are generated by the dressed operators that we will define in Eq. (B7).

Then let us try to get an idea of the form of ρ_* via a heuristic reasoning based on the symmetry of the Hamiltonian (A1). If we consider a quantum state ρ and the position operator Q (that symbolizes either the operator q or any of the operators q_ω) and the momentum operator P (that symbolizes either p or p_ω) in the usual case we will have

$$(\Delta Q)^2 = \mathrm{Tr}(Q^2\rho) - [\mathrm{Tr}(Q\rho)]^2 = \langle Q^2\rangle - \langle Q\rangle^2.$$

$$(\Delta P)^2 = \mathrm{Tr}(P^2\rho) - [\mathrm{Tr}(P\rho)]^2 = \langle P^2\rangle - \langle P\rangle^2. \tag{A3}$$

When ρ is a functional, we can generalize these equation as

$$(\Delta Q)^2=(\rho|Q^2)-[(\rho|Q)]^2=\langle Q^2\rangle-\langle Q\rangle^2,$$

$$(\Delta P)^2=(\rho|P^2)-[(\rho|P)]^2=\langle P^2\rangle-\langle P\rangle^2, \qquad \text{(A4)}$$

and in general $(\Delta Q)^2\neq(\Delta P)^2$. But if ρ is the diagonal state ρ_* of Eq. (18) (or the states $(\omega_0|$, $(\omega|$) we will have

$$(\rho|Q)=(\omega_0|Q), \quad (\rho|Q^2)=(\omega_0|Q^2),$$

$$(\rho|P)=(\omega_0|P), \quad (\rho|P^2)=(\omega_0|P^2). \qquad \text{(A5)}$$

So, as H has a q-p symmetry everything is symmetric under the transformation $p\leftrightarrow q$ (or $q\rightarrow -i\partial/\partial q, -i\partial/\partial q\rightarrow q$) and therefore $(\Delta Q)^2=(\Delta P)^2$. This would not be the case if ρ would not be diagonal in the basis where H is diagonal, since Q and P are not diagonal in this basis, e.g., ρ could commute with P but not with Q showing, in this case, a clear asymmetry $P\leftrightarrow Q$. Thus our diagonal states are states such that $\Delta Q=\Delta P$. Namely, for the ground state we have $\Delta p=\Delta q$, a well-known fact. If now we introduce in (A1) a small asymmetric interaction $\lambda' W$ ($\lambda'\ll 1$) we will have $\Delta Q\cong\Delta P$. On the contrary if the interaction is $\lambda' W(q,q_\omega)$ ($\lambda'\gg 1$) the Hamiltonian H can be neglected and the diagonal states will be position eigenvalues (so our results coincide with those of Refs. [1] and [22], see a detailed example below).

APPENDIX B: AN EXAMPLE OF CORRELATIONS AND LOCALIZATION

There are systems, e.g., the one of Appendix A, with variables Q and P and a bath, where the interaction is such that Q and P become correlated. Namely, the evolution makes both ΔQ and ΔP bounded, and a wave packet appears that eventually becomes a minimal uncertainty wave packet, when $t\rightarrow\infty$, then maximal localization appears in the usual way as mentioned in Sec. IV. As an example, let us now find the correlations between Q and P in the model of Hamiltonian (A1) using our method as explained above and in the paper [4]. Let us first write the Hamiltonian (A1) using creation and annihilation operators

$$H=\Omega b^\dagger b+\int d\mathbf{k}\omega_k a_\mathbf{k}^\dagger a_\mathbf{k}+\int d\mathbf{k}V_k(a_\mathbf{k}^\dagger b+b^\dagger a_\mathbf{k}),$$

$$\omega_k=k, \quad k=|\mathbf{k}|. \qquad \text{(B1)}$$

The coordinate q and the momentum p of the oscillator can be expressed as a function of the $b^\dagger$ and b as

$$q=\left(\frac{\hbar}{2m\Omega}\right)^{1/2}(b^\dagger+b); \quad p=i\left(\frac{m\hbar\Omega}{2}\right)^{1/2}(b^\dagger-b). \qquad \text{(B2)}$$

We can adimensionalize the last equation defining Q and P such that

$$q=\left(\frac{\hbar}{m\Omega}\right)^{1/2}Q, \quad p=(m\hbar\Omega)^{1/2}P. \qquad \text{(B3)}$$

