Local environment can enhance fidelity of quantum teleportation

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We show how an interaction with the environment can enhance fidelity of quantum teleportation. To this end, we present examples of states which cannot be made useful for teleportation by any local unitary transformations; nevertheless, after being subjected to a dissipative interaction with the local environment, the states allow for teleportation with genuinely quantum fidelity. The surprising fact here is that the necessary interaction does not require any intelligent action from the parties sharing the states. In passing, we produce some general results regarding optimization of teleportation fidelity by local action. We show that bistochastic processes cannot improve fidelity of two-qubit states. We also show that in order to have their fidelity improvable by a local process, the bipartite states must violate the so-called reduction criterion of separability.

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I. INTRODUCTION

Quantum teleportation [1] is fundamentally important as an operational test of the presence and the strength of entanglement. Moreover, a recent series of beautiful experiments [2], which realized teleportation in practice, opened a window for a wide range of its possible technological applications.

In this paper, teleportation is understood as any strategy which uses local quantum operations and classical communication (LOCC) [3] to transmit an unknown state via a shared pair of particles. In an ideal teleportation scheme, the electron pair is in a pure, maximally entangled bipartite state:

$$\psi_- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{1}$$

The state is shared by a sender (Alice) and a receiver (Bob). By use of ψ_- with Alice, Bob can produce an *exact* replica of another (input) state originally held by Alice. In reality, however, interactions with the environment and imperfections of preparation result in Alice and Bob sharing a state which is always mixed. Consequently, at Bob's end, the teleported state can only be a distorted copy of the input initially held by Alice. Moreover, if the bipartite state is mixed too much, it will not provide for any better transmission fidelity than that of an ordinary classical communication channel [4]. To do better than a classical channel, the shared quantum state must be entangled. A natural question then is [4]: can any entangled state provide better than classical fidelity of teleportation?

Early attempts to answer this question concentrated on the characterization of the states which can offer nonclassical fidelity within the original teleportation scheme supplemented by local unitary rotations. Henceforth we will call such a scheme the *standard teleportation scheme* (STS). Fidelity of teleportation achievable in STS is uniquely determined by the bipartite state's *fully entangled fraction*. It was defined in Ref. [5] as

$$f(\varrho) = \max_{\psi} \langle \psi | \varrho | \psi \rangle. \tag{2}$$

In the definition, the maximum is taken over all maximally entangled states ψ, i.e., over $\psi = U_1 \otimes U_2 \psi_+$, where

$$\psi_+ = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle; \tag{3}$$

U_1 and U_2 are unitary transformations. Later, it was shown that in order to be useful for STS, the states acting on a Hilbert space $C^d \otimes C^d$ must have $f > 1/d$ [6,7]. Moreover, it was shown that no *bound entangled* state (see Ref. [8]) can offer better fidelity than classical communication [9,7]. Somewhat earlier, in Refs. [10] and [11], the authors identified a class of states which do not permit any increase of f, neither by any trace preserving (TP) LOCC nor even by some less restricted non-TP LOCC actions. Mixtures of a maximally mixed state and ψ_+ [4,12] belong, among others, to this class.

One could then be tempted to speculate that f could not be increased by any TP LOCC operations. If so, then STS would be a unique teleportation scheme in the sense that no other scheme would provide better fidelity than STS. On the other hand, one could still suspect that by some intelligent, sophisticated LOCC operation, Alice and Bob would be able to increase f for some states anyway. An important question was then to be answered: *Is it possible to design a teleportation scheme, for which at least some states with $f \leq 1/d$ would give nonclassical fidelity?*

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In this paper, we answer this question by presenting a class of two-qubit states with $f \leqslant 1/2$, which can, nevertheless, be used for teleportation with nonclassical fidelity. For that, however, one has to allow for some *dissipative* interaction between the states and their local environment first. This means that dissipation, which is usually associated with decoherence and destruction of teleportation, increases f of some initially nonteleporting states to above 1/2. In other words, some states can produce nonclassical fidelity within the original teleportation scheme but only after being ‘‘corrupted’’ by the environment!

To our knowledge, this is a previously unknown effect. In particular, it is different than that used in the so called *filtering* method of improving some of the states' parameters [13,14]. Filtering includes a *selection* process based on a *readout* of measurement outcomes. In our examples, on the other hand, Alice and Bob do not need to know the outcomes at all. Hence, in particular, unlike filtering, the actions in our examples are entirely trace preserving.

