

## Inequalities for electron-field correlation functions

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We show that there exists a class of inequalities between correlation functions of different orders of a chaotic electron field. These inequalities lead to the antibunching effect and are a consequence of the fact that electrons are fermions: indistinguishable particles with antisymmetric states. The derivation of the inequalities is based on the known form of the correlation functions for the chaotic state and on the properties of matrices and determinants.

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### I. INTRODUCTION

In 1956 Hanbury Brown and Twiss observed a correlation of photocurrents from two detectors aimed on the same star [1]. They explained this phenomenon using the classical electromagnetic theory of light. A more proper treatment of the problem shows that there must be a correlation between the photons coming from the star. Namely, photons are more likely to arrive in groups (“bunches”) rather than alone, which results in an enhanced shot noise with respect to randomly arriving (Poisson) particles. This phenomenon is called bunching and it is a typical behavior of photons emitted from thermal sources. It is caused by the fact that photons are not distinguishable in principle and their quantum state is symmetrical with respect to a permutation of two photons. In terms of the probability theory, bunching is expressed by the fact that the probability of detecting two photons at the two detectors shortly after one another is larger than the product of probabilities of the two individual detections.

For the case of electrons, a similar correlation has been predicted also in 1956 [2]. As an electron state is antisymmetrical with respect to a permutation of two particles, electrons avoid coming in pairs, which results in a reduced shot noise. This phenomenon, called antibunching, has been observed experimentally only recently [3]. In analogy to the case of photons, antibunching is equivalent to the fact that the probability of detecting two electrons at two detectors shortly after one another is less than the product of probabilities of the two individual detections.

In this way, the typical behaviors of photons and electrons can be expressed in terms of certain inequalities between the detection probabilities. There is a question whether there is maybe a whole class of inequalities between some physical quantities that would describe the bosonic or fermionic behavior of photons and electrons, respectively. We will show that such inequalities exist, at least for electron chaotic states. To do this, we first introduce correlation functions of the electron field.

### II. CORRELATION FUNCTIONS

Suppose we have an electron field with the density operator  $\hat{\rho}$  and an electron detector with the quantum efficiency  $\eta$

and the cross section  $S$  localized at the point  $\mathbf{r}$ , which is able to detect single electrons. The probability of detecting an electron at the detector during a short time interval  $\langle t, t + \Delta t \rangle$  can be then expressed as

$$P(\mathbf{r}, t, \Delta t) = G^{(1)}(\mathbf{r}, t) \eta S \Delta t, \quad (1)$$

where  $G^{(1)}(\mathbf{r}, t)$  is the so-called one-electron correlation function defined by the relation

$$G^{(1)}(\mathbf{r}, t) = \text{Tr}\{\hat{\rho} \hat{\phi}^\dagger(\mathbf{r}, t) \hat{\phi}(\mathbf{r}, t)\}, \quad (2)$$

$\hat{\phi}(\mathbf{r}, t)$  and  $\hat{\phi}^\dagger(\mathbf{r}, t)$  being the flux annihilation and creation operators of the electron at the space-time point  $(\mathbf{r}, t)$  (see [4]).

Now, suppose we have  $k$  detectors at different points  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$  and inquire what is the probability that we detect an electron at the first detector during the time interval  $\langle t_1, t_1 + \Delta t \rangle$ , another electron at the second detector during the time interval  $\langle t_2, t_2 + \Delta t \rangle$ , etc., and the  $k$ th electron at the last detector during the time interval  $\langle t_k, t_k + \Delta t \rangle$ . This probability is now equal to

$$P(\mathbf{r}_1, \dots, \mathbf{r}_k, t_1, \dots, t_k, \Delta t) = G^{(k)}(\mathbf{r}_1, \dots, \mathbf{r}_k, t_1, \dots, t_k) \times (\eta S \Delta t)^k \quad (3)$$

with the  $k$ -electron correlation function

$$G_{1,2,\dots,k}^{(k)} \equiv G^{(k)}(\mathbf{r}_1, \dots, \mathbf{r}_k, t_1, \dots, t_k) = \text{Tr}\{\hat{\rho} \hat{\phi}^\dagger(\mathbf{r}_1, t_1) \dots \hat{\phi}^\dagger(\mathbf{r}_k, t_k) \hat{\phi}(\mathbf{r}_k, t_k) \dots \hat{\phi}(\mathbf{r}_1, t_1)\}. \quad (4)$$

In principle, it is possible to evaluate the correlation functions for any electron field, according to Eq. (4). However, the calculation sometimes can be very difficult and correlation functions are nowadays known for relatively few electron states [5–8]. We will concentrate on an electron chaotic state in the following that is quite explored and the explicit form of correlation functions is known for it.