Then

$$Q=\frac{1}{\sqrt{2}}(b^\dagger+b), \quad P=i\frac{1}{\sqrt{2}}(b^\dagger-b), \qquad \text{(B4)}$$

$$b=\frac{1}{\sqrt{2}}(Q+iP), \quad b^\dagger=\frac{1}{\sqrt{2}}(Q-iP). \qquad \text{(B5)}$$

In the Heisenberg representation the operator Q evolves as

$$Q(t)=\frac{1}{\sqrt{2}}[b^\dagger(t)+b(t)]$$

$$=\frac{1}{\sqrt{2}}\int d\mathbf{k}V_k\left(\frac{1}{\eta_-(k)}A_\mathbf{k}^\dagger e^{i\omega_k t}+\frac{1}{\eta_+(k)}A_\mathbf{k}e^{-i\omega_k t}\right), \qquad \text{(B6)}$$

where

$$A_\mathbf{k}^\dagger=a_\mathbf{k}^\dagger+\frac{V_k}{\eta_+(k)}\left(b^\dagger+\int\frac{d\mathbf{k}'V_{k'}a_{\mathbf{k}'}^\dagger}{\omega_k-\omega_{k'}+i0}\right),$$

$$A_\mathbf{k}=a_\mathbf{k}+\frac{V_k}{\eta_-(k)}\left(b+\int\frac{d\mathbf{k}'V_{k'}a_{\mathbf{k}'}}{\omega_k-\omega_{k'}+i0}\right). \qquad \text{(B7)}$$

The functions $\eta_-(k)$ and $\eta_+(k)$ and all the details of the calculations can be found in paper [4].

Let us consider the initial conditions

$$\langle Q\rangle_{t=0}=Q_0, \quad \langle P\rangle_{t=0}=P_0 \qquad \text{(B8)}$$

for the oscillator, and also

$$\langle a_\mathbf{k}^\dagger\rangle_{t=0}=\langle a_\mathbf{k}\rangle_{t=0}=0, \qquad \text{(B9)}$$

which corresponds to the field being initially in its ground state. Therefore

$$\langle b\rangle_{t=0}=\frac{1}{\sqrt{2}}(Q_0+iP_0), \quad \langle b^\dagger\rangle_{t=0}=\frac{1}{\sqrt{2}}(Q_0-iP_0) \qquad \text{(B10)}$$

and for the time evolution of the mean value of the coordinate and momentum of the oscillator we obtain

$$\langle Q\rangle_t=\frac{1}{\sqrt{2}}\int d\mathbf{k}\frac{V_k^2}{\eta_-(k)\,\eta_+(k)}(e^{i\omega_k t}\langle b^\dagger\rangle_{t=0}+e^{-i\omega_k t}\langle b\rangle_{t=0}), \qquad \text{(B11)}$$

$$\langle P\rangle_t=\frac{i}{\sqrt{2}}\int d\mathbf{k}\frac{V_k^2}{\eta_-(k)\,\eta_+(k)}(e^{i\omega_k t}\langle b^\dagger\rangle_{t=0}-e^{-i\omega_k t}\langle b\rangle_{t=0}). \qquad \text{(B12)}$$

The oscillating time dependent factors inside the integrals produce the vanishing of both $\langle Q\rangle_t$ and $\langle P\rangle_t$ for very long times. We can study the poles of the analytic extension of the

factor $V_k^2/\eta_-(k)\eta_+(k)$, as in [10] and prove that the trajectory of $\langle Q\rangle_t$ and $\langle P\rangle_t$ in the phase space of the oscillator is a spiral ending at $\langle Q\rangle=\langle P\rangle=0$.

Now we would like to compute Δq and Δp as a function of time. In addition to Eqs. (B8) and (B9) let us assume the following initial conditions for the oscillator:

$$\langle bb^\dagger\rangle_{t=0}=\beta,\quad \langle bb\rangle_{t=0}=\alpha,\quad \langle b^\dagger b\rangle_{t=0}=1-\beta,$$
$$\langle b^\dagger b^\dagger\rangle_{t=0}=\alpha^*, \tag{B13}$$

being α and β some arbitrary constants. If the field is in its ground state, we also have

$$\langle a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\rangle_{t=0}=\delta^3(\mathbf{k}-\mathbf{k}'),$$
$$\langle a^\dagger_{\mathbf{k}}a_{\mathbf{k}'}\rangle_{t=0}=\langle a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\rangle_{t=0}=\langle a_{\mathbf{k}}a_{\mathbf{k}'}\rangle_{t=0}=0. \tag{B14}$$

All other initial mean values of products of pairs of creation or annihilation operators are zero. This means that we have taken the oscillator in an arbitrary state and the field in the ground state as initial conditions.

