Building multiparticle states with teleportation

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We describe a protocol which can be used to generate any *N*-partite pure quantum state using Einstein-Podolsky-Rosen pairs. This protocol employs only local operations and classical communication between the *N* parties (*N*-LOCC). In particular, we rely on quantum data compression and teleportation to create the desired state. The protocol can be used to obtain upper bounds for the bipartite entanglement of formation of an arbitrary *N*-partite pure state, in the asymptotic limit of many copies. We apply it to a few multipartite states of interest, showing that in some cases it is not optimal. Generalizations of the protocol are developed which are optimal for some of the examples we consider, but which may still be inefficient for arbitrary states.

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I. INTRODUCTION

Quantum entanglement represents a resource that can be used for many applications, such as teleportation [1], superdense coding [2], quantum cryptographic key distribution [3], and entanglement-enhanced communication complexity [4]. Much progress has been made in the analysis of bipartite entanglement of pure and mixed states. The characterization of multiparticle entangled states is, however, much harder to attain, as some key theorems that apply to the bipartite states cannot be easily extended to the multipartite case.

In the case of bipartite pure states a single satisfactory measure of entanglement has been found [5,6]: the partial von Neumann entropy of one of the parties. This quantity equals both the state's distillable entanglement [the number of Einstein-Podolsky-Rosen (EPR) pairs that can be asymptotically obtained from it using only classically coordinated local operations] and the state's entanglement of formation (the number of EPR pairs necessary to asymptotically build the state with classically coordinated local operations). The characterization of entanglement in multiparticle states is still an elusive goal. There seems to be qualitatively different kinds of entanglement in *N*-partite states, making it hard to even find higher bounds for the number of different kinds of entangled states necessary to build N-partite states reversibly [7].

In this work we describe a protocol for creating any *N*-partite pure state $|\psi\rangle$ using only local operations and classical communication between the *N* parties (*N*-LOCC), and a certain number of EPR pairs. As discussed before [8], any measure of entanglement must be nonincreasing under *N*-LOCC. This will enable us to identify the number of EPR pairs used as an upper bound for the bipartite entanglement of formation $E_F(|\psi\rangle)$. It is likely that multipartite states will be built from EPR pairs in future practical applications of multiparticle entanglement; our protocol would then provide bounds for the efficiency of such a procedure. We also hope that our approach may help clarify the relations between bipartite and multipartite entanglements, still a problematic theoretical issue.

Our protocol needs to rely on an asymptotically large number of EPR pairs, creating a correspondingly large number of copies of the desired *N*-partite state $|\psi\rangle$. $E_F(|\psi\rangle)$ will then be bounded from above by the number of EPR pairs used per copy of $|\psi\rangle$ generated. We note that in general the procedure will be irreversible.

II. PROTOCOL P1

In this section we describe a protocol that builds a large number of copies of an arbitrary *N*-partite pure state using *N*-LOCC and EPR pairs only. Let us first see how it would work for an arbitrary three-qubit pure state $|\psi\rangle$.

The basic idea is very simple, and consists of distributing the state using a series of teleportations, keeping track of the amount of EPR pairs used in each step. Let us suppose that we start with a large number *M* of copies of $|\psi\rangle$ at Charlie's laboratory. Let us label the three qubits according to their von Neumann entropies, qubit 1 being the one with least entropy S_1, and so on in ascending order. The first step is to do quantum data compression [9] on subensembles corresponding to qubits 1 and 2, resulting in two groups of particles asymptotically containing MS_1 and MS_2 maximally compressed qubits.

Now Charlie teleports the set corresponding to qubit 1 to Alice and the other set to Bob. Thanks to the data compression we performed, these two teleportation steps will require only MS_1 EPR pairs shared with Alice and MS_2 with Bob. To end the process we decompress the sets at Alice and Bob, asymptotically recovering *M* qubits at each party with negligible error. The resulting state consists of *M* copies of $|\psi\rangle$ distributed among the three parties. The above discussion shows us that an arbitrary three-partite pure state can be built using 3-LOCC and a bipartite entanglement of S_1+S_2 EPR pairs, if we consider the manipulation of a large number of copies of the state.

