Unitary relation between a harmonic oscillator of time-dependent frequency and a simple harmonic oscillator with and without an inverse-square potential

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The unitary operator that transforms a harmonic oscillator system of time-dependent frequency into that of a simple harmonic oscillator of different timescale is found, with and without an inverse-square potential. It is shown that for both cases, this operator can be used in finding complete sets of wave functions of a generalized harmonic oscillator system from the well-known sets of the simple harmonic oscillator. Exact invariants of the time-dependent systems can also be obtained from the constant Hamiltonians of unit mass and frequency by making use of this unitary transformation. The geometric phases for the wave functions of a generalized harmonic oscillator with an inverse-square potential are given.

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It is certainly of importance to find a complete set of wave functions for a system of the time-dependent Hamiltonians [1–17]. It has long been known that a harmonic oscillator of time-dependent frequency with or without an inverse-square potential is a system of practical applications (see, e.g., Ref. [3]), where the wave functions are described in terms of solutions of the classical equation of motion of the oscillator without the inverse-square potential [4–8]. In Ref. [9] it has been shown that, for a generalized harmonic oscillator system, the kernel of the system is determined by the classical action. This is one of the basic reasons for the fact that the wave functions are described by the classical solutions. On the other hand, it has long been noticed that there exist classical canonical transformations that relate the (driven) harmonic oscillators of different parameters (see, e.g., Refs. [8,10]). Recently, in Ref. [11], it has been shown that a driven harmonic oscillator of time-dependent frequency is related, through canonical transformations, to the simple harmonic oscillator of unit mass and unit frequency but with a different timescale [12,10]. This fact has been used to find the wave functions of a driven system that exactly agree with the known results [9,13].

In this Brief Report, we will show that for both oscillators with and without an inverse-square potential, there is a unitary operator that transforms the harmonic oscillator systems of time-dependent frequency into those of the unit-mass and unit-frequency oscillators with different timescales. This unitary operator can be used to find complete sets of wave functions of the systems with time-dependent parameters from the well-known sets of wave functions of the simple harmonic oscillator with [14] or without an inverse-square potential. It has been known that [4–8] there exist *exact* invariants in the systems of time-dependent parameters that have long been used to find the wave functions [4–8,15–17]. As might have been implied by the classical treatments through canonical transformations, it will also be shown that the *exact* quantum-mechanical invariants in oscillator systems of *time-dependent* parameters can be obtained from the *constant* Hamiltonians of unit mass and frequency (which, certainly, are invariants in their systems, respectively), through the unitary transformation given here and those in Refs. [9,13].

The unit-mass harmonic oscillator of time-dependent frequency $w_0(t)$ is described by the Hamiltonian

$$H_0(x,p,t)=p^2/2+[w_0^2(t)/2]x^2, \tag{1}$$

with the classical equation of motion

$$\ddot{x}+w_0^2(t)x=0. \tag{2}$$

If we denote the two linearly independent solutions of Eq. (2) as $u_0(t)$ and $v_0(t)$, the $\rho_0(t)$ defined by $\rho_0(t)=\sqrt{u_0^2+v_0^2}$ should satisfy

$$(d^2/dt^2)\,\rho_0+w_0^2(t)\rho_0-\Omega_0^2/\rho_0^3=0, \tag{3}$$

with a time-constant $\Omega_0(\equiv \dot{v}_0u_0-\dot{u}_0v_0)$. Without losing generality, we assume that Ω_0 is positive. The wave functions $\psi_n^0(x,t)$ of the system should satisfy the Schrödinger equation

$$O_0(t)\psi_n^0(x,t)=0, \tag{4}$$

where $O_0(t)=-ih(\partial/\partial t)+H_0(x,p,t)$. For the simple harmonic oscillator system of unit mass and frequency whose time is τ, the wave functions $\psi_n^s(x,\tau)$ should satisfy

$$O_s(\tau)\psi_n^s(x,\tau)=0, \tag{5}$$

where $O_s(\tau)=-ih(\partial/\partial\tau)+H^s$ with $H^s=\frac{1}{2}(p^2+x^2)$. If t and τ are related through the relation

$$d\tau=(\Omega_0/\rho_0^2)\,dt, \tag{6}$$

by defining the unitary operator

$$U_{w0}(\rho_0,\Omega_0)=\exp[(i\dot{\rho}_0/2\hbar\rho_0)\,x^2]\times\exp[-(i/2\hbar)\,(\ln\rho_0/\sqrt{\Omega_0})(xp+px)], \tag{7}$$