Therefore we have

$$\langle A^\dagger_{\mathbf{k}}A^\dagger_{\mathbf{k}'}\rangle_{t=0}=\frac{V_kV_{k'}}{\eta_+(k)\eta_+(k')}\alpha^*,$$
$$\langle A^\dagger_{\mathbf{k}}A_{\mathbf{k}'}\rangle_{t=0}=\frac{V_kV_{k'}}{\eta_+(k)\eta_-(k')}(\beta-1),$$
$$\langle A_{\mathbf{k}}A_{\mathbf{k}'}\rangle_{t=0}=\frac{V_kV_{k'}}{\eta_-(k)\eta_-(k')}\alpha, \tag{B15}$$
$$\langle A_{\mathbf{k}}A^\dagger_{\mathbf{k}'}\rangle_{t=0}=\delta^3(\mathbf{k}-\mathbf{k}')+\frac{V_{k'}V_k}{\eta_+(k')(\omega_{k'}-\omega_k+i0)}+\frac{V_kV_{k'}}{\eta_-(k)(\omega_k-\omega_{k'}-i0)}+\frac{V_kV_{k'}}{\eta_-(k)\eta_+(k')}\beta+\frac{V_kV_{k'}}{\eta_-(k)\eta_+(k')}\times\int\frac{d\mathbf{k}''V^2_{k''}}{(\omega_k-\omega_{k''}-i0)(\omega_{k'}-\omega_{k''}+i0)}.$$

The time evolution of the mean value of $Q(t)^2$ is given by

$$\langle Q(t)^2\rangle=\frac{1}{2}\left\langle\int d\mathbf{k}V_k\left(\frac{1}{\eta_-(k)}A^\dagger_{\mathbf{k}}e^{i\omega_kt}+\frac{1}{\eta_+(k)}A_{\mathbf{k}}e^{-i\omega_kt}\right)\times\int d\mathbf{k}'V_{k'}\left(\frac{1}{\eta_-(k')}A^\dagger_{\mathbf{k}'}e^{i\omega_{k'}t}+\frac{1}{\eta_+(k')}A_{\mathbf{k}'}e^{-i\omega_{k'}t}\right)\right\rangle. \tag{B16}$$

Then, replacing Eqs. (B15) in Eq. (B16) and always using the Riemann-Lebesgue theorem we have

$$\lim_{t\to\infty}\langle Q(t)^2\rangle=\frac{1}{2}\int d\mathbf{k}\frac{V_k^2}{\eta_-(k)\eta_+(k)}=\frac{1}{2}, \tag{B17}$$

and therefore

$$\lim_{t\to\infty}[\Delta Q(t)]^2=\lim_{t\to\infty}\langle(Q(t)-\langle Q(t)\rangle)^2\rangle=\frac{1}{2}.$$

Making an analogous calculation for P, we have

$$\lim_{t\to\infty}\Delta Q=\lim_{t\to\infty}\Delta P=\frac{1}{\sqrt{2}}, \tag{B18}$$

so the wave packet around the spiral trajectory evolves to a minimal uncertainty symmetrical wave packet, showing the localization process in the usual way. This proves the presence of correlations in our model. Reestablishing the units, when $t\to\infty$, and introducing the velocity v we have

$$\Delta q=\left(\frac{\hbar^2}{2m\Omega}\right)^{1/2},\quad \Delta v=\left(\frac{\hbar^2\Omega}{2m}\right)^{1/2}. \tag{B19}$$

This fact shows that the wave packet is more peaked for big a m than for small a m. Then, in some models big mass particles can be considered as classical while other remain quantum. Moreover if $\hbar\to0$ the uncertainties disappear. Also the classical limit of the oscillator has a spiral motion in phase space. In fact, if using Eq. (25) we compute the Wigner function corresponding to the matrix density, we will find the motion of this classical density that will be centered in the spiral trajectory and having, when $t\to\infty$, a symmetrical circle of diameter $(\frac{1}{2}\hbar)^{1/2}$ as support.[12]

APPENDIX C: DECOHERENCE OF HISTORIES

From Sec. V A we can conclude that our notion of history of the system is essentially contained in the state $\rho(t)$. This history ends in the final equilibrium state ρ_*. In this Appendix we will study the relation of this notion with the usual histories formalism [18] and compare the results. The computation will turn out to be very simple for two reasons.