We begin our presentation by recalling some of the general results on optimal teleportation fidelity in Sec. II (cf. Ref. [7]). This allows us to conclude that an optimal teleportation scheme should include maximization of f by means of TP LOCC operations. Then, in Sec. III we put the problem in the context of increasing f by the maps of the form $I \otimes \Lambda$. We can limit the possible successful maps by showing that, e.g., for two qubits, the bistochastic processes cannot do the job. We also show that the states with f improvable by $I \otimes \Lambda$ action must violate the so-called *reduction criterion*. Subsequently, in Sec. IV we present the examples of states, for which f can be nontrivially increased by TP LOCC operations. The paper ends with the summary of the results and the conclusions in Sec. V.

II. OPTIMAL FIDELITY IN A GENERAL TELEPORTATION SCHEME

Let Alice and Bob share a pair of particles in a given state ϱ acting on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B = C^d \otimes C^d$. Additionally, let Alice have a third particle in an unknown pure state $\psi \in \mathcal{H}_C = C^d$ to be teleported. In the most general teleportation scheme, Bob and Alice apply some trace preserving (TP) (hence without selection of the ensemble) LOCC operation $\mathcal{T}$ to the particles which they share and to the third (Alice's) particle. After the operation is completed, the final state of Bob's particle (from the pair) is

$$\varrho_{Bob}^{\psi} = \mathrm{Tr}_{A,C}[\mathcal{T}(|\psi\rangle\langle\psi| \otimes \varrho)]. \tag{4}$$

The resulting mapping of the input state (the state of the third particle) onto $\varrho_{Bob}(\psi)$ establishes a *teleportation channel* Λ (it depends on both, $\mathcal{T}$ and ϱ):

$$\Lambda(|\psi\rangle\langle\psi|) = \varrho_{Bob}(\psi). \tag{5}$$

The aim of teleportation is to obtain $\varrho_{Bob}(\psi)$ as close to $|\psi\rangle\langle\psi|$ as possible. A useful measure of the quality of teleportation is then provided by teleportation's *fidelity* [4]

$$\mathcal{F} = \overline{\langle\psi|\varrho_{Bob}(\psi)|\psi\rangle}. \tag{6}$$

Fidelity is a function of map Λ and, like Λ, it depends on both, teleporting state ϱ and the strategy of teleportation $\mathcal{T}$. One can show [7] that in the standard teleportation scheme, the maximal fidelity achievable from a given bipartite state ϱ is

$$\mathcal{F} = \frac{fd+1}{d+1}, \tag{7}$$

where f is the fully entangled fraction of ρ given by formula (2). To achieve this fidelity, Alice and Bob have to rotate their respective parts of the teleporting state ρ so that the maximum of formula (2) is attained on singlet ψ_-. The original teleportation scheme applied with the rotated bipartite state ρ will now produce the maximal fidelity (7).

If, on the other hand, Alice and Bob do not share any quantum state, then their best strategy is [4]: (i) Alice performs an optimal measurement of the system to be teleported and sends the outcome to Bob (classically). (ii) On the basis of her results, Bob tries to reconstruct the state.

The optimal teleportation fidelity for this strategy is equal to the optimal fidelity of the state estimation for a single system. It is given by [15,7]

$$\mathcal{F}_{cl} = \frac{2}{1+d}. \tag{8}$$

One can easily see now that, in order to perform better than classical communication, STS needs bipartite states with $f > 1/d$. With $f \leqslant 1/d$, Alice and Bob can just as well discard their bipartite state and rise classical communication alone.

There is no reason why STS should represent the most efficient teleportation scheme using states with $f > 1/d$. One can show, however, that the optimal teleportation scheme (OTS) is a generalization of STS [7]. OTS consists of two steps: (i) Alice and Bob try to maximize f by applying TP LOCC (not necessarily unitary) operations to the original state ϱ. (ii) They apply STS using the transformed state.

Let then $f_{max}(\varrho)$ denote the maximal f attainable from ϱ by means of TP LOCC operations. The maximal teleportation fidelity from state ϱ is then given by [7]

$$\mathcal{F}_{max} = \frac{f_{max}d+1}{d+1}. \tag{9}$$

Thus, to find the optimal teleportation fidelity for a given bipartite state ρ, one must find f_{max}. In other words, the fidelity of STS can be improved if: (i) f can be increased by LOCC, (ii) The final f is in quantum region, i.e., it is greater than $1/d$.