This protocol can be simply generalized for any number of parties $N>3$. To see how, let us first recall the subadditivity property of von Neumann entropies: if *A* and *B* are subsystems of a larger system *AB*, their entropies must obey the inequality

$$S_A+S_B \geqslant S_{AB}.$$

In order to create the *N*-partite distributed state $|\psi\rangle$ using our protocol we will need to perform $N-1$ teleportations. Each

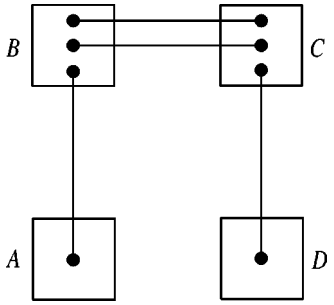

FIG. 1. This four-partite pure state consists of four EPR pairs shared between parties *A*, *B*, *C*, and *D*. A naive approach would involve teleporting each party's subsystem from party *C*, using a total of $S_A+S_B+S_D=1+3+1=5$ EPR pairs per copy. Protocol P1 invokes the most economical sequence of teleportations, one of which in this case is: first subsystem *AB* from site *C*, and then the other two teleportations, spending the optimal amount of $S_{AB}+S_A+S_D=2+1+1=4$ EPR pairs.

teleportation takes the compressed subensemble corresponding to subsystem ρ_X (which may include one or more of the final parties' subsystems we want to distribute) from one party to another, using S_X EPR pairs in the process. Due to the subadditivity property of entropy, the most economical series of teleportations will, in general, involve teleporting larger subparts of $|\psi\rangle$ first, in order to save on the number of EPR pairs used in total. As an example, in order to distribute a state among four parties *A*, *B*, *C*, and *D*, it may be advantageous to first teleport the compressed subsystem *CD* from site *A* to site *C*, and then teleport subsystems *B* and *D* to their respective sites.

We are interested in finding the best series of teleportations from a list of all possible sequences of $N-1$ teleportations that distribute the *N*-partite state $|\psi\rangle$ from a single location. This optimal series of teleportations can be found through an exhaustive search and constitutes what we shall call protocol P1 for creating $|\psi\rangle$; the number of EPR pairs used per copy of state $|\psi\rangle$ created will be called $\mathrm{P1}(|\psi\rangle)$ for short. An example of the use of protocol P1 for a simple four-partite system can be seen in Fig. 1.

III. RESULTS FOR SOME CLASSES OF STATES

In this section we calculate $\mathrm{P1}(|\psi\rangle)$ for some multipartite states of interest. We will see that protocol P1 is not optimal, at least for some classes of states. $\mathrm{P1}(|\psi\rangle)$ will then represent an upper bound for $E_F(|\psi\rangle)$. $\mathrm{P1}(|\psi\rangle)$ will also be an upper bound for the amount of bipartite entanglement distillable from $|\psi\rangle$, as this quantity is bounded from above by E_F.

The tripartite case is especially simple: if the three subsystems of $|\psi\rangle$ have entropies S_1, S_2, and S_3 (labeling the systems such that $S_1 \leqslant S_2 \leqslant S_3$), then $\mathrm{P1}(|\psi\rangle)=S_1+S_2$ EPR pairs. In the case of an *N*-partite state $|\phi\rangle$ $(N>3)$ there is not such a closed formula for $\mathrm{P1}(|\phi\rangle)$; as we discussed above, in order to find $\mathrm{P1}(|\phi\rangle)$ we will need to find the most economical series of $N-1$ teleportations that will distribute $|\phi\rangle$.

A. *N*-GHZ states

Let us apply protocol P1 to build a generalized *N*-GHZ state

$$|N\text{-GHZ}\rangle=\frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle+|1^{\otimes N}\rangle), \tag{1}$$

which is a maximally entangled state of *N* qubits, also known as a *N*-cat state. If *X* is a nontrivial subset of the parties, then the reduced density matrix of subset *X* is defined as

$$\rho_X(|\psi\rangle)=\mathrm{Tr}_{\overline{X}}(|\psi\rangle\langle\psi|), \tag{2}$$

where $\overline{X}$ denotes the complement of set *X*.