(i) As in all the paper we will work only in the limit $t\to\infty$, where the existence of an exact final pointer basis will make all the computations quite trivial.

(ii) Also we will only consider one space of observables $\mathcal{O}$ and therefore just one set of final consistent histories [24]. The case of many sets will be considered elsewhere.

Nevertheless we think that the results are of some interest since (i) for times $t\gg\gamma^{-1}$ all the exact result obtained in the limit $t\to\infty$ can be considered as good approximations; (ii) the existence of a space of observables $\mathcal{O}$, where we can use

[12]The cosmological models of papers [23] are other examples that we will further develop elsewhere.

the Riemann-Lebesgue theorem, perhaps can be considered as a selection principle to choose the physically relevant consistent set [25].

So let us begin giving the main definitions. Let us consider a time dependent basis of $\mathcal{H}$: $\{|\alpha(t)\rangle\}$, and the projectors

$$P_\alpha(t)=|\alpha(t)\rangle\langle\alpha(t)|, \tag{C1}$$

such that they represent exhaustive and exclusive alternatives

$$\sum_\alpha P_\alpha=1, \quad P_\alpha P_\beta=\delta_{\alpha\beta}P_\alpha. \tag{C2}$$

We will call (fine-grained) *history* $\vec{\alpha}$ to a string of time dependent projectors:[13]

$$C_{\vec{\alpha}}=P_{\alpha_1}(t_1),\ldots,P_{\alpha_n}(t_n), \quad t_1<\cdots<t_n. \tag{C3}$$

For a state ρ we will call *decoherence matrix*

$$\begin{aligned} M(\vec{\alpha},\vec{\alpha}')&=C^\dagger_{\vec{\alpha}}\rho C_{\vec{\alpha}'} \\ &=P_{\alpha_n}(t_n),\ldots,P_{\alpha_1}(t_1)\rho P_{\alpha_1'}(t_1'),\ldots,P_{\alpha_n'}(t_n'). \end{aligned} \tag{C4}$$

We introduce this matrix because we consider it as the natural generalization of the usual density matrix to the case where single projectors are changed by histories.

We will call *decoherence functional*

$$D(\vec{\alpha},\vec{\alpha}')=\mathrm{Tr}\,M(\vec{\alpha},\vec{\alpha}'), \tag{C5}$$

which would be the generalization of the trace of an ordinary matrix.[14]

We will call *candidate probability for the history* $\vec{\alpha}$

$$p(\vec{\alpha})=\mathrm{Tr}\,M(\vec{\alpha},\vec{\alpha}), \tag{C6}$$

which would be the generalization of the usual probability. It is only a ''candidate probability'' because, at this stage, it does not satisfy the axioms of the usual Boolean probability theory.

If

$$\mathrm{Re}\,D(\vec{\alpha},\vec{\alpha}')=0 \tag{C7}$$

for $\vec{\alpha}\neq\vec{\alpha}'$, we will say that the set of histories is *consistent or weakly decoherent*. In this case it is proved that the set can be in principle submitted to the ordinary boolean logic [27], and the candidate probability can be considered as the probability of each history.

If

$$D(\vec{\alpha},\vec{\alpha}')=0 \tag{C8}$$

for $\vec{\alpha}\neq\vec{\alpha}'$, we will say that the set has *medium decoherence*. Theorems about records can be proved if the set of histories has this type of decoherence [26].

If

$$M(\vec{\alpha},\vec{\alpha}')=0 \tag{C9}$$

for $\vec{\alpha}\neq\vec{\alpha}'$ we will say that the set is *intrinsically consistent* [1] or that it has *matrix decoherence*. Of course matrix decoherence implies medium decoherence, and medium decoherence implies weak decoherence.