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one may find the relation

$$(\Omega_0/\rho_0^2)\, U_{w0} O_s(\tau) U_{w0}^{\dagger}\big|_{\tau=\tau(t)} = O_0(t). \tag{8}$$

In Eq. (7), the overdot denotes the differentiation with respect to time t, while in Eq. (8) the notation "$|_{\tau=\tau(t)}$" is to mention that τ should be replaced by the function of t satisfying the relation (6). In a different vein, the relation (8) has also been noticed in Ref. [18]. Eqs. (5) and (8) imply the following relation in wave functions;

$$\psi_n^0(x,t) = U_{w0}\psi_n^s\big|_{\tau=\tau(t)}. \tag{9}$$

As is well known [1], the simplest choice of $\{\psi_n^s | n = 0,1,2,\ldots\}$ may be given as

$$\psi_n^s(x,\tau)\big|_{\tau=\tau(t)} = \frac{1}{(2^n n!\sqrt{\pi\hbar})^{1/2}} e^{-i(n+1/2)\tau} \times \exp\left[-\frac{x^2}{2\hbar}\right] H_n\left(\frac{1}{\sqrt{\hbar}}x\right)\Bigg|_{\tau=\tau(t)} \tag{10}$$

$$= \frac{1}{(2^n n!\sqrt{\pi\hbar})^{1/2}} \left(\frac{u_0(t) - i v_0(t)}{\rho_0(t)}\right)^{n+1/2} \times \exp\left[-\frac{x^2}{2\hbar} + i c_0\right] H_n\left(\frac{1}{\sqrt{\hbar}}x\right), \tag{11}$$

where c_0 is an arbitrary real number that will be set to zero from now on. In obtaining Eq. (11), we make use of the fact that

$$d\tau = (\Omega_0/\rho_0^2)\, dt = i[(\dot{u}_0 - i\dot{v}_0)/(u_0 - i v_0) - \dot{\rho}_0/\rho_0]dt. \tag{12}$$

In order to find a general expression of $\psi_n^0(x,t)$, we consider another unitary transformation. By defining $\delta_{u_1}(t)$ through the relations

$$\dot{\delta}_{u_1} = \tfrac{1}{2} w_0^2 u_1^2 - \tfrac{1}{2}\dot{u}_1^2, \tag{13}$$

where u_1 is a linear combination of $u_0(t)$ and $v_0(t)$, one may find that the unitary operator U_f given as [13]

$$U_f = \exp\{(i/\hbar)\,[\dot{u}_1 x + \delta_{u_1}(t)]\}\exp[-(i/\hbar)\, u_1 p] \tag{14}$$

satisfies the following relation:

$$U_f O_0 U_f^{\dagger} = O_0. \tag{15}$$

Therefore, the wave functions ψ_n^0 satisfying the Schrödinger equation of Eq. (4) may in general be written as

$$\psi_n^0(x,t) = U_f U_{w0}\psi_n^s(x,\tau)\big|_{\tau=\tau(t)} = \frac{1}{\sqrt{2^n n!\,\rho_0(t)}}\left(\frac{\Omega_0}{\pi\hbar}\right)^{1/4}\left(\frac{u_0(t) - i v_0(t)}{\rho_0(t)}\right)^{n+1/2} \times \exp\left[\frac{i}{\hbar}[\dot{u}_1(t)x + \delta_{u_1}(t)]\right] \times \exp\left[\frac{[x-u_1(t)]^2}{2\hbar}\left(-\frac{\Omega_0}{\rho_0^2(t)} + i\frac{\dot{\rho}_0}{\rho_0}\right)\right] \times H_n\left(\sqrt{\frac{\Omega_0}{\hbar}}\frac{x-u_1(t)}{\rho_0(t)}\right). \tag{16}$$

This wave function, of course, agrees with the known one [9,13,11] if we consider u_1 as a (fictitious) particular solution. If $u_1 = 0$, the wave function given in Eq. (16) also agrees with that in Refs. [7,15–17].

