Higher-order conductivity corrections to the Casimir force

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The finite conductivity corrections to the Casimir force in two configurations are calculated in the third and fourth orders in the relative penetration depth of electromagnetic zero oscillations into the metal. The obtained analytical perturbation results are compared with recent computations. Applications to the experiments on precision Casimir force measurements are discussed.

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Considerable recent attention has been focused on experimental investigation of the Casimir force between metallic surfaces [1–5]. In addition, Casimir energies for some pairs of conductors of different shape were studied theoretically [6]. The obtained experimental results and the extent of their agreement with theory were used to establish stronger constraints for the parameters of hypothetical long-range interactions predicted by the unified gauge theories, supersymmetry, and supergravity [7–9].

To be confident that data fit theory at a level of about several percent, a variety of corrections to the ideal expression for the Casimir force should be taken into account. The main contribution is given by the corrections due to finite conductivity of the boundary metal, its roughness, and that due to nonzero temperature (see [10] for review).

The subject of the present paper is the analytical calculation of higher-order finite conductivity corrections to the Casimir force in relative penetration depth of electromagnetic zero oscillations into the metal. We consider configurations of two plane-parallel plates and a sphere above a plate. The first-order finite conductivity correction was found in [11] for configuration of two plane-parallel plates (see also [12]). Later this result was recalculated in [13]. Second-order correction was first found in [14] (see also [10]). The first- and second-order corrections were modified for the configuration of a sphere above a disk in [1] and [15], respectively, by the use of the proximity force theorem (PFT) [16]. The results for the Casimir force up to the second power in the relative penetration depth are now commonly used when discussing the recent experiments. They are not sufficient, however, when an accuracy of about several percent is needed. In [5,8] the third- and the fourth-order corrections were obtained approximately from an interpolation formula. They allowed to achieve the excellent agreement between theory and experiment [5]. But the exact values of the third- and fourth-order corrections remained unknown.

In [17], numerical calculations of the Casimir force with account of finite conductivity have been attempted based on the tabulated data for the complex refractive index as a function of frequency. The same computation was repeated in [18] with conflicting results. Our analytical calculation of higher-order conductivity corrections agrees with the results of [18] in the application range of the perturbative approach. As is shown below, the perturbation results obtained in the context of the plasma model are valid with rather high accuracy when the distance between the test bodies is larger (not much larger) than the plasma wavelength.

Let us consider two semi-infinite solids with dielectric permittivity $\varepsilon(\omega)$ separated by a plane-parallel gap of width a. The surfaces of the bodies are planes $z=0,a$. The Casimir force per unit area acting between these bodies can be found most simply following [19–21],

$$F_p(a)=-\frac{\hbar c}{32\pi^2 a^4}\int_0^\infty x^3 dx\int_1^\infty\frac{dp}{p^2}\left\{\left[\frac{(s+p\varepsilon)^2}{(s-p\varepsilon)^2}e^x-1\right]^{-1}+\left[\frac{(s+p)^2}{(s-p)^2}e^x-1\right]^{-1}\right\},\tag{1}$$

where $s\equiv\sqrt{\varepsilon-1+p^2}$ and $\varepsilon=\varepsilon(i\xi)=\varepsilon(icx/2pa)$ is a dielectric permittivity on the imaginary frequency axis $\omega=i\xi$.

It is common knowledge that the dominant contribution to the Casimir force comes from frequencies $\xi\sim c/a$. We consider the micrometer domain with a from a few tenths of a micrometer to around a hundred micrometers. Here the dominant frequencies are of visible light and infrared optics. In this domain, the plasma model works well and the dielectric permittivity of a metal can be represented as

$$\varepsilon(\omega)=1-\omega_p^2/\omega^2,\quad \varepsilon(i\xi)=1+\omega_p^2/\xi^2,\tag{2}$$

where the plasma frequency ω_p is different for different metals. Note that the plasma model does not take into account relaxation processes. The relaxation parameter, however, is much smaller than the plasma frequency. That is why relaxation could play some role only for large distances between plates $a\gg\lambda_p=2\pi c/\omega_p$, where the corrections to the Casimir force due to finite conductivity are very small.