Let us now compare all these concepts with our formalism. We choose:

$$|\alpha(t_1)\rangle=|x,r_1,\ldots,r_N;t_1\rangle, \tag{C10}$$

where we have used the shorthand notation introduced above. The set of operators $P_{\alpha(t)}=|\alpha(t)\rangle\langle\alpha(t)|=|x,r_1,\ldots,r_N;t\rangle\langle x,r_1,\ldots,r_N;t|$ will be our ''final pool of operators'' if we use the language of [12]. The evolution of these operators will be

$$\begin{aligned} P_\alpha(t)&=e^{-iH(t-t_1)}P_\alpha(t_1)e^{iH(t-t_1)} \\ &=e^{-ix(t-t_1)}P_\alpha(t_1)e^{ix(t-t_1)} \\ &=P_\alpha(t_1)=P_\alpha=|\alpha(0)\rangle\langle\alpha(0)|. \end{aligned} \tag{C11}$$

i.e., these operators are constant. Then the projectors are time constant and

$$C_{\vec{\alpha}}=P_\alpha, \tag{C12}$$

and these histories can be labeled with the ordinary α instead of the $\vec{\alpha}$ with the arrow.

In more detail let us first study our ''pool'' of projectors to compute Eq. (C4) in our formalism and when $t\to\infty$,

$$P_{\alpha(t)}=P_\alpha=|x,r_1,\ldots,r_N\rangle\langle x,r_1,\ldots,r_N|=|x,r_1,\ldots,r_N). \tag{C13}$$

(1) $r_1,\ldots,r_N$ are discrete indices and the final stationary state ρ_* is diagonal in these indices, so this part of the problem is trivial.

(2) x symbolizes (ω_0,ω) where only ω is continuous, so the treatment of ω_0 is also trivial.

The problem is only ω so, for simplicity, let us only consider this index. The projector reads:

$$P_\omega=|\omega\rangle\langle\omega|=|\omega). \tag{C14}$$

So let us compute

$$P_\omega\rho_* P_{\omega'}=|\omega\rangle\langle\omega|\rho_*|\omega'\rangle\langle\omega'|, \tag{C15}$$

but first we must find the meaning of this symbol. In the discrete case we have

$$|a\rangle\langle b|\rho|c\rangle\langle d|=|a\rangle\mathrm{Tr}(\rho|c\rangle\langle b|)\langle d|, \tag{C16}$$

[13]We can consider a more general case were the exclusive and exhausting set of histories is different at every time t_i and therefore the projectors are $P^i_{\alpha_i}(t_i)$. But this is not the usual case.

[14]If some of the α are continuous indices, for them we must use the generalization of the trace introduced in [2].

which can be generalized to the continuous case as

$$|a\rangle\langle b|\rho|c\rangle\langle d| = |a\rangle(\rho||c\rangle\langle b|)\langle d|. \tag{C17}$$

Thus

$$P_\omega \rho_* P_{\omega'} = ||\omega\rangle\langle\omega|\left[\int \rho_{\omega''}(\omega''|d\omega'']|\omega'\rangle\langle\omega'|\right.$$
$$= |\omega\rangle\left[\int \rho_{\omega''}(\omega''|\omega',\omega)d\omega''\right]\langle\omega'|. \tag{C18}$$

So, from Eqs. (11) we have (1) If $\omega \neq \omega'$, it is $P_\omega \rho_* P_{\omega'} = 0$. (2) If $\omega = \omega'$, it is

$$P_\omega \rho_* P_{\omega'} = |\omega\rangle\left[\int \rho_{\omega''}(\omega''|\omega)d\omega''\right]\langle\omega|$$
$$= |\omega\rangle\left[\int \rho_{\omega''}\delta(\omega''-\omega)d\omega''\right]\langle\omega|$$
$$= \rho_\omega|\omega\rangle\langle\omega|. \tag{C19}$$

So with a symbolic obvious notation (that we will use from now on) we can say that

$$P_\omega \rho_* P_{\omega'} = |\omega\rangle\rho_\omega\delta_{\omega\omega'}\langle\omega'|. \tag{C20}$$