Henceforth, when referring to a process of increasing f, we will understand it as increasing so that the final value is above $1/d$ (within the range $f \leqslant 1/d$, the fully entangled fraction can be increased relatively easily. This, however, does not produce any better fidelity than $\mathcal{F}_{cl}$).

III. SOME GENERAL RESULTS ON IMPROVING $\mathcal{F}$ BY LOCAL INTERACTIONS

A. Simplified formula for maximal f attainable by local interaction

When local TP transformations are used to increase f of a general bipartite state $\varrho \in C^d \otimes C^d$, then the best attainable result is

$$f_A = \max_{\Lambda} \mathrm{Tr}[(\Lambda \otimes I)\varrho P_+]. \tag{10}$$

The maximum is here taken over all TP completely positive (CP) maps Λ and $P_+ = |\psi_+\rangle\langle\psi_+|$, with ψ_+ given by Eq. (3). Stinespring decomposition of Λ gives [16]

$$\Lambda(\cdot) = \sum_i V_i(\cdot)V_i^\dagger \tag{11}$$

with $\Sigma_i V_i^\dagger V_i = I$. Moreover, we can utilize the fact that $A \otimes I\psi_+ = I \otimes A^T\psi_+$ [17] (superscript T denotes transposition in basis $\{|i\rangle\}$) and rewrite formula (10) as

$$f_A = \max_{\Gamma} \mathrm{Tr}[\varrho(I \otimes \Gamma)P_+], \tag{12}$$

with

$$\Gamma(\cdot) = \sum_i W_i(\cdot)W_i^\dagger \tag{13}$$

and $W_i = V_i^*$ (the star denotes complex conjugation). Naturally, like Λ, Γ is trace preserving, too.

We can now recall that there is an isomorphism between the TP CP maps and the bipartite states with one subsystem maximally mixed. The isomorphism is given by

$$\varrho' = (I \otimes \Lambda)P_+. \tag{14}$$

Thus, for any TP CP map, the corresponding state has a maximally mixed subsystem A and for any state with a maximally mixed subsystem A, there exists a map that realizes it via the above formula. Consequently, we can obtain the following form for f_A:

$$f_A(\varrho) = \max_{\varrho'} \mathrm{Tr}(\varrho\varrho'), \tag{15}$$

where the maximum is taken over all states ϱ' with maximally mixed subsystem A. An analogous formula holds for f_B. In general, the values f_A and f_B are likely to be different from one another.

Formula (15) allows for identification of those maps which definitely cannot improve f. Take, for instance, the maps describing the action of random external fields [18]. They are of the form

$$\Lambda(\cdot) = \sum_i p_i U_i(\cdot)U_i^\dagger, \tag{16}$$

with U_i denoting unitary transformations occurring with probability p_i. The corresponding $\varrho' = (I \otimes \Lambda)P_+$ is a mixture of maximally entangled vectors. Consequently, $\mathrm{Tr}(\varrho\varrho')$ cannot exceed $f(\varrho)$ which is equal to the maximal overlap of ϱ with one maximally entangled vector.

In addition to preserving trace, maps (16) preserve the identity, i.e., $\Lambda(I) = I$. Maps preserving both the trace and the identity are called bistochastic. In general, the class of bistochastic maps can be wider than the class specified by Eq. (16). For two qubits, however, the two classes coincide. To see this, one can note that, in general, the set of states corresponding to the set of bistochastic maps via the isomorphism consists of the states with *both* subsystems maximally mixed. For two-qubit systems such states are mixtures of maximally entangled vectors [19]. Each such vector can be written as $I \otimes U\psi_+$ for some unitary U. Hence the maps corresponding to mixtures of such vectors are mixtures of unitary maps. Thus for two qubits the bistochastic maps cannot increase f. One may conjecture that this should be the case in higher dimensions, too.