Let us expand the expression under the integral with respect to p in Eq. (1) in powers of a small parameter,

$$\alpha\equiv\xi/\omega_p=(c/2\omega_p a)\,(x/p)=(\delta_0/a)\,(x/2p),\tag{3}$$

where $\delta_0=\lambda_p/(2\pi)$ is the effective penetration depth of the electromagnetic zero-point oscillations into the metal. Note that in terms of this parameter $\varepsilon(\omega)=1+(1/\alpha^2)$.

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After straightforward calculations one obtains

$$\left[\frac{(s+p\varepsilon)^2}{(s-p\varepsilon)^2}e^x-1\right]^{-1}=\frac{1}{e^x-1}\left[1-\frac{4A}{p}\alpha+\frac{8A}{p^2}(2A-1)\alpha^2+\frac{2A}{p^3}(-6+32A-32A^2+2p^2-p^4)\alpha^3+\frac{8A}{p^4}(2A-1)(2-16A+16A^2-2p^2+p^4)\alpha^4+O(\alpha^5)\right], \tag{4}$$

where $A\equiv e^x/(e^x-1)$.

In perfect analogy, the other contribution from Eq. (1) is

$$\left[\frac{(s+p)^2}{(s-p)^2}e^x-1\right]^{-1}=\frac{1}{e^x-1}[1-4Ap\alpha+8A(2A-1)p^2\alpha^2+2A(-5+32A-32A^2)p^3\alpha^3+8A(1+18A-48A^2+32A^3)p^4\alpha^4+O(\alpha^5)] \tag{5}$$

[note that this expression actually does not depend on p due to Eq. (3)].

After substitution of Eqs. (4) and (5) into Eq. (1), all integrals with respect to p have the form $\int_0^\infty dp\,p^{-k}$ with $k\geqslant 2$ and are calculated immediately. The integrals with respect to x have the form

$$\int_0^\infty dx\frac{x^n e^{mx}}{(e^x-1)^{m+1}} \tag{6}$$

and can be easily calculated with the help of [22]. Substituting their values into Eq. (1), we obtain after some transformations the Casimir force between metallic plates with finite conductivity corrections up to the fourth power in the relative penetration depth,

$$F_p(a)=F_p^{(0)}(a)\left[1-\frac{16}{3}\frac{\delta_0}{a}+24\frac{\delta_0^2}{a^2}-\frac{640}{7}\left(1-\frac{\pi^2}{210}\right)\frac{\delta_0^3}{a^3}+\frac{2800}{9}\left(1-\frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4}\right], \tag{7}$$

where $F_p^{(0)}(a)\equiv-(\pi^2\hbar c)/(240a^4)$.

Now let us turn to the configuration of a lens or a sphere above a plate. Using the result of [20] for the Casimir energy density between plates and applying the PFT, we get the Casimir force,

$$F_l(a)=\frac{\hbar cR}{16\pi a^3}\int_0^\infty x^2dx\int_1^\infty\frac{dp}{p^2}\left\{\ln\left[1-\frac{(s-p\varepsilon)^2}{(s+p\varepsilon)^2}e^{-x}\right]+\ln\left[1-\frac{(s-p)^2}{(s+p)^2}e^{-x}\right]\right\}. \tag{8}$$

Bearing in mind the further expansions, it is convenient to perform in Eq. (8) integration by parts with respect to x. The result is

$$F_l(a)=-\frac{\hbar cR}{48\pi a^3}\int_0^\infty x^3dx\times\int_1^\infty\frac{dp}{p^2}\left[\frac{(s-p\varepsilon)^2-(s+p\varepsilon)^2\frac{\partial}{\partial x}\frac{(s-p\varepsilon)^2}{(s+p\varepsilon)^2}}{(s+p\varepsilon)^2e^x-(s-p\varepsilon)^2}+\frac{(s-p)^2-(s+p)^2\frac{\partial}{\partial x}\frac{(s-p)^2}{(s+p)^2}}{(s+p)^2e^x-(s-p)^2}\right]. \tag{9}$$