If now we repeat the reasoning including all the trivial discrete indices we will obtain the same result since ρ_* is diagonal in these indices. Then, when $t\to\infty$ we have that

$$M(\vec{\alpha},\vec{\alpha}') \to \delta_{\alpha\alpha'}\rho_\alpha|\alpha\rangle\langle\alpha| \tag{C21}$$

and therefore we have final matrix decoherence in a time long enough. Then we have found the final ''statistical classical domain or realm'' of Gell-Mann and Hartle. In this way final classical behavior emerges from quantum behavior and transition (a) of the Introduction appears in the histories formalism. Essentially we have used the weak limit of Eq. (19) and the fact that it is the only possible limit we can use, since ρ is a functional over the space $\mathcal{O}$. But the choice of Eq. (C10) has an extra bonus: it decomposes the density matrix just in the way that was announced in the Introduction.

From the matrix decoherence we have medium decoherence and weak decoherence, so we have proved that any quantum system, fulfilling the conditions required in Sec. II, has a set of final intrinsically consistent histories, the essential conditions being the continuous spectrum and the existence of just one ground state. Classically these histories will be the $\rho^W_{xr}(q,p)$ of Eq. (46). This exact final decoherence has being obtained using the basis $\{|xr\rangle\}$, other near bases obviously yield final approximate decoherence. Also basis $\{|xr\rangle\}$ will give approximate decoherence in a time long enough.

But we must observe that in all cases where $P_\alpha(t) = P_\alpha = \text{const}$ [even if Eq. (C10) is not satisfied] we can immediately prove medium decoherence with no reference to matrix decoherence. In fact, if $P_\beta = |\beta\rangle\langle\beta| = \text{const}$ we have

$$D(\vec{\beta},\vec{\beta}') = D(\beta,\beta')$$
$$= \mathrm{Tr}(|\beta\rangle\langle\beta|\rho|\beta'\rangle\langle\beta'|)$$
$$= \langle\beta|\beta'\rangle\langle\beta|\rho|\beta'\rangle$$
$$= \delta_{\beta\beta'}p(\beta). \tag{C22}$$

These would be the case with the P_α of this section and also for any constant P_β. This result seems very trivial but it is not. The essential property of projector (C13) is that it is time constant, but our formalism contains other time-constant projectors. If we go back to Sec. II A we find

$$P_{\beta(t)} = P_\beta = |x,m_1,\ldots,m_N\rangle\langle x,m_1,\ldots,m_N|$$
$$= |x,m_1,\ldots,m_N), \tag{C23}$$

namely, the projectors related with the basis $|x,m\rangle$ before the diagonalization (20) that yields the basis $|x,r\rangle$. The P_β are also time constants and yield medium decoherence (only the P_a yield matrix decoherence). The main fact is that in order to reach the classical statistical mechanics of Sec. III we must use the basis $|x,r\rangle$ that diagonalize ρ_* in *all indices* [see Eqs. (30)–(36)]. Thus, since our demonstration is based in the matrix decoherence in the basis $|x,r\rangle$, these objects are essential for us. Only after this demonstration we can speak of classical constants of the motion and classical trajectories because only then we can pass from the quantum formulas to the classical ones.

Then the last result can be translated as follows.

(1) There is final matrix decoherence between any pair of different sets of constants (x,r) i.e., between any pair of sets of classical trajectories in the phase space. This set of sets of trajectories is intrinsically consistent [see Eq. (21)].

(2) But, of course, any set of functions of the ''r,'' such as the ''m,'' will define equally well the set of classical trajectories. But the ''m'' do not provide a basis with good defined probabilities, as the ''r'' does, since in the basis ''m'' the ρ_* is not diagonal [see Eq. (18)]. In this case the set of histories is consistent but not intrinsically consistent.

So our point of view is that, even if all sets endowed of medium decoherence can be considered as consistent sets, there is only one with physical importance, the one with matrix decoherence, the only one which is an ''intrinsically consistent set.'' This idea may help to find the selection principle searched for in [25].

Finally, if the potential and the initial conditions are such to privilege a history (as in Appendix B) the locations process (b) of the Introduction will take place and we will have a unique classical object with a unique history P_α. Then we would find the final ''classical domain or realm'' of Gell-Mann and Hartle.

We will end this section showing how several requirements necessary for a efficient histories decoherence are satisfied by our formalism.