It may be interesting to find how many free parameters are in the wave function $\psi_n^0(x,t)$. First, there are two parameters in determining $u_1(t)$. In the case of $u_1 = 0$, one may think that there are four parameters that come from determining $u_0(t), v_0(t)$. However, one of them is not a free parameter, since the wave functions are invariant under the multiplication of $u_0(t)$ and $v_0(t)$ with the same constant factor. For the simple harmonic oscillator of time-translational symmetry, one of the remaining three parameters of $u_1 = 0$ is simply related to a time shifting of the wave function. This can be seen from the fact that, for the unit frequency case, the u_0 and v_0 can be taken as $\cos(t+t_0)$ and $C\sin(t+\beta+t_0)$, respectively, with real constants C, β, t_0.

If one considers $\rho_s(\tau)$ satisfying

$$\frac{d^2}{d\tau^2}\rho_s + \rho_s - \frac{\Omega_s^2}{\rho_s^3} = 0, \tag{17}$$

and a simple harmonic oscillator of unit mass and frequency and with time τ' that is related to τ as

$$d\tau' = \frac{\Omega_s}{\rho_s^2} d\tau,$$

by defining

$$U_s = \exp\left(\frac{i d\rho_s/d\tau}{2\hbar\rho_s}x^2\right)\exp\left[-\frac{i}{2\hbar}\left(\ln\frac{\rho_s}{\sqrt{\Omega_s}}\right)(xp + px)\right], \tag{18}$$

one may find that

$$(\Omega_s/\rho_s^2)\, U_s O_s(\tau') U_s^{\dagger}\big|_{\tau'=\tau'(\tau)} = O_s(\tau). \tag{19}$$

The wave functions $\tilde{\psi}_n^s(\tau)$ defined by

$$\tilde{\psi}_n^s(\tau) \equiv U_s \psi_n^s(x,\tau')\big|_{\tau'=\tau'(\tau)}$$

then satisfy the Schrödinger equation $O_s(\tau)\tilde{\psi}_n^s = 0$. In fact, $\tilde{\psi}_n^s(\tau)$ is closely related to the wave functions of the squeezed states [19,20,2].

One may think that a more general expression of the unitary operator, U_{w0}, may be obtained by combining use of U_{w0} and U_s. This, however, is not the case as can be seen from the relation

$$U_{w0}(\rho_0,\Omega_0)U_s|_{\tau=\tau(t)}=U_{w0}(\rho_0\rho_s,\Omega_0\Omega_s)|_{\tau=\tau(t)}, \quad (20)$$

which is in accordance with the number counting of free parameters in $\psi_n^0(x,t)$.

The harmonic oscillator of unit mass and frequency with an inverse-square potential is described by the Hamiltonian [14]

$$H_{in}^s=p^2/2+x^2/2+g/x^2. \quad (21)$$

We only consider the case of $g>-\hbar^2/8$, and the region of $x>0$. By defining $\alpha=\frac{1}{2}(1+8g/\hbar^2)^{1/2}$ and

$$O_s^{in}(\tau)=-i\hbar\,(\partial/\partial\tau)+H_{in}^s, \quad (22)$$

the wave function ϕ_n^s satisfying the Schrödinger equation $O_s^{in}(\tau)\phi_n^s=0$ is given as [14]

$$\begin{aligned}\phi_n^s&\equiv\langle x|\phi_n^s\rangle\\&=\left(\frac{4}{\hbar}\right)^{1/4}\left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2}e^{-i(2n+\alpha+1)\tau}\\&\quad\times\left(\frac{x^2}{\hbar}\right)^{(2\alpha+1)/4}\exp\left(-\frac{x^2}{2\hbar}\right)L_n^\alpha\left(\frac{x^2}{\hbar}\right). \quad (23)\end{aligned}$$