The chaotic state is a generalization of a thermal state and it is believed to be produced by the most coherent electron source nowadays available, the field-emission gun [8,9]. It is defined to have a maximum entropy if certain parameters (the mean number of particles and the energy spectrum) are

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fixed at given values. In other words, for these fixed parameters the chaotic electron field is as random as possible. One of the interesting properties of this state is that if there is some correlation in the chaotic field, it must have its origin in the indistinguishableness of particles, i.e., in the Pauli principle. Indeed, distinguishable chaotic particles would come to a detector completely uncorrelated, which means that any joint detection probability would factorize into a product of the individual detection probabilities, i.e., it would imply  $G_{1,2,\dots,k}^{(k)} = G_1^{(1)}G_2^{(1)}\dots G_k^{(1)}$ . In this way, any aberration from this equation has its origin in the fermionic nature of electrons.

According to [5,8], the correlation function of a spin-polarized chaotic state has the form of the determinant

$$G_{1,2,\dots,k}^{(k)} = \begin{vmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1k} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k1} & \Gamma_{k2} & \cdots & \Gamma_{kk} \end{vmatrix}, \quad (5)$$

where  $\Gamma_{ij} = \text{Tr}\{\hat{\rho}\hat{\phi}^\dagger(\mathbf{r}_i, t_i)\hat{\phi}(\mathbf{r}_j, t_j)\}$  is the cross-correlation function of the electron field at the space-time points  $(\mathbf{r}_i, t_i)$  and  $(\mathbf{r}_j, t_j)$ .

It is useful to introduce the complex degree of coherence by the relation

$$\gamma_{ij} = \frac{\Gamma_{ij}}{\sqrt{\Gamma_{ii}\Gamma_{jj}}} \quad (6)$$

(we suppose that  $\Gamma_{ii} \neq 0$  for all  $i$ ; the opposite case is not very interesting since some of the detectors are then not illuminated by electrons at all). An analogous physical quantity has been known in optics for a long time that expresses the mutual coherence of the electromagnetic field at two space-time points.<sup>1</sup> Similarly,  $\gamma_{ij}$  expresses the mutual coherence of the electron field at the space-time points  $(\mathbf{r}_i, t_i)$  and  $(\mathbf{r}_j, t_j)$  and contains information about both the temporal and spatial coherence of the field. As we will see later, the matrix  $\Gamma^{(k)}$  composed of the cross-correlation functions  $\Gamma_{ij}$  is either positive-definite or positive-semidefinite, from which it follows that  $\Gamma_{ij}\Gamma_{ji} \leq \Gamma_{ii}\Gamma_{jj}$  and  $|\gamma_{ij}| \leq 1$  for all  $i, j$ . The case  $|\gamma_{ij}| = 1$  corresponds to the complete mutual coherence of the electron field at the points  $(\mathbf{r}_i, t_i), (\mathbf{r}_j, t_j)$ , while  $|\gamma_{ij}| = 0$  corresponds to the complete incoherence. Thus for  $|\gamma_{ij}| > 0$ , some properties of the electron field at the point  $\mathbf{r}_j$  at the time  $t_j$  can be determined from the knowledge of the electron field at the point  $\mathbf{r}_i$  at the time  $t_i$ . On the other hand, if  $|\gamma_{ij}| = 0$ , even if the properties of the field at the point  $\mathbf{r}_i$  at the time  $t_i$  are known completely, we cannot say anything about the field at the point  $\mathbf{r}_j$  at the time  $t_j$ .

<sup>1</sup>One usually speaks about coherence of light but not about mutual coherence. The coherence expresses the ability of light to interfere. In a similar way, if there is a mutual coherence of the electromagnetic field at two points, there would occur interference if we brought the light from these two points together.