The expansion of the first term under the integral in powers of the parameter α introduced in Eq. (3) is

$$\frac{(s-p\varepsilon)^2-(s+p\varepsilon)^2\frac{\partial}{\partial x}\frac{(s-p\varepsilon)^2}{(s+p\varepsilon)^2}}{(s+p\varepsilon)^2e^x-(s-p\varepsilon)^2}=\frac{1}{e^x-1}\left\{1+\frac{4}{px}(1-Ax)\alpha+\frac{8A}{p^2x}(-2-x+2Ax)\alpha^2+\frac{2}{p^3x}[2-6p^2+3p^4+Ax(-6+32A-32A^2+2p^2-p^4)+16A(2A-1)]\alpha^3+\frac{8A}{p^4x}[-8+32A-32A^2+8p^2-4p^4+x(2A-1)(2-16A+16A^2-2p^2+p^4)]\alpha^4+O(\alpha^5)\right\}. \tag{10}$$

In the same way for the second term under the integral of Eq. (9), one obtains

$$\frac{(s-p)^2-(s+p)^2\frac{\partial}{\partial x}\frac{(s-p)^2}{(s+p)^2}}{(s+p)^2e^x-(s-p)^2}=\frac{1}{e^x-1}\left[1+\frac{4}{x}(1-Ax)p\alpha+\frac{8A}{x}(-2-x+2Ax)p^2\alpha^2+\frac{2}{x}(-1-16A+32A^2-5Ax+32A^2x-32A^3x)p^3\alpha^3\right.$$

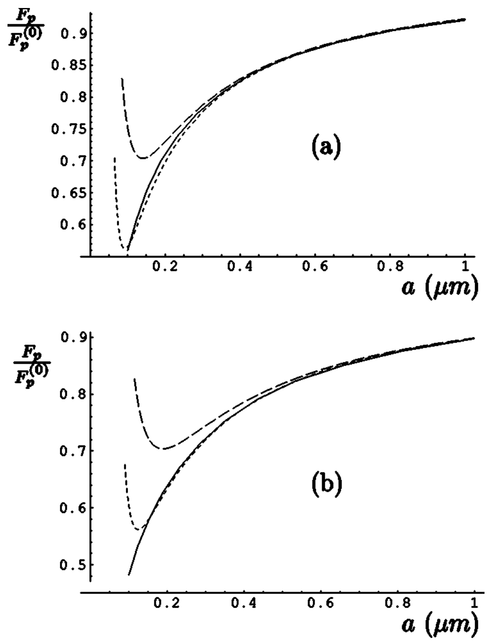

FIG. 1. Correction factors to the Casimir force in the configuration of two plane-parallel plates for Al (a) and Cu or Au (b) bodies as a function of the distance. Solid lines represent the results of computations [18]; short- and long-dashed lines are obtained from the fourth- and second-order results of Eq. (7), respectively.

$$+\frac{8A}{x}(-4+32A-32A^2-x+18Ax$$

$$-48A^2x \quad +32A^3x)p^4\alpha^4+O(\alpha^5)\Bigg]. \qquad (11)$$

Substituting Eqs. (10) and (11) into Eq. (9), we first calculate integrals with respect to p. All integrals with respect to x are of the form (6). Calculating them, we come to the following result after long but straightforward calculations:

$$F_l(a)=F_l^{(0)}(a)\left[1-4\frac{\delta_0}{a}+\frac{72}{5}\frac{\delta_0^2}{a^2}-\frac{320}{7}\left(1-\frac{\pi^2}{210}\right)\frac{\delta_0^3}{a^3}\right.$$

$$\left.+\frac{400}{3}\left(1-\frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4}\right], \qquad (12)$$

where $F_l^{(0)}(a)\equiv-(\pi^3\hbar cR)/(360a^3)$.