The *N*-GHZ states have the property that for every nontrivial subset *X* we have $S_X(|\psi\rangle)=1$. This means that we use exactly one EPR pair in each teleportation step, no matter which subsystem of $|N\text{-GHZ}\rangle$ we choose to teleport. Therefore, all sequences of teleportations that distribute the state $|N\text{-GHZ}\rangle$ will use exactly the same amount of EPR pairs, this number being

$$\mathrm{P1}(|N\text{-GHZ}\rangle)=(N-1). \tag{3}$$

Since all teleportation sequences are equivalent, we may as well choose one in particular: let us consider the one that distributes each subsystem directly to its definitive party, thus teleporting $N-1$ qubits from a single location. Since each qubit is in a maximally mixed state, the quantum data compression would be of no use, making it unnecessary for us to operate on multiple copies of the state.

This establishes the correspondence between protocol P1 and the one by Zeilinger *et al.* [10] for building a single copy of a *N*-GHZ state of *N* qubits from $N-1$ EPR pairs. Protocol P1 is as efficient as Zeilinger *et al.*'s but has a different physical meaning: we are simply teleporting each party's subsystem from a single location, using a total of $N-1$ EPR pairs.

The above discussion generalizes trivially for the case of *N*-GHZ states in which each party possesses *k*-level subsystems instead of qubits. Incidentally, the *N*-GHZ states represent the worst case scenario for protocol P1, in the sense that a single copy of *any N*-partite state of *N* qubits can be prepared locally and distributed using $P1(|N\text{-GHZ}\rangle)$ EPR pairs. It is not known whether protocol P1 is optimal for *N*-GHZ states.

B. *N*-partite Schmidt decomposable states

These are states which can be written in Schmidt form,

$$|\psi^{ABC\cdots N}\rangle=\sum_i a_i|i^A\rangle|i^B\rangle|i^C\rangle\cdots|i^N\rangle, \tag{4}$$

where $|i^X\rangle$ represents an orthonormal basis in party *X*'s Hilbert space. Note that the coefficients a_i may be made to be real by convenient redefinition of the basis vectors. For these states all reduced density matrices ρ_X have the same von Neumann entropy of $S_{Schmidt}=-\Sigma_i a_i^2 \log_2 a_i^2$, and therefore

$$P1(|\psi^{ABC\cdots N}\rangle)=(N-1)S_{Schmidt}. \tag{5}$$

Bennett *et al.* [7] showed that a state $|\psi^{ABC\cdots N}\rangle$ can (in the asymptotic limit of many copies) be transformed reversibly by N-LOCC into a set of $S_{Schmidt}$ N-GHZ states of N qubits. Thus we may achieve the creation of $|\psi^{ABC\cdots N}\rangle$ by two different routes: either using protocol P1 directly or first creating $S_{Schmidt}$ N-GHZ states with protocol P1 and then using Bennett *et al.*'s procedure to turn them into $|\psi^{ABC\cdots N}\rangle$. We have seen above that $P1(|N\text{-GHZ}\rangle)=(N-1)$ EPR pairs, which means that the two routes demand the same number of EPR pairs for creating the state. In view of this, we see that protocol P1's optimality for creating N-GHZ states (if it can be proved) would mean that protocol P1 would be an optimal protocol for Schmidt decomposable N-partite states as well.

C. Symmetrical N-partite states of $\binom{N}{2}$ EPR pairs

Let us consider the N-partite state $|N\text{-shared}\rangle$ consisting of $\binom{N}{2}=[N(N-1)]/2$ shared EPR pairs, one between each pair of parties. (This state has also been referred to as an N-toast state after the custom of clinking glasses during a toast [7].) We choose these states here as an example of the inefficiency of protocol P1 for some classes of states.

The most economical series of teleportations that builds $|N\text{-shared}\rangle$ consists again of teleporting each party's subsystem directly from an initial party. In order to see this, let us calculate the entropies of the reduced density matrices corresponding to all nontrivial sets of parties X.