Using the properties of determinants and the fact that  $\Gamma_{ii} = G_i^{(1)}$ , it is possible to rewrite Eq. (5) in terms of the  $\gamma$ 's:

$$G_{1,2,\dots,k}^{(k)} = G_1^{(1)}G_2^{(1)}\dots G_k^{(1)} \begin{vmatrix} 1 & \gamma_{12} & \cdots & \gamma_{1k} \\ \gamma_{21} & 1 & \cdots & \gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \gamma_{k1} & \gamma_{k2} & \cdots & 1 \end{vmatrix}. \quad (7)$$

### III. INEQUALITY BETWEEN ONE- AND TWO-ELECTRON CORRELATION FUNCTIONS

We will first investigate the two-electron correlation function. According to Eq. (7), it follows that

$$G_{1,2}^{(2)} = G_1^{(1)}G_2^{(1)}(1 - \gamma_{12}\gamma_{21}) = G_1^{(1)}G_2^{(1)}(1 - |\gamma_{12}|^2). \quad (8)$$

Here we used the fact that  $\gamma_{12} = \gamma_{21}^*$  that will be proved later. Equation (8) shows that

$$G_{1,2}^{(2)} \leq G_1^{(1)}G_2^{(1)}, \quad (9)$$

so the joint detection probability is less than or equal to the product of the individual detection probabilities. It means that one is not likely to detect two electrons at the space-time points where the electron field is mutually coherent. In usual electron fields, this happens if the spatial separation of the two points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$  is not larger than the coherence length  $l_c$  of the electrons and if the time difference  $t_2 - t_1$  is not larger than the coherence time  $T_c$ .<sup>2</sup> From this it follows that a detection of two electrons at the same detector with a time separation less than  $T_c$  is not likely because the term  $1 - |\gamma_{12}|^2$  is then small. On the other hand, the detection probability of two electrons with a time separation much more than  $T_c$  (when  $\gamma_{12}$  is already equal to zero) is simply equal to the product of the individual detection probabilities and there is therefore no correlation. So it seems that at the typical time scale of  $T_c$ , the electrons avoid coming in pairs (or groups) to a detector and prefer coming alone. This effect is called antibunching (see Fig. 1). Thus, we can say that antibunching is a consequence of the fact that the probability of detecting two electrons at two detectors shortly after one another is less than the product of the probabilities of the two individual detections, or more generally, that it is a consequence of inequality (9).

In the extreme case when  $|\gamma_{12}| = 1$ , the two-electron correlation function turns into zero. Then, no two electrons can be found at the space-time points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$  simultaneously. This reminds one of the Pauli principle: the latter prohibits two electrons to be in the same quantum state, while Eq. (8) prohibits two electrons to be at the space-time points  $(\mathbf{r}_1, t_1)$  and  $(\mathbf{r}_2, t_2)$  where the electron field is mutually completely coherent.

<sup>2</sup>Strictly speaking, in quasimonochromatic fields  $\gamma_{12} \neq 0$  holds if  $\mathbf{r}_2 - \mathbf{r}_1 \approx \mathbf{v}(t_2 - t_1)$ , where  $\mathbf{v}$  is the group velocity of the electrons.

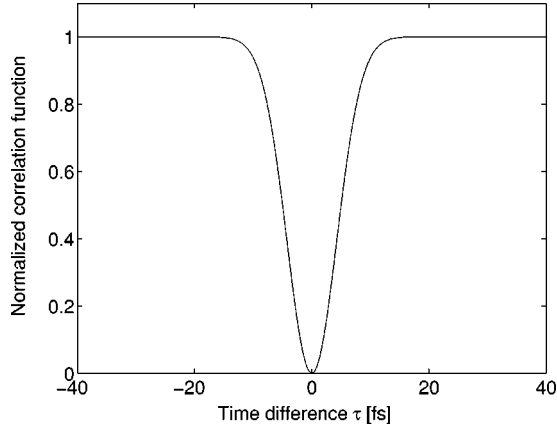


FIG. 1. The normalized two-electron correlation function  $G_{1,2}^{(2)}/[G_1^{(1)}G_2^{(1)}]=1-|\gamma_{12}|^2$  for  $\mathbf{r}_1=\mathbf{r}_2$  as a function of the time difference  $\tau=t_2-t_1$  for spin-polarized quasimonochromatic electrons with the Gaussian energy spectrum. The width of the peak is approximately equal to the coherence time of the electrons. The probability that two electrons come after one another within the coherence time is less than the probability that they come longer after one another, which is called antibunching.