1. Griffiths-Omnès condition

The Griffiths-Omnès condition for consistency [27], [28] is automatically satisfied since

$$\mathrm{Re\,Tr}[|\alpha\rangle\langle\alpha|\rho(1-|\alpha\rangle\langle\alpha|)|\alpha\rangle\langle\alpha|]=0. \qquad \text{(C24)}$$

2. Permanence of the past

If we take our projectors from the pool of the projectors $|\alpha\rangle\langle\alpha|$ the condition of permanence of the past [12] is trivially satisfied, since a chain with an certain number of $|\alpha\rangle\langle\alpha|$ can only be continued repeating this projector. This is the most important property required in [25].

3. Insensitivity

While quantum states are modified by the measurement processes, classical states are not sensitive to these measurements. This property of classical states is called insensitivity [1]. The projector $P_{\alpha_i}=|\alpha_i\rangle\langle\alpha_i|$ can be considered as a measurement operator, so if ρ_{before} is the state before the measurement and ρ_{after} is the state after the measurement, we will have

$$\rho_{after}=\sum_i P_{\alpha_i}\rho_{before}P_{\alpha_i}=\sum_i |\alpha_i\rangle\langle\alpha_i|\rho_{befor}|\alpha_i\rangle\langle\alpha_i|$$

$$=\sum_i P_{\alpha_i}\rho_{before}P_{\alpha_i}, \qquad \text{(C25)}$$

where p_i is the probability to measure α_i. Now if, after the decoherence process, ρ_{before} is a diagonal matrix, precisely ρ_*, i.e.,

$$\rho_{before}=\sum_i p_i|\alpha_i\rangle\langle\alpha_i| \qquad \text{(C26)}$$

and we only measure the observers in the CSCO $\{H,P_1,\ldots P_n\}$, so the P_{α_i} are just the $P_\alpha=|\alpha\rangle\langle\alpha|$. We have

$$\rho_{after}=\sum_i |\alpha_i\rangle\langle\alpha_i|\left(\sum_j p_j|\beta_j\rangle\langle\beta_j|\right)|\alpha_i\rangle\langle\alpha_i|$$

$$=\sum_i p_i|\alpha_i\rangle\langle\alpha_i|=\rho_{before}. \qquad \text{(C27)}$$

In fact, the matrix ρ_* is insensitive to the measurement of the CSCO $\{H,P_1,\ldots P_n\}$ (and also the CSCO $\{H,O_1,\ldots,O_N\}$ where the operators O are related with the constants m). This is the maximum insensitivity we can get.

4. Strong decoherence and records

If for any history $\vec{\alpha}$ there is a projector R_α such that $\{R_\alpha\}$ is not necessarily a complete set of projectors in $\mathcal{H}$, in the sense that $\{R_\alpha|\psi\rangle\}$ is not necessarily a basis of $\mathcal{H}$, and for any state ρ it is

$$C_{\vec{\alpha}}\rho=R_\alpha\rho, \quad R_\alpha R_\beta=\delta_{\alpha\beta}P_\alpha. \qquad \text{(C28)}$$

We will say that we have *strong decoherence* ([29], Eq. (2.4)). As the R_α are timeless entities and as $C_{\vec{\alpha}}\to R_\alpha$, R_α can be considered as the *record* of the history $\vec{\alpha}$, R_α can also be considered the record of not one but several decohered histories, associated by unitary transformations [12]. So really R_α is the record of an equivalent class of histories.

It is clear that if these records exist we have medium decoherence. In fact,

$$D(\vec{\alpha},\vec{\alpha}')=\mathrm{Tr}(C^\dagger_{\vec{\alpha}}\rho C_{\vec{\alpha}'})$$

$$=\mathrm{Tr}(R_\alpha\rho R_{\alpha'})$$

$$=\mathrm{Tr}(\rho R_{\alpha'}R_\alpha)$$

$$=\delta_{\alpha\alpha'}p(\vec{\alpha}). \qquad \text{(C29)}$$

So strong decoherence implies medium decoherence.

In our case these final R_α exist and they are

$$R_\alpha=|\alpha\rangle\langle\alpha|=|xr\rangle\langle xr|=P_\alpha. \qquad \text{(C30)}$$

Thus the numbers $x,r_1,\ldots,r_N$ can be considered as the record of the corresponding final history.

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