By defining O_0^{in} as

$$O_0^{in}(t)=-i\hbar\frac{\partial}{\partial t}+\frac{p^2}{2}+w_0^2(t)\frac{x^2}{2}+\frac{g}{x^2}, \quad (24)$$

as in the case without the inverse-square term, one may find the relation

$$(\Omega_0/\rho_0^2)\,U_{w0}O_s^{in}(\tau)U_{w0}^\dagger|_{\tau=\tau(t)}=O_0^{in}(t). \quad (25)$$

In deriving Eq. (25), we make use of the commutator relation

$$\left[xp+px,\frac{1}{x^2}\right]=4i\hbar\frac{1}{x^2}. \quad (26)$$

For a further generalization, we define a unitary operator

$$U_g=\exp\left[\frac{i}{\hbar}\left(Max^2-\frac{\dot{M}}{4}\right)x^2\right]\exp\left[i\frac{\ln M}{4\hbar}(xp+px)\right], \quad (27)$$

where M is a positive function of t, and $a(t)$ is a real function. One may then easily find the relation

$$\begin{aligned}U_gO_0^{in}U_g^\dagger&\equiv O^{in} \quad (28)\\&=-i\hbar\,(\partial/\partial t)+H_{in}, \quad (29)\end{aligned}$$

where (see Ref. [9])

$$H_{in}=\frac{p^2}{2M(t)}-a(t)(xp+px)+\frac{1}{2}M(t)c(t)x^2+\frac{g}{M(t)}\frac{1}{x^2} \quad (30)$$

with

$$c(t)=w_0^2(t)+\frac{1}{\sqrt{M}}\frac{d^2\sqrt{M}}{dt^2}+4a^2-2\frac{1}{M}\frac{d}{dt}(Ma).$$

For convenience [13], we consider the equation

$$(d/dt)\,(M\dot{x})+w^2(t)x=0, \quad (31)$$

where $w^2(t)=w_0^2(t)+(1/\sqrt{M})(d^2\sqrt{M}/dt^2)$. The two linearly independent solutions $u(t),v(t)$ of Eq. (31) can be given from $u_0(t),v_0(t)$ as $u(t)=u_0/\sqrt{M}$, $v(t)=v_0/\sqrt{M}$, so that one may find the relation $\Omega_0=M(\dot{v}u-\dot{u}v)$. We also define the $\rho(t)$ as $\rho(t)=\rho_0/\sqrt{M}$. The wave function ϕ_n of the system described by the Hamiltonian $H_{in}(x,p,t)$ can then be obtained as

$$\begin{aligned}\phi_n&=U_G\phi_n^s(\tau)|_{\tau=\tau(t)} \quad (32)\\&=\left(\frac{4\Omega_0}{\hbar\rho^2}\right)^{1/4}\left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2}\\&\quad\times\left(\frac{u-iv}{\rho}\right)^{(2n+\alpha+1)}\left(\frac{\Omega_0x^2}{\hbar\rho^2}\right)^{(2\alpha+1)/4}\\&\quad\times\exp\left[-\frac{x^2}{2\hbar}\left(\frac{\Omega_0}{\rho^2}-iM\frac{\dot{\rho}}{\rho}-2iMa\right)\right]L_n^\alpha\left(\frac{\Omega_0x^2}{\hbar\rho^2}\right), \quad (33)\end{aligned}$$

where

$$U_G=U_gU_{w0}. \quad (34)$$

For $a=0$, the wave functions ϕ_n agree with those in Refs. [7,17]. As in Ref. [9], by considering the kernel of the system [7], it may be easy to see that the wave functions $\phi_n(x,t)$ form a complete set. The form of ϕ_n in Eq. (33) indicates that, even for the system described by the constant Hamiltonian H_{in}^s given in Eq. (21), there are wave functions whose probability density distributions pulsate as in those of the squeezed states.