Inequality (9) holds between the one- and two-electron correlation functions of a chaotic electron field. Now, the question is whether it would be possible to find a similar inequality also between correlation functions of higher orders. The answer is yes. A possible generalization of Eq. (9) that comes to mind is  $G_{1,2,\dots,k}^{(k)}\leq G_1^{(1)}G_2^{(1)}\dots G_k^{(1)}$ . If this should hold, then the determinant in Eq. (7) would have to be less than or equal to unity. In the following, we will show that it is indeed so by using the well-known properties of matrices and determinants. Moreover, we will prove an even more general inequality between the correlation functions of different orders.

#### IV. GENERAL INEQUALITY BETWEEN CORRELATION FUNCTIONS

First, we note that the matrix composed of the cross-correlation functions

$$\Gamma^{(k)} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1k} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2k} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k1} & \Gamma_{k2} & \cdots & \Gamma_{kk} \end{pmatrix} \quad (10)$$

is Hermitian and either positive-definite or positive-semidefinite. The Hermiticity of  $\Gamma^{(k)}$  follows simply from the Hermiticity of the density operator  $\hat{\rho}$  and from the invariance of the trace under a commutation of operators:

$$\Gamma_{ij} = \text{Tr}\{\hat{\rho}\hat{\phi}_i^\dagger\hat{\phi}_j\} = [\text{Tr}\{\hat{\phi}_j^\dagger\hat{\phi}_i\hat{\rho}^\dagger\}]^* = [\text{Tr}\{\hat{\rho}\hat{\phi}_j^\dagger\hat{\phi}_i\}]^* = \Gamma_{ji}^*. \quad (11)$$

Of course, from Eq. (11) it follows also that  $\gamma_{ij}=\gamma_{ji}^*$ , i.e., the matrix  $\gamma^{(k)}$  composed of the complex degrees of coherence is also Hermitian.

The second property can be proved in a similar way as an analogous statement in the quantum optics (see [10], p. 585). Let  $\hat{O}$  be the operator defined as

$$\hat{O} = \sum_{i=1}^k \lambda_i \hat{\phi}_i, \quad (12)$$

where  $\lambda_1, \dots, \lambda_k$  are arbitrary complex numbers. It holds

$$\text{Tr}\{\hat{\rho}, \hat{O}^\dagger \hat{O}\} = \sum_{i,j=1}^k \lambda_i^* \lambda_j \text{Tr}\{\hat{\rho} \hat{\phi}_i^\dagger \hat{\phi}_j\} = \sum_{i,j=1}^k \lambda_i^* \lambda_j \Gamma_{ij}. \quad (13)$$

At the same time,  $\text{Tr}\{\hat{\rho} \hat{O}^\dagger \hat{O}\}$  is a non-negative number. As the right-hand side of Eq. (13) is a quadratic form in the  $\lambda$ 's with the coefficients  $\Gamma_{ij}$ , the matrix  $\Gamma^{(k)}$  must be either positive-definite or positive-semidefinite. A similar statement can be proved also for the matrix  $\gamma^{(k)}$  using  $\hat{O} = \sum_{i=1}^k \lambda_i (\Gamma_{ii} \Gamma_{jj})^{-1/2} \hat{\phi}_i$ .

As we will see in the following, from the Hermiticity and definiteness of the matrices  $\Gamma^{(k)}$  and  $\gamma^{(k)}$ , the desired inequality follows directly. However, first it will be necessary to prove the following lemma.

*Lemma:* The determinant of any positive-definite or positive-semidefinite Hermitian matrix  $A=(A_{ij})$  with non-zero diagonal elements cannot exceed the product of the diagonal elements of  $A$ , i.e.,  $\det(A)\leq A_{11}A_{22}\dots A_{kk}$ , and the equality takes place if and only if  $A$  is diagonal.