Let ρ_X denote the reduced density matrix corresponding to an arbitrary set of M parties $(1\leqslant M\leqslant N-1)$. Given this partition of the set of parties, the $\binom{N}{2}$ EPR pairs in state $|N\text{-shared}\rangle$ can be grouped in three categories: $\binom{M}{2}$ shared between parties in X; $\binom{N-M}{2}$ shared between parties in $\overline{X}$; and $M(N-M)$ shared between a party in X and another in $\overline{X}$. The trace operation over the $N-M$ parties will result in a ρ_X that is a tensor product of the $\binom{M}{2}$ EPR pairs within X, and the identity matrix corresponding to the EPR pairs that were ''broken'' between X and $\overline{X}$. Thus we have

$$\rho_X=\frac{\mathbf{I}}{2^{M(N-M)}}\otimes\left|\binom{M}{2}\text{EPR's}\right\rangle\left\langle\binom{M}{2}\text{EPR's}\right|, \tag{6}$$

where $\mathbf{I}$ denotes the identity matrix. The entropy of ρ_X can then be easily evaluated to be $S_X=M(N-M)$. S_X corresponds to the number of EPR pairs necessary to teleport subsystem ρ_X as part of protocol P1. Since protocol P1 consists of $N-1$ teleportations, we cannot hope to perform protocol P1 and use less than

$$(N-1)\min\{S_X\}=(N-1)^2$$

EPR pairs (where the minimum is taken over all possible X). This is actually achievable, as we may teleport each party's subsystem directly from an initial one, using a total number of EPR pairs equal to

$$P1(|N\text{-shared}\rangle)=(N-1)^2. \tag{7}$$

The fact that state $|N\text{-shared}\rangle$ consists of a collection of $\binom{N}{2}$ EPR pairs makes it obvious that $E_F(|N\text{-shared}\rangle)=\binom{N}{2}$, which is clearly below $P1(|N\text{-shared}\rangle)$. For these states

$$\frac{P1(|N\text{-shared}\rangle)}{E_F(|N\text{-shared}\rangle)}=\frac{(N-1)^2}{\binom{N}{2}}=\frac{2(N-1)}{N}, \tag{8}$$

which shows that our protocol is particularly inefficient for $|N\text{-shared}\rangle$ states: $P1(|N\text{-shared}\rangle)$ is as high as twice the actual bipartite entanglement of formation for large N.

IV. OTHER BOUNDS FOR E_F

We have just seen that protocol P1 is not optimal. This inefficiency prompts us to obtain bounds for E_F using other methods, as a way of assessing the protocol's shortcomings. This can be done for the specific case of N-GHZ states by appealing to a recent theorem proved by Julia Kempe [11]:

Definition: Two N-partite states are said to be N-LOCC incommensurate when they cannot be transformed into each other either way by any N-LOCC protocol.

Theorem (Kempe): Two N-partite states $|\psi\rangle$ and $|\phi\rangle$ are N-LOCC incommensurate if and only if they are not equivalent under local unitary transformations at each party.

Kempe applied this theorem to prove that $N-1$ N-GHZ states and an $|N\text{-shared}\rangle$ state are N-LOCC incommensurate. In both cases the density matrices of each party are identical and equal to

$$\rho=\frac{\mathbf{I}}{2^{(N-1)}}. \tag{9}$$

The $|N\text{-shared}\rangle$ state has inseparable ρ_X for any choice of X other than a single party [see Eq. (6)]. The state consisting of $N-1$ N-GHZ's has separable ρ_X for all choices of X. Using Kempe's theorem and the fact that local unitaries cannot increase entanglement between parties, we prove that the states under consideration are N-LOCC incommensurate. A similar result was also obtained by Bennett *et al.* in Ref. [7].