Although the results (7) and (12) for two configurations were obtained independently, they can be tied by the use of the PFT. By way of example, the energy density associated with the fourth-order contribution in Eq. (7) is

$$E_p^{(4)}(a)=\int_a^\infty F_p^{(4)}(a)da=-\frac{5\pi^2\hbar c}{27}\left(1-\frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^7}. \qquad (13)$$

Then the fourth-order contribution to the force between a plate and a lens

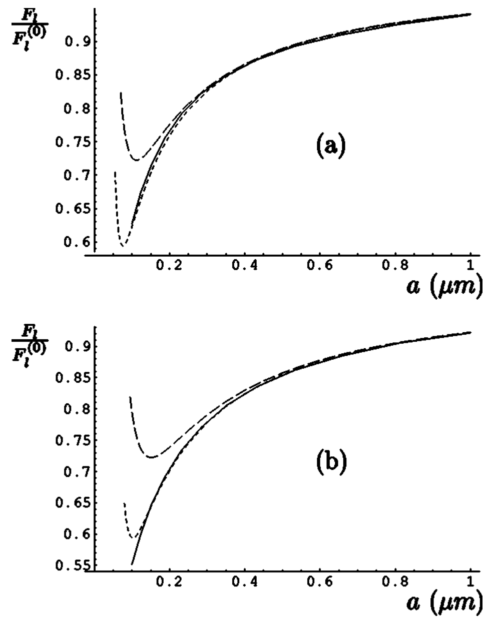

FIG. 2. Correction factors to the Casimir force in the configuration of a lens (sphere) above a plate for Al (a) and Cu or Au (b) bodies as a function of the distance. Solid lines represent the results of computations [18]; short- and long-dashed lines are obtained from the fourth- and second-order results of Eq. (12), respectively.

$$F_l^{(4)}(a)=2\pi RE_p^{(4)}(a)=-\frac{10\pi^3\hbar cR}{27a^3}\left(1-\frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4} \qquad (14)$$

agrees with Eq. (12). The other coefficients of Eq. (12) can be verified in the same way.

We now consider the application range of the expressions (7) and (12). Let us compare the correction to the force between two plane-parallel plates given by Eq. (7) with computations. These computations were performed in [18] for three metals (Au, Cu, and Al) by the numerical integration of the formulas which are equivalent to Eqs. (1) and (8). In doing so, the tabulated data [23] for the complex refractive index were used. The quantity $\varepsilon(i\xi)$ was obtained through the imaginary part of the dielectric permittivity by the use of the dispersion relation [21].

In Fig. 1(a), the solid line represents the computational results of Ref. [18] for $F_p/F_p^{(0)}$ in the case of Al depending on the distance between the plates a. The short-dashed line is obtained from Eq. (7) with the value of the plasma wavelength $\lambda_p^{\rm Al}=98$ nm; the long-dashed line takes account the terms of Eq. (7) up to the second power only. It is seen that Eq. (7) is in excellent agreement with the computational results of [18] for all $a\geqslant\lambda_p^{\rm Al}$. For example, for $a=0.1$ μm, 0.5 μm, and 3 μm it follows from Eq. (7) that $F_p/F_p^{(0)}=0.56$, 0.85, and 0.97, which can be compared with the computations of [18]: 0.55, 0.85, and 0.96, respectively.

In Fig. 1(b), the analogical results for Cu and Au are shown. The dashed lines were obtained with $\lambda_p^{\rm Cu,Au}=132$ nm. For the typical distances indicated above, it follows from Eq. (7) that $F_p/F_p^{(0)}=0.60$, 0.81, and 0.96, which can be compared with the values 0.48, 0.81, and 0.96 [18].