*Proof:* As all the diagonal elements  $A_{ii}$  of the matrix  $A$  are positive, we can define the matrix  $a=(a_{ij})$  with elements  $a_{ij}=A_{ij}/\sqrt{A_{ii}A_{jj}}$  (in analogy with defining the matrix  $\gamma^{(k)}$  with the help of  $\Gamma^{(k)}$ ). Thanks to the Hermiticity of the matrix  $a$ , it is possible to transform it into the diagonal form with a unitary transformation, i.e., there exists a unitary matrix  $U$  for which the matrix  $b=UaU^\dagger$  is diagonal. This transformation changes neither the determinant nor the trace of the matrix because it is a unitary transformation. If we denote the diagonal elements of the matrix  $b$  as  $b_i$ , then  $\text{Tr}(a)=\text{Tr}(b)=\sum_{i=1}^k b_i$  and  $\det(a)=\det(b)=\prod_{i=1}^k b_i$  evidently hold. At the same time,  $\text{Tr}(a)=k$  holds due to the fact that  $a_{ii}=1$  for all  $i$ . To find out what is the maximal possible value of  $\det(a)$ , we will use now the inequality between the arithmetical and geometrical averages. The arithmetical average of the numbers  $b_i$  is  $\alpha=\sum_{i=1}^k b_i/k=1$  and their geometrical average is  $\beta=\sqrt[k]{\prod_{i=1}^k b_i}$ . As the numbers  $b_i$  are non-negative, the inequality  $\beta\leq\alpha$  holds, from which it then follows that  $\det(a)=\prod_{i=1}^k b_i\leq 1$ . As is known, the equality  $\beta=\alpha$  takes place if and only if  $b_1=b_2=\dots=b_k$ . In this case, the matrix  $b$  is the unit matrix, from which it follows that  $a$  is also the unit matrix and  $a_{ij}=\delta(i,j)$ . Thus,  $\det(a)\leq 1$  holds and the equality takes place only when all the nondiagonal elements of the matrix  $a$  vanish. Expressing the

determinant of the original matrix  $A$  with the help of  $\det(a)$  as  $\det(A) = A_{11}A_{22} \cdots A_{kk} \det(a)$ , we get from the inequality  $\det(a) \leq 1$  that

$$\det(A) \leq A_{11}A_{22} \cdots A_{kk}. \quad (14)$$

Moreover,  $A$  is diagonal if and only if  $a$  is diagonal. Therefore, the equality in Eq. (14) takes place if and only if the matrix  $A$  is diagonal.

If we identify the matrix  $a$  with  $\gamma^{(k)}$ , then from Eq. (7) and the proof above it follows immediately that

$$G_{1,2,\dots,k}^{(k)} \leq G_1^{(1)} G_2^{(1)} \cdots G_k^{(1)}. \quad (15)$$

This is a generalization of the inequality (8) for a correlation function of arbitrary order. An even stronger generalization would be evidently

$$G_{1,2,\dots,k}^{(k)} \leq G_{1,2,\dots,l}^{(l)} G_{l+1,l+2,\dots,k}^{(k-l)}. \quad (16)$$

As we will see now, this inequality indeed holds.

### V. PROOF OF THE INEQUALITY (16)

First we will define a matrix  $\Gamma'$  of the type  $k/k$  in the following block form:

$$\Gamma' = \begin{pmatrix} \Gamma^{(l)} & 0 \\ 0 & \Gamma^{(m)} \end{pmatrix}. \quad (17)$$

Here 0 stands for the zero matrices of the type  $l/m$  or  $m/l$  (we have denoted  $m = k - l$ ) and  $\Gamma^{(l)}, \Gamma^{(m)}$  are the matrices of the type  $l/l$  and  $m/m$ , respectively, corresponding to the correlation functions  $G_{1,\dots,l}^{(l)}$  and  $G_{l+1,\dots,k}^{(m)}$ :

$$\Gamma^{(l)} = \begin{pmatrix} \Gamma_{1,1} & \cdots & \Gamma_{1,l} \\ \vdots & & \vdots \\ \Gamma_{l,1} & \cdots & \Gamma_{l,l} \end{pmatrix}, \quad (18)$$

$$\Gamma^{(m)} = \begin{pmatrix} \Gamma_{l+1,l+1} & \cdots & \Gamma_{l+1,k} \\ \vdots & & \vdots \\ \Gamma_{k,l+1} & \cdots & \Gamma_{k,k} \end{pmatrix}.$$

Due to Eq. (5) and the block form of  $\Gamma$ , it holds

$$G_{1,\dots,k}^{(k)} = \det(\Gamma), \quad (19)$$

$$G_{1,\dots,l}^{(l)} G_{l+1,\dots,k}^{(m)} = \det(\Gamma^{(l)}) \det(\Gamma^{(m)}) = \det(\Gamma').$$

Now, we know that the matrix  $\Gamma \equiv \Gamma^{(k)}$  is either positive-definite or positive-semidefinite. In the latter case, the inequality (16) is satisfied trivially because then  $\det(\Gamma) = 0$  and  $\det(\Gamma^{(l)}), \det(\Gamma^{(m)})$  are both non-negative due to their definiteness. Therefore, in the following we will discuss the case when  $\Gamma$  is positive-definite.