B. Increasing f by local actions and the reduction criterion for separability

Let us now derive some constraints for the states with f improvable by local interaction. A state suitable for a teleportation channel must be entangled, i.e., it must be impossible to represent it by a mixture of product states [12]:

$$\varrho \neq \sum_i p_i \varrho_i \otimes \tilde{\varrho}_i. \tag{17}$$

Such states violate different separability criteria. Here, we consider the so-called *reduction criterion* for separability. It is given by the following conditions satisfied by all separable states [20,21]:

$$\varrho_A \otimes I - \varrho \geqslant 0, \quad I \otimes \varrho_B - \varrho \geqslant 0. \tag{18}$$

The inequalities mean that the operators on the left-hand sides must be *positive*, i.e., they must have non-negative eigenvalues only. In a two-qubit case, the reduction criterion is equivalent to separability (hence it is also a sufficient condition for separability), while it becomes a weaker ''detector'' of entanglement in higher dimensions. In other words, there exist nonseparable (entangled) states in higher dimensions which do not violate the reduction criterion.

Suppose now that for some state ϱ one has $f_A(\varrho) > f(\varrho)$, i.e., f can be improved by a local TP operation on subsystem A. Naturally, we require that the improvement is nontrivial, i.e., $f_A > 1/d$. We will show now that this condition implies violation of the reduction criterion. Indeed, since $f_A > 1/d$, then there exists a state ϱ' whose one subsystem (say, ϱ'_A) has maximal entropy and

$$\mathrm{Tr}(\varrho\varrho') > 1/d. \tag{19}$$

Maximum entropy means that $\varrho'_A = I/d$. This implies $\mathrm{Tr}[(\varrho_A \otimes I)\varrho'] = \mathrm{Tr}(\varrho_A\varrho'_A) = 1/d$. By putting this into inequality (19), we obtain

$$\mathrm{Tr}[(\varrho_A\otimes I-\varrho)\varrho']<0. \tag{20}$$

The trace of a composition of two positive operators is nonnegative. Operator ϱ' is positive. Consequently, in order to satisfy the last inequality, the operator $\varrho_A\otimes I-\varrho$ cannot be positive.

Since all the entangled two-qubit states violate the reduction criterion, the condition for improvability of f derived above, does not put any new restrictions on the class of states with improvable f here [10,11]. Nevertheless, the condition should be useful while investigating bipartite states in higher dimensions. This is because not all the entangled states there violate the reduction criterion.

IV. BEATING THE STANDARD TELEPORTATION SCHEME

Before showing how to do better than STS, we will still need to introduce some methods of dealing with the fully entangled fraction of two-qubit states.

A. Fully entangled fraction in the Hilbert-Schmidt representation

An arbitrary state of a two-qubit system can be represented as

$$\varrho=\frac{1}{4}\left(I\otimes I+\boldsymbol{r}\cdot\boldsymbol{\sigma}\otimes I+I\otimes\boldsymbol{s}\cdot\boldsymbol{\sigma}+\sum_{m,n=1}^{3}t_{nm}\sigma_n\otimes\sigma_m\right). \tag{21}$$

Here, I stands for the identity operator, $\boldsymbol{r}$ and $\boldsymbol{s}$ belong to R^3,$\{\sigma_n\}_{n=1}^3$ are standard Pauli matrices, $\boldsymbol{r}\cdot\boldsymbol{\sigma}=\sum_{i=1}^3 r_i\sigma_i$. Coefficients $t_{mn}=\mathrm{Tr}(\rho\sigma_n\otimes\sigma_m)$ form a real 3×3 matrix later denoted by T. Note that $\boldsymbol{r}$ and $\boldsymbol{s}$ are local parameters as they determine the reductions of ϱ:

$$\varrho_1\equiv\mathrm{Tr}_{\mathcal{H}_2}\varrho=\frac{1}{2}(I+\boldsymbol{r}\cdot\boldsymbol{\sigma}),$$
$$\varrho_2\equiv\mathrm{Tr}_{\mathcal{H}_1}\varrho=\frac{1}{2}(I+\boldsymbol{s}\cdot\boldsymbol{\sigma}). \tag{22}$$

Matrix T, on the other hand, is responsible for the correlations

$$E(\boldsymbol{a},\boldsymbol{b})\equiv\mathrm{Tr}(\varrho\,\boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\boldsymbol{b}\cdot\boldsymbol{\sigma})=(\boldsymbol{a},T\boldsymbol{b}). \tag{23}$$

One can notice now, that for any two-qubit state ϱ, one can find a product unitary transformation $U_1\otimes U_2$ which will transform ϱ to a form with *diagonal* T. This statement follows from the fact that for any 2×2 unitary transformation U, there is a unique 3×3 rotation O such that [22]

$$U\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}U^\dagger=(O\hat{\boldsymbol{n}})\cdot\boldsymbol{\sigma}. \tag{24}$$