For the system of the Hamiltonian H_{in}, if $M(t),w_0^2(t)$, and $a(t)$ are periodic with a period T, one may study the nonadiabatic geometric phases [21,22]. The wave function ϕ_n is (quasi)periodic, only if $\rho(t)$ is periodic. The condition for periodic $\rho(t)$ with the period $T'(=T$ or $2T)$ has been analyzed in Ref. [23]. Here, we only consider the case of such a periodic $\rho(t)$. The overall phase change of ϕ_n under the T' evolution is given as

$$\chi_n=-(2n+\alpha+1)\int_0^{T'}\frac{\Omega_0}{\rho_0^2(t)}dt.$$

The expectation value of the H_{in} can be evaluated by making use of the relation

$$H_{in}\phi_n=\left(i\hbar\frac{\partial}{\partial t}U^G\right)\phi_n^s+U_G i\hbar\frac{\partial}{\partial t}\phi_n^s. \tag{35}$$

From the fact that

$$i\hbar\frac{\partial}{\partial t}\phi_n^s=i\hbar\frac{d\tau}{dt}\frac{\partial}{\partial\tau}\phi_n^s=(2n+\alpha+1)\hbar\frac{\Omega_0}{\rho_0^2(t)}\phi_n^s, \tag{36}$$

one may find that the geometric phase γ_n for the wave function ϕ_n under the T' evolution is written as

$$\begin{aligned}\gamma_n&=\chi_n+\frac{1}{\hbar}\int_0^{T'}\langle\phi_n|H_{in}|\phi_n\rangle dt\\&=\frac{1}{\hbar\Omega_0}\int_0^{T'}(M\dot{\rho}^2+2Ma\rho\dot{\rho})dt\langle\phi_n^s|x^2|\phi_n^s\rangle\\&=(2n+\alpha+1)\frac{1}{\Omega_0}\int_0^{T'}(M\dot{\rho}^2+2Ma\rho\dot{\rho})dt.\end{aligned} \tag{37}$$

The unitary operators can be used in finding the *exact* invariants for the cases without and with the inverse-square potential from H^s and H_{in}^s, respectively. First of all, it is clear that H^s and H_{in}^s are invariants in the systems they describe, respectively. For the system described by $H_0(x,p,t)$, if we only consider the case of $u_1=0$, the invariant I_0 is obtained by applying the unitary transformation to the invariant H^s,

$$\begin{aligned}I_0&=U_{w0}H^sU_{w0}^\dagger\\&=\frac{1}{2\Omega_0}\left[\left(\frac{\Omega_0 x}{\rho_0}\right)^2+(\rho_0 p-\dot{\rho}x)^2\right],\end{aligned} \tag{38}$$

which agrees with those in Refs. [4–7,15,16]. For the system described by $H_{in}(x,p,t)$, the invariant is again given from the invariant H_{in}^s as

$$\begin{aligned}I_{in}=U_GH_{in}^sU_G^\dagger=\frac{1}{2\Omega_0}\Bigg[&\left(\frac{\Omega_0 x}{\rho}\right)^2\\&+\{\rho p-(M\dot{\rho}+2Ma\rho)x\}^2+2\rho^2\frac{g}{x^2}\Bigg].\end{aligned} \tag{39}$$

For the case of $a=0$, the invariant I_{in} reduces to the known one [17]. One can explicitly check that the invariant I_{in} indeed satisfies the relation

$$i\hbar\,(\partial I_{in}/\partial t)+[I_{in},H_{in}]=0. \tag{40}$$

Alternatively, making uses of Eqs. (35) and (36) and relying on the completeness of the set $\{\phi_n^s|\ n=0,1,2,\ldots\}$, a simple proof of Eq. (40) may also be possible.

In summary, we have found a unitary operator that transforms a harmonic oscillator system of time-dependent frequency into that of a simple harmonic oscillator of different timescale, with and without the inverse-square potential. Making use of the unitary operator, the exact invariants and wave functions of the time-dependent systems have been evaluated from the well-known results in the corresponding system of constant Hamiltonians. It should be mentioned, however, that the classical solutions of the time-dependent harmonic oscillator system must be found for actual applications, while the classical equation [see Eq. (2)] is formally equivalent to a one-dimensional time-independent Schrödinger equation (of arbitrary potential). The classical correspondent of unitary transformation is the canonical transformation that has been studied in the model [11,10]. It would be interesting if the relationship could be used in finding relations among the quantities in classical and quantum mechanics such as that between the geometric phases and the Hannay's angle (see Ref. [23]).

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