As the matrices  $\Gamma^{(l)}$  and  $\Gamma^{(m)}$  are Hermitian, it is possible to transform each of them into the diagonal form with a unitary transformation. Let  $U^{(l)}$  and  $U^{(m)}$  denote the corre-

sponding unitary transformational matrices, so that the matrices  $D^{(l)} = U^{(l)} \Gamma^{(l)} U^{(l)\dagger}$  and  $D^{(m)} = U^{(m)} \Gamma^{(m)} U^{(m)\dagger}$  are both diagonal. Then evidently the unitary matrix

$$U = \begin{pmatrix} U^{(l)} & 0 \\ 0 & U^{(m)} \end{pmatrix} \quad (20)$$

transforms the matrix  $\Gamma'$  into the diagonal form, so that  $D' = U \Gamma' U^\dagger$  is diagonal. Let  $D$  denote the matrix obtained from  $\Gamma$  by the same unitary transformation, i.e.,  $D = U \Gamma U^\dagger$ . Thanks to the block form of the matrix  $U$ , the matrix  $D$  has the form

$$D = \begin{pmatrix} D^{(l)} & D^{(lm)} \\ D^{(ml)} & D^{(m)} \end{pmatrix}, \quad (21)$$

where  $D^{(lm)}$  and  $D^{(ml)}$  are some mutually Hermitian-conjugate matrices of the type  $l/m$  and  $m/l$ , respectively. Applying now the Lemma to the matrix  $D$  (we can do that because  $D$  is positive-definite and Hermitian; the latter follows from the unitarity of the matrix  $U$ ), we see that  $\det(D) \leq \det(D')$  because the diagonal elements of the matrices  $D$  and  $D'$  are identical and  $D'$  is diagonal. Combining this with the equations that hold due to the unitarity of the matrix  $U$ ,

$$\det(D) = \det(\Gamma), \quad \det(D') = \det(\Gamma') = \det(\Gamma^{(l)}) \det(\Gamma^{(m)}), \quad (22)$$

and with Eq. (19), we finally obtain the inequality (16). Now, the inequality  $\det(D) \leq \det(D')$  changes into equality if and only if the matrix  $D$  is diagonal, i.e., if  $D^{(lm)}$  and  $D^{(ml)}$  are the zero matrices. Then, again due to the block form of the transformation matrix  $U$ , also the matrices

$$\Gamma^{(lm)} = \begin{pmatrix} \Gamma_{1,l+1} & \cdots & \Gamma_{1,k} \\ \vdots & & \vdots \\ \Gamma_{l,l+1} & \cdots & \Gamma_{l,k} \end{pmatrix}, \quad (23)$$

$$\Gamma^{(ml)} = \begin{pmatrix} \Gamma_{l+1,1} & \cdots & \Gamma_{l+1,l} \\ \vdots & & \vdots \\ \Gamma_{k,1} & \cdots & \Gamma_{k,l} \end{pmatrix}$$

are the zero matrices. Thus, we can conclude that the inequality (16) holds and it changes into equality if and only if all the cross-correlation functions  $\Gamma_{i,j}$  vanish for  $i = 1, \dots, l$  and  $j = l+1, \dots, k$ .