We will now argue that this result holds true for exact transformations between arbitrarily large numbers of copies of N-GHZ and EPR states. Let us consider the two following N-partite systems: the state $|\text{GHZ}\rangle$ consisting of a tensor product of $k(N-1)$ N-GHZ states of N qubits, each party possessing a subsystem with $k(N-1)$ qubits in state

$$\rho_{GHZ}=\frac{\mathbf{I}}{2^{k(N-1)}}; \tag{10}$$

and state $|\phi\rangle$ consisting of the tensor product $|N\text{-shared}\rangle^{\otimes n}$, with each party's subsystem in state

$$\rho_\phi=\frac{\mathbf{I}}{2^{n(N-1)}}. \tag{11}$$

We will have equal density matrices at each party for $|\text{GHZ}\rangle$ and $|\phi\rangle$ if $k/n=1$, making it possible for us to apply the same reasoning as we did above for the case with two GHZ

states and three EPR states. Thus we have established that $|N\text{-GHZ}\rangle^{\otimes k(N-1)}$ and $|N\text{-shared}\rangle^{\otimes n}$ are incommensurate states. In particular, this means that we need strictly more than $\binom{N}{2}/(N-1)=N/2$ EPR pairs to create one N-GHZ state, no matter how clever our N-LOCC protocol is. In all of the above we have been considering exact transformations between the states. In Refs. [5,7] the authors defined larger equivalence classes between N-partite states, allowing for slightly imperfect fidelity $(1-\varepsilon)$ in the transformation and for k/n that deviates from 1 by a small quantity δ. The question of whether the bound of $N/2$ EPR pairs per N-GHZ state created holds for the transformations considered in Refs. [5,7] is open.

Combining the lower bound of $N/2$ EPR pairs per N-GHZ with the higher bound given by protocol P1, we can write

$$\frac{N}{2}<E_F(|N\text{-GHZ}\rangle)\leqslant N-1. \tag{12}$$

It is interesting to note that the above result implies that there is a fundamental irreversibility when we transform N-partite entanglement into bipartite, and vice versa, using exact transformations of a large number of copies. For example, a 3-GHZ cannot be asymptotically converted into 3/2 of an EPR pair, yet we need strictly more than 3/2 of an EPR pair to asymptotically recreate a 3-GHZ.

Protocol P1 is obviously optimal for a trivial set of states, such as that in Fig. 1. It is, however, inefficient for states like that of Sec. III C above. For N-GHZ states, as far as we know, there is no known protocol more efficient than P1, and inequality (12) represents what seems to be the best bounds as yet for their bipartite entanglement of formation. If it can be proved that protocol P1 is optimal for N-GHZ states, then it would be optimal for Schmidt decomposable N-partite states as well, as we discussed in Sec. III B above.

V. OTHER PROTOCOLS

Protocol P1 is not the most general protocol using quantum data compression and teleportation to build multipartite states. In this section we will briefly describe three successive generalizations. These generalizations all involve taking advantage of the fact that each party's system may be considered as consisting of subsystems which can be compressed and teleported independently.

First we describe protocol P2. Take an N-partite state in the Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C\otimes\cdots\otimes\mathcal{H}_Z$ (where $A,B,\ldots,Z$ are the parties). Now, it may be the case that some or all of the spaces $\mathcal{H}_A,\mathcal{H}_B,\ldots,\mathcal{H}_Z$ can themselves be written as tensor products of smaller spaces. For example, we may write the space $\mathcal{H}_A$ as a tensor product of m spaces:

$$\mathcal{H}_A=\mathcal{H}_A^1\otimes\mathcal{H}_A^2\otimes\cdots\otimes\mathcal{H}_A^m.$$

Writing the Hilbert space in this way corresponds to dividing each party's system into subsystems. The most general way of doing this for a party A is the following. We first write $\dim(\mathcal{H}_A)$ as the product of its prime factors. If there are m such prime factors then we can regard A as consisting of m subsystems, each having the dimension of one of the prime factors. All possible subsystems of system A consist either of these m prime-dimensional subsystems, or of the composite-dimensional subsystems obtained by grouping these in different ways. To obtain the most general way of identifying subsystems we can consider performing unitary transformations at each party, either passively (on the Hilbert space) or actively on the system itself (in which case we would need to undo the transformation at a later stage).

Now let us see the steps that constitute protocol P2. We start by preparing many copies of the state we wish to build at one location. Then we consider all possible ways in which the systems $A,B,\ldots,Z$ (which must be distributed to each party) can be decomposed in a tensor product of prime-dimensional subsystems. The idea now is to analyze all possible ways in which these subsystems, or groups of them, can be successively compressed at one party, teleported and decompressed at another party. Protocol P2 is taken to be the most efficient way of building the N-partite state by such a series of teleportations of subsystems.