Now, if a state is subjected to a $U_1\otimes U_2$ transformation, the parameters $\boldsymbol{r}$,$\boldsymbol{s}$ and T are transformed into

$$\boldsymbol{r}'=O_1\boldsymbol{r},$$
$$\boldsymbol{s}'=O_2\boldsymbol{s}, \tag{25}$$
$$T'=O_1TO_2^\dagger,$$

with O_i's corresponding to U_i's via formula (24). Thus, for every two-qubit state ρ, we can always find such U_1 and U_2 so that the corresponding rotations will diagonalize T [23]. Moreover, by selecting suitable rotations, one can make t_{11} and t_{22} nonpositive. In what follows, the states with diagonal T and $t_{11},t_{22}\leqslant0$ will be called *canonical.*

For the states with diagonal matrix T (hence also for the canonical states), the fully entangled fraction is given by (cf. Ref. [24])

$$f=\begin{cases}\frac{1}{4}(1+\sum_i|t_{ii}|) & \text{if } \det T\leqslant0\\ \frac{1}{4}[1+\max\limits_{i\neq k\neq j}(|t_{ii}|+|t_{jj}|-|t_{kk}|)] & \text{if } \det T>0.\end{cases} \tag{26}$$

One can show now [19,24] that if $\det T\geqslant0$, then $f\leqslant1/2$, i.e., f belongs to the classical region. Thus, while analyzing f in the quantum region, it will be convenient to investigate a relatively simple function $N(\varrho)$, instead of a more involved matrix T. Function $N(\varrho)$ is given by

$$N(\varrho)=\sum_i|t_{ii}|. \tag{27}$$

It has the following important properties: (i) $f(\varrho)=\frac{1}{4}[1+N(\varrho)]$ for $f\geqslant\frac{1}{2}$; (ii) $N(\varrho)\leqslant1$ if and only if $f\leqslant\frac{1}{2}$. It then contains all the information necessary to analyze f.

B. Canonical form in terms of the matrix elements

By applying the formula for t_{ij}, one can easily show that diagonality of T is equivalent to the following conditions for the matrix elements of ϱ written in the standard basis ($|1\rangle=|00\rangle,|2\rangle=|01\rangle$, etc.):

$$\varrho_{12}=\varrho_{34}, \tag{28}$$

$$\varrho_{14}=\varrho_{32}, \tag{29}$$

$$\varrho_{23}\quad\text{and}\quad\varrho_{14}\quad\text{are real.} \tag{30}$$

Moreover, since $t_{11}=2(\varrho_{14}+\varrho_{23})$ and $t_{22}=2(\varrho_{23}-\varrho_{14})$, the condition $t_{11},t_{22}\leqslant0$ is equivalent to

$$\varrho_{23}\leqslant0, \tag{31}$$

$$|\varrho_{23}|\geqslant|\varrho_{14}|. \tag{32}$$

Thus any state ϱ can be locally rotated to a form with matrix elements satisfying the above constraints. This gives the following expression for $N(\varrho)$:

$$N(\varrho)=|1-2(\varrho_{22}+\varrho_{33})|-2\varrho_{23}. \tag{33}$$

Now, for

$$\varrho_{22}+\varrho_{33}\geqslant\frac{1}{2}, \tag{34}$$

we have $t_{33}\leqslant 0$ hence $\det T\leqslant 0$. Consequently, by Eq. (26) the fully entangled fraction is given by

$$f(\varrho)=\frac{1}{4}[1+N(\varrho)]=\frac{1}{2}(\varrho_{22}+\varrho_{33}-2\varrho_{23}). \tag{35}$$

Then, with $-2\varrho_{23}$ large enough, one has $f\geqslant 1/2$ and f is attained on singlet $\psi_-: f=\langle\psi_-|\varrho|\psi_-\rangle$.

C. Local action which improves *f*

With the canonical form of ϱ at hand, it is not all that difficult to eventually find examples of states with improvable *f*. After some trials, we focused our attention on a simple family of states which in their canonical form have $\varrho_{24}=\varrho_{13}=0$:

$$\varrho=\begin{bmatrix}\varrho_{11} & 0 & 0 & \varrho_{14}\\ 0 & \varrho_{22} & -p_{23} & 0\\ 0 & -p_{23} & \varrho_{33} & 0\\ \varrho_{14} & 0 & 0 & \varrho_{44}\end{bmatrix}. \tag{36}$$

Here $p_{23}\geqslant 0$ and ϱ_{14} are real. We assumed also that ϱ satisfies the condition (34) and that $p_{23}\geqslant(1-\varrho_{22}-\varrho_{33})/2$, so that the state has $f=\langle\psi_-|\varrho|\psi_-\rangle\geqslant 1/2$. Explicitly, *f* is given by

$$f(\varrho)=\frac{1}{2}(\varrho_{22}+\varrho_{33}+2p_{23}). \tag{37}$$

We know (see Sec. III) that bistochastic maps cannot improve *f*. So, to improve it, we must try a nonbistochastic map. A possible simple candidate is, e.g., a map which acts on Bob's qubit and transforms it as follows:

$$\varrho_B\rightarrow\tilde{\varrho}_B=\Lambda(\varrho)=W_0\varrho_B W_0^\dagger+W_1\varrho_B W_1^\dagger, \tag{38}$$

where the operators W_i are given by

$$W_1=\begin{bmatrix}1 & 0\\ 0 & \sqrt{p}\end{bmatrix},\qquad W_2=\begin{bmatrix}0 & \sqrt{1-p}\\ 0 & 0\end{bmatrix}. \tag{39}$$

It is easy to check that W_i's satisfy $W_1^\dagger W_1+W_2^\dagger W_2=I$, hence the operation is trace preserving. Moreover, one can notice that Λ can be regarded as resulting from the interaction of a two-level atom (Bob's qubit) with electromagnetic field (an environment). Such an interaction produces the following transitions:

$$|0\rangle_a|0\rangle_e\rightarrow|0\rangle_a|0\rangle_e, \tag{40}$$

$$|1\rangle_a|0\rangle_e\rightarrow\sqrt{p}|0\rangle_a|1\rangle_e+\sqrt{1-p}|1\rangle_a|0\rangle_e, \tag{41}$$

where the subscripts *a* and *e* denote atomic and field states, respectively. The parameter *p* is then interpreted as the probability of photon emission from the atom in its upper state $|1\rangle_a$. This kind of interaction is called the *amplitude damping channel* and one can check [25] that, if repeatedly applied to a qubit, it produces an exponential decay characteristic to spontaneous emission. The completely positive map Λ is then obtained from the amplitude damping channel by tracing out the environment variables [16].

Let us then put $\sqrt{p}=\sin\theta$ and apply transformation (38) to Bob's part of the total (two-qubit) system. The two-qubit operator corresponding to W_i is $A_i\equiv I\otimes W_i$ and, consequently, we obtain

$$\varrho\rightarrow\varrho'=A_1\varrho A_1^\dagger+A_2\varrho A_2^\dagger \tag{42}$$

with

$$A_1=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \cos\theta\end{bmatrix} \tag{43}$$

and

$$A_2=\begin{bmatrix}0 & \sin\theta & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \sin\theta\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{44}$$

Note that like the original state ϱ, the new state $\tilde{\varrho}$ is in its canonical form, too.

$$\tilde{\varrho}=\begin{bmatrix}\varrho_{11}+\varrho_{22}\sin^2\theta & 0 & 0 & \varrho_{14}\cos\theta\\ 0 & \varrho_{22}\cos^2\theta & -p_{23}\cos^2\theta & 0\\ 0 & -p_{23}\cos^2\theta & \varrho_{33}+\varrho_{44}\sin^2\theta & 0\\ \varrho_{14}\cos\theta & 0 & 0 & \varrho_{44}\cos^2\theta\end{bmatrix}. \tag{45}$$

The change of *f* associated with the transformation is now given by $\Delta_B=\langle\psi_-|\tilde{\varrho}|\psi_-\rangle-f(\varrho)$. A simple calculation shows that

$$\Delta_B=(1-\cos\theta)\left[\frac{1+\cos\theta}{2}(\varrho_{44}-\varrho_{22})-p_{23}\right]. \tag{46}$$

Here, the index *B* indicates that Bob's qubit has been transformed. One can check that if one transforms Alice's qubit instead of Bob's then the resulting Δ_A is given by

$$\Delta_A=(1-\cos\theta)\left[\frac{1+\cos\theta}{2}(\varrho_{44}-\varrho_{33})-p_{23}\right]. \tag{47}$$