### VI. FERMIONIC NATURE OF ELECTRON CORRELATIONS

Let us see what the inequality (16) that we just proved really means. We denote the detection of  $l$  electrons at the space-time points  $(\mathbf{r}_1, t_1), \dots, (\mathbf{r}_l, t_l)$  as event  $A$  and the detection of  $k-l$  electrons at the points  $(\mathbf{r}_{l+1}, t_{l+1}), \dots, (\mathbf{r}_k, t_k)$  as event  $B$ . Then the inequality (16)

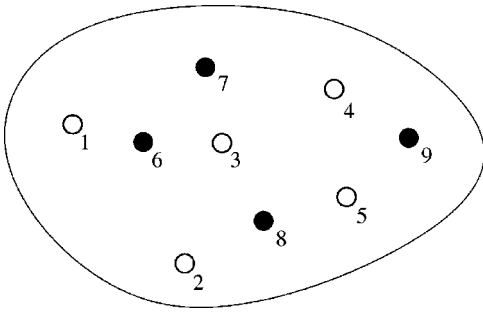


FIG. 2. An example of inequality (16) for  $k=9$ ,  $l=5$ . The correlation function corresponding to all the points is less than or equal to the product of correlation functions corresponding to the ‘‘white’’ and ‘‘black’’ points. The equality takes place if and only if  $\gamma_{ij}=0$  for each pair containing one white and one black point.

says that the probability that both events  $A$  and  $B$  happen is less than or equal to the product of probabilities of events  $A$  and  $B$  (see Fig. 2). In this way, the inequality (16) generalizes the inequality (9) also to multiple electron detection processes.

We have seen that the inequality (9) leads to the antibunching effect as a consequence of the indistinguishability of electrons. Similarly, the inequality (16) reflects the same principle for more complicated detection processes and leads to more general correlations in electron fields. It is a fundamental statement that expresses the fermionic behavior of electrons in a very compact way.

As follows from Sec. V, the case of equality in Eq. (16) corresponds to the situation when  $\Gamma_{ij}=0$  (and hence  $\gamma_{ij}=0$ ) for all  $i=1, \dots, l$  and  $j=l+1, \dots, k$ . Then the electron field at any point of the first set of points  $S_l = \{(r_i, t_i) | i=1, \dots, l\}$  is incoherent with the field at any point of the second set  $S_m = \{(r_j, t_j) | j=l+1, \dots, k\}$ . The equality in Eq. (16) is then very reasonable; if the fields at the points corresponding to the both sets  $S_l, S_m$  are mutually completely incoherent, the detections at the points of the two sets are mutually independent and therefore total detection probability factorizes into the product of the detection probabilities corresponding to the individual sets.

Of course, the inequality (16) can be applied repeatedly and the points  $(r_1, t_1), \dots, (r_k, t_k)$  can be interchanged arbitrarily to obtain a whole class of inequalities. We will write just an example for illustration:

$$G_{1,2,3,4,5,6,7}^{(7)} \leq G_{1,2}^{(2)} G_{3,5,7}^{(3)} G_4^{(1)} G_6^{(1)}. \quad (24)$$

## VII. CONCLUSION

We have proved a relatively simple inequality between correlation functions of different orders for chaotic electrons. As any correlation in a chaotic electron field originates from the fermionic character of the electrons, the inequality (16) is a direct consequence of the Pauli principle. It demonstrates the aversion of the electrons to staying and coming to a detector in groups. The inequality (16) determines a set of conditions that must be fulfilled on the hierarchy of the chaotic correlation functions. We must point out that the inequalities in Eq. (16) do not hold for all electron fields. There are electron states that show even bunching instead of antibunching [11,12]. However, these states are quite rare and the chaotic state remains the most important and widespread state in electron beams.

From the experimental point of view, the observation of correlations is limited especially by an extremely short coherence time of available electron beams [13,14]. The coherence time is related to the energy bandwidth of the beam by the relation  $T_c \approx h/\Delta E$ , which yields  $T_c \approx 2 \times 10^{-14}$  s for a typical field-emission beam for which  $\Delta E \approx 0.2$  eV. The measurement of correlations with such a characteristic time requires very fast detectors and coincidence electronics and even under optimum conditions the experimental resolution time exceeds the coherence time by three orders of magnitude. The signal-to-noise ratio is therefore very small and it is not surprising that a two-electron correlation was observed in the last year only. Observation of higher-order correlations would require a more complicated experimental setup and I believe that it will not be possible until electron sources with a much longer resolution time become available. On the other hand, if the resolution and coherence time became comparable, the highest order of observable correlations would be limited by the fidelity of the coincidence electronics. Thus we must conclude that the inequality (9) is the only candidate for an experimental verification from the whole class of inequalities (16) at the present time.

## ACKNOWLEDGMENT

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