This protocol will be optimal for any state that consists of a tensor product of shared bipartite states, in particular for N-shared states. This becomes clear if we consider a simple example. Take a 3-shared state

$$|3\text{-shared}\rangle=\frac{1}{\sqrt{2}}(|0_{A1}0_{B1}\rangle+|1_{A1}1_{B1}\rangle)\otimes\frac{1}{\sqrt{2}}(|0_{A2}0_{C1}\rangle+|1_{A2}1_{C1}\rangle)\otimes\frac{1}{\sqrt{2}}(|0_{B2}0_{C2}\rangle+|1_{B2}1_{C2}\rangle). \tag{13}$$

We have seen that protocol P1 is inefficient at creating N-shared states; protocol P1 would take four EPR pairs to create this tensor product of three EPR pairs. We write system A as being composed of two two-dimensional subsystems A_1 and A_2, system B as being composed of subsystems B_1 and B_2, and system C as being composed of subsystem of C_1 and C_2. This state can be built in three teleportation steps. We start with many copies of $|3\text{-shared}\rangle$ at location A. First we teleport system C_1 to location C (since subsystem C_1 is maximally mixed, there is no need to compress it). Then we take subsystems B_1, B_2, and C_2 and, regarding them as one subsystem, we compress and teleport them to location B, where they can be decompressed. Finally, we teleport subsystem C_2 from location B to location C (again, there is no need for compression). Each of these three steps requires one EPR pair per copy of $|3\text{-shared}\rangle$, and hence this procedure is optimal, constituting an example of protocol P2. It is clear that this protocol will be optimal for any state consisting of shared bipartite states.

Protocol P2 reduces to protocol P1 in the case where all systems $A,B,\ldots,Z$ have a prime number of dimensions (thus being not decomposable). This observation motivates protocol P3. In protocol P3 we allow the introduction of ancillary systems $R_A,R_B,\ldots,R_Z$ prepared in some known state. Let system A' consist of A and R_A, and similarly for $B',C',\ldots Z'$. Protocol P3 consists of applying protocol P2

to the primed systems, and also optimizing over all possible ancilla choices. The introduction of ancillas allows for greater flexibility of manipulation of the prime-dimensional subsystems, as unprimed systems with prime dimension may result in primed decomposable systems.

It is easy to find examples of states for which protocol P3 is more efficient than both protocols P1 and P2. For example, let us consider the following tripartite state closely resembling the $|3\text{-shared}\rangle$ state, but in a $5\times5\times5$ dimensional Hilbert space:

$$|\phi\rangle=\sqrt{1-\varepsilon}|3\text{-shared}\rangle+\sqrt{\varepsilon}|4,4,4\rangle$$

$$=\sqrt{1-\varepsilon}\frac{1}{\sqrt{8}}\begin{bmatrix}|0,0,0\rangle+|0,1,1\rangle+|1,0,2\rangle+|1,1,3\rangle\\+|2,2,0\rangle+|2,3,1\rangle+|3,2,2\rangle+|3,3,3\rangle\end{bmatrix}$$

$$+\sqrt{\varepsilon}|4,4,4\rangle, \tag{14}$$

where we have rewritten the $|3\text{-shared}\rangle$ state in a different form, adding a term proportional to $\sqrt{\varepsilon}$ and orthogonal to the space spanned by the $|3\text{-shared}\rangle$ state.

Eigenvalues of density matrices that are continuous functions of a parameter ε are themselves continuous functions of ε [12]. Hence the entropies of the partial density matrices obtained from ρ_ϕ will differ from those of $|3\text{-shared}\rangle\langle 3\text{-shared}|$ by arbitrarily small amounts, for suitable values of ε. We are now dealing with subsystems in a prime-dimensional [Eq. (5)] Hilbert space, which means that we are unable to identify subsystems in each party's system and apply protocol P2. Therefore,

$$\mathrm{P2}(|\phi\rangle)=\mathrm{P1}(|\phi\rangle)=S(\rho_A)+S(\rho_B),$$

which is arbitrarily close to 4, for small ε.