The difference in the first values is due to $\lambda_p^{\mathrm{Cu,Au}} > 100$ nm, i.e., Eq. (7) is not applicable for $a=100$ nm in the case of Cu and Au. For $a \geqslant \lambda_p^{\mathrm{Cu,Au}}$, the results agree perfectly well. Note that the values of the plasma wavelength $\lambda_p = c\sqrt{\pi m}/(e\sqrt{N})$, where m is the effective mass of the conduction electrons and N is their density, are not known very precisely. For Al, usually $\lambda_p^{\mathrm{Al}} = 100$ nm is used [23]. For Au and Cu, the value $\lambda_p^{\mathrm{Cu,Au}} = 136$ nm was estimated recently [18]. We have chosen λ_p based on the smallest rms deviation between the computational results and the ones obtained from Eq. (7) (we use $\lambda_p^{\mathrm{Cu}} = 132$ nm as it was found in [24]). It should be noted that the values of $F_p/F_p^{(0)}$ are insensitive to 2–3 % uncertainties in λ_p for the range of the distances considered here.

Now let us turn to the Casimir force between a plate and a lens. The numerical results were obtained in [18] by the integration of an equation equivalent to Eq. (8). In Fig. 2(a), the results for Al bodies are shown, and in Fig. 2(b), the results for Cu or Au bodies are shown. Solid lines represent the computations of [18]; short- and long-dashed ones are obtained from Eq. (12) used in full or up to the second-order terms. In both figures, the fourth-order perturbation results are in excellent agreement with computations for all $a \geqslant \lambda_p$. At the distances $a=0.1$ μm, 0.5 μm, and 3 μm in the case of Al, we have $F_l/F_l^{(0)} = 0.62$, 0.89, 0.98 from Eq. (12) and 0.63, 0.88, 0.97 from [18]. For Cu and Au, Eq. (12) gives $F_l/F_l^{(0)} = 0.59$, 0.85, 0.97 in agreement with the values 0.55, 0.85, 0.97 [18] (note that for the first value $\lambda_p^{\mathrm{Cu,Au}} > 0.1$ μm).

It should be emphasized that our analytical results are in disagreement with the computations of [17]. By way of example, at $a=0.5$ μm for Au and Cu one can find in [17] $F_p/F_p^{(0)} = 0.657$ and 0.837, respectively, whereas according to our results $F_p/F_p^{(0)} = 0.81$ for both metals. At the same distance and metals for a lens above a plate $F_l/F_l^{(0)} = 0.719$ and 0.874 [17], whereas from Eq. (12) one gets $F_l/F_l^{(0)} = 0.85$. As shown above, our results, however, are in good agreement with the alternative computations of [18].

It is also useful to compare the exact third- and fourth-order conductivity corrections obtained above with the approximate ones obtained by the use of the interpolation formula [5,8]. To take one example, for the force between a lens and a plate the coefficients near the third- and fourth-order corrections in the interpolation formula are -50.67 and $+177.33$ [compare with -43.57 and $+104.13$ from Eq. (12)]. For the smallest separations $a=120$ nm in experiment [2] and $\delta_0/a \approx 0.13$ for Al, this leads to the 0.5% difference only in the results obtained by the interpolation formula [5] and by Eq. (12).

In conclusion, we note that the results (7) and (12) can be reliably used even for the distances a less than the characteristic absorption wavelength λ_0 if $\lambda_p < \lambda_0$ (this is a case, e.g., for Au and Cu, which are characterized by $\lambda_0 \approx 500$ nm or for Cr with $\lambda_p \approx 314$ nm and $\lambda_0 \approx 600$ nm [25]). They give the possibility to calculate the finite conductivity corrections to the Casimir force in a simple way. The accuracy of the obtained analytical results is rather high and cumbersome computations are not used. The generalization of the above perturbative calculations for the case of more complicated geometries (e.g., for cubes) is of much importance.

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