Finally, one can swap places of 1 and $\cos\theta$ on the diagonal of the first transformation matrix A_1 and adjust A_2 accordingly. This, translated into changes of *f*, result in expressions like Eq. (46) and (47) but with ϱ_{44} substituted by ϱ_{11}. In other words, single qubit, trace preserving transformations like that defined by Eq. (42) can improve fidelity of states in form (29) provided that

$$[\max(\varrho_{11},\varrho_{44})-\min(\varrho_{22},\varrho_{33})]-p_{23}\geqslant 0. \quad (48)$$

The maximal increase $\Delta=\max\{\Delta_A,\Delta_B\}$ achievable in this way is

$$\Delta=\frac{[\max(\varrho_{11},\varrho_{44})-\min(\varrho_{22},\varrho_{33})-p_{23}]^2}{2[\max(\varrho_{11},\varrho_{44})-\min(\varrho_{22},\varrho_{33})]}. \quad (49)$$

To obtain a more clear picture of the situation, let us write the diagonal elements of ϱ as

$$\varrho_{11}=\frac{1-\varepsilon-\gamma}{4} \quad \varrho_{44}=\frac{1-\varepsilon+\gamma}{4}, \quad (50)$$

$$\varrho_{22}=\frac{1+\varepsilon-\delta}{4} \quad \varrho_{33}=\frac{1+\varepsilon+\delta}{4}. \quad (51)$$

To satisfy $(\varrho_{22}+\varrho_{33}+2p_{23})\geqslant 1$ [so that $f(\varrho)=\langle\psi_-|\varrho|\psi_-\rangle\geqslant 1/2$], one needs a non-negative ε and

$$\frac{1-\varepsilon}{4}\leqslant p_{23}\leqslant\frac{1}{4}\sqrt{(1+\varepsilon)^2-\delta^2} \quad (52)$$

(the upper limit for p_{23} guaranties positivity of ϱ). Thus the method improves f on states with $0<\varepsilon<1$ and $|\gamma|+|\delta|-2\varepsilon>4p_{23}$. One can easily check that in this class, the ''most improvable'' border state ($4p_{23}=1-\varepsilon$, i.e., $f=1/2$) is

$$\varrho=\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&3-2\sqrt{2}&1-\sqrt{2}&0\\0&1-\sqrt{2}&1&0\\0&0&0&2\sqrt{2}-2\end{bmatrix}. \quad (53)$$

Since $f(\varrho)=1/2$ then standard teleportation scheme using ϱ does not offer any better fidelity than classical. On the other hand, if we transform ϱ by transformation (42) with $\cos\theta=(\sqrt{2}-1)/(4\sqrt{2}-5)$ (this choice maximizes Δ), then the new state still satisfies the condition (34), and we obtain $f(\tilde{\varrho})\approx 0.53>1/2$. The new state can then be used for teleportation with nonclassical fidelity

$$\mathcal{F}\approx\frac{2.06}{3}>\frac{2}{3}. \quad (54)$$

In other words, the state ϱ gets ''better'' when corrupted by environment. The improvement is small, nevertheless it is significant. It changes the character of the state: from non-teleporting to teleporting.

While analyzing this result, one may notice that the states with the fully entangled fraction improvable by the map (42) form a rather restricted class. In particular, this map cannot increase the entangled fraction of states like

$$\varrho=\frac{1}{2}|\psi_-\rangle\langle\psi_-|+\frac{1}{2}|00\rangle\langle 00|.$$

It would then be very interesting to provide a complete characterization of the class of states which allow to improve fidelity by some local process, as well as the class of local processes capable to improve fidelity for some states. This task is, however, beyond the scope of this paper.

V. CONCLUSIONS

We have examined the problem of optimal teleportation fidelity with given bipartite quantum states. To this end, we investigated a possibility of increasing the fully entangled fraction by means of trace preserving LOCC operations and discovered a class of LOCC operations which nontrivially increase f on some of the two-qubit states. To a surprise, the successful operations do not represent any sophisticated action of Alice or Bob. Instead, they result from a common (dissipative) interaction between the teleporting state and the local environment. The unexpected conclusion then is that a dissipative interaction, normally associated with the destruction of quantum teleportation, can sometimes facilitate it.

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