Protocol P3 improves on this by adding a three-qubit ancilla to each five-level system. We then map the state of each five-level system into its ancilla through the unitary operation that takes (for Alice's system)

$$|0_{Alice}\rangle|0_{R1},0_{R2},0_{R3}\rangle\rightarrow|0_{Alice}\rangle|0_{R1},0_{R2},0_{R3}\rangle,$$

$$|1_{Alice}\rangle|0_{R1},0_{R2},0_{R3}\rangle\rightarrow|0_{Alice}\rangle|0_{R1},1_{R2},0_{R3}\rangle,$$

$$|2_{Alice}\rangle|0_{R1},0_{R2},0_{R3}\rangle\rightarrow|0_{Alice}\rangle|1_{R1},0_{R2},0_{R3}\rangle,$$

$$|3_{Alice}\rangle|0_{R1},0_{R2},0_{R3}\rangle\rightarrow|0_{Alice}\rangle|1_{R1},1_{R2},0_{R3}\rangle,$$

$$|4_{Alice}\rangle|0_{R1},0_{R2},0_{R3}\rangle\rightarrow|0_{Alice}\rangle|0_{R1},0_{R2},1_{R3}\rangle,$$

and similarly for Bob's and Charlie's systems. This amounts basically to increasing the dimensionality of each system from 5 to 8, as we may discard the original systems and deal with the ancillas instead. Since 8 is a composite number, now we can apply protocol P3, which consists of finding the optimal way of distributing this nine-qubit system among Alice, Bob, and Charlie through teleportation. It suffices to say that the optimal way will be similar to the one we adopted when we applied protocol P2 to the $|3\text{-shared}\rangle$ state, with the difference that now we have one extra qubit at each party (like ancillary bit $R3$ in the example above) which is minimally entangled with the rest of the system. By the same continuity argument we see that $\mathrm{P3}(|\phi\rangle)$ is as close as we like to $\mathrm{P2}(|3\text{-shared}\rangle)=3$, for small ε. This provides us with an example for which protocol P3 is better than both protocols P1 and P2.

The introduction of the ancillas in protocol P3 was not done in the most general way possible. We considered local operations at each party that entangled ancillas to the systems corresponding to a single copy of the state we want to prepare. As we have argued, this allows for a more general manipulation of the state, and perhaps to a more efficient series of teleportations that build it. We may, however, operate with a large number of copies of the state, introducing ancillas which couple to a certain number g of copies, instead of a single one. Then our basic units would consist of one ancilla for each set of g copies of system A, one for each g copies of system B, and so on up to the last system Z. Protocol P4 can then be defined as the application of protocol P2 to these larger units, optimizing over g and ancilla sizes. Protocol P4 is more general than protocol P3, and may be more efficient for some states.

VI. CONCLUSIONS

We have described a protocol that can be used (in the asymptotic limit of many copies) to build any N-partite pure quantum state. Protocol P1 uses quantum data compression and teleportation to distribute a large number of copies of the state among the N parties, consuming bipartite entanglement in the form of EPR pairs.

The efficiency of the protocol depends on the state we want to create. It is optimal for some simple states (such as that in Fig. 1), but can be shown to be inefficient for N-shared states, discussed in Sec. III C. Its efficiency for arbitrary states is hard to evaluate, as there are not many results for the exact bipartite entanglement of formation E_F of N-partite states. Lower bounds for E_F of N-GHZ states are presented for comparison with protocol P1's results. We also show that proof of optimality of protocol P1 for N-GHZ states would imply its optimality for Schmidt decomposable states as well.

We consider some generalizations of protocol P1, which can be shown to be more efficient than protocol P1 for some classes of states (such as N-shared states). It would be interesting to investigate in more depth the bounds for E_F obtained by the proposed generalizations.

Note added: Recently, some new interesting results about multiparticle states were obtained; in particular, it was proved that the conversion of EPR pairs to N-partite states ($N>2$) is an irreversible process, even if we consider slightly imperfect transformations in the asymptotic limit [7